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Compensated integrability on tori; *a priori* estimate for space-periodic gas flows

Intégrabilité par compensation sur un tore ; Estimation a priori *pour les écoulements gazeux périodiques en espace*

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Abstract. We extend our theory of Compensated Integrability of positive symmetric tensors, to the case where the domain is the product of a linear space $\mathbb{R}^k$ and of a torus $\mathbb{R}^m/\Lambda$, Λ being a lattice of $\mathbb{R}^m$. We apply our abstract results in two contexts, for which $k = 1$ is associated with a time variable, while $m = d$ is a space dimension. On the one hand to d-dimensional inviscid gas dynamics, governed by the Euler equations, when the initial data is space-periodic; we obtain an *a priori* space-time estimate of our beloved quantity $\rho^{\frac{1}{d}}p$. On the other hand to hard spheres dynamics in a periodic box $L\mathbb{T}_d$. We obtain a weighted estimate of the average number of collisions per unit time, provided that the "linear density" Na/L (N particles of radius a) is smaller than some threshold.

Résumé. Nous étendons notre théorie d'Intégrabilité par Compensation au cas des domaines $\mathbb{R}^k \times (\mathbb{R}^m/\Lambda)$, produits d'un facteur linéaire et d'un tore plat. Nous appliquons les résultats abstraits à deux contextes, pour lesquels $k = 1$ est associé à une variable de temps, tandis que $m = d$ est la dimension de l'espace physique ambiant. Le premier est la dynamique des gaz non visqueux, gouvernée par les équations d'Euler, lorsque les données initiales sont périodiques en espace. Nous obtenons une estimation *a priori* de notre quantité favorite $\rho^{\frac{1}{d}}p$. Le second est la dynamique des sphères dures, dans une boîte périodique $L\mathbb{T}_d$. Nous obtenons une estimation pondérée du nombre moyen de collisions par unité de temps, pourvu que la « densité linéique » Na/L (N particules de rayon a) soit inférieure à un certain seuil.

Keywords. Compensated integrability, perfect gas, billiard, periodic data.

Mots-clés. Intégrabilité par compensation, gaz parfait, billard, données périodiques.

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Notations

The space of $n \times n$ symmetric matrices with real entries is $\mathbf{Sym}_n$. The open cone of positive definite symmetric matrices is $\mathbf{SPD}_n$, its closure being $\mathbf{Sym}_n^+$. The latter defines the natural order $\prec$ in $\mathbf{Sym}_n$. Transposition of matrices and vectors is written $A \mapsto A^T$. The cofactor matrix of $M \in \mathbf{M}_n(\mathbb{R})$ is $\widehat{M}$; we recall that $\widehat{M}M^T = M^T\widehat{M} = (\det M)I_n$, and $\det\widehat{M} = (\det M)^{n-1}$. When $a, b \in \mathbb{R}^n$, $a \otimes b$ is the $n \times n$ matrix with entries $a_i b_j$. The operator norm of an endomorphism u is $|u|_{\rm op}$.

Partial derivatives in $\mathbb{R}^n$ are denoted $\partial_1, \dots, \partial_n$. The inner product of $\mathbb{R}^n$ is either $x \cdot y$ or $\langle x, y\rangle$, and the unit sphere is $\mathbb{S}_{n-1}$. The Hessian matrix of a function $\theta : \mathbb{R}^n \to \mathbb{R}$ is $\mathrm{D}^2\theta$.

Given an open domain $U \subset \mathbb{R}^n$, the space $\mathscr{M}(U)$ of finite Radon measures over U is equipped with the norm

$$\|\mu\|_{\mathscr{M}} = \sup\{\mu(\phi) \mid \phi \in \mathscr{C}_b(U) \text{ and } \forall x \in U, |\phi(x)| \le 1\}.$$

The notation $\mathscr{M}$ is also used in the context of vector-valued measures. The cone of non-negative test functions in $\mathscr{D}(U)$ is $\mathscr{D}_+(U)$. The norm in $L^p(U)$ is denoted $\|\cdot\|_p$.

An inequality $F(X) \le_n G(X)$ between functionals F and G means that there exists a constant $C(n) \in (0, +\infty)$ such that $F(X) \le C(n)G(X)$ for every argument X under consideration. The parameter n is in general the dimension of some underlying space.

The canonical m-dimensional torus is $\mathbb{T}_m = (\mathbb{R}/2\pi\mathbb{Z})^m$. If $\Lambda \subset \mathbb{R}^n$ is a lattice, and the measurable function $f : \mathbb{R}^n \to \mathbb{R}_+$ is Λ-periodic, then $\fint f(x)\,dx$ denotes its mean value over the torus $\mathbb{R}^n/\Lambda$.

1. Introduction and statements

1.1. *State of the art*

Let $n \ge 2$ be an integer. The objects under consideration in Compensated Integrability (in short CI) are symmetric tensors

$$A = (a_{ij})_{1\le i,j\le n}, \qquad a_{ji} = a_{ij},$$

where the entries are finite Radon measures over an open domain $U \subset \mathbb{R}^n$. Two properties are at stake in the theory:

Divergence control. The coordinates of the row-wise Divergence,

$$(\mathrm{Div}\, A)_i := \sum_{j=1}^n \frac{\partial a_{ij}}{\partial x_j},$$

are finite measures too.

When this condition is met, we say that A is *Divergence-BV*. The space $\mathrm{DivBV}(U)$ of such tensors is equipped with the norm

$$\|A\|_{\mathscr{M}} + \|\mathrm{Div}\, A\|_{\mathscr{M}},$$

which makes it a Banach space.

If actually $\mathrm{Div}\, A \equiv 0$, then we say that A is *Div-free*.

Positive semi-definiteness. For every $\xi \in \mathbb{R}^n$, the measure

$$\xi^T A \xi = \sum_{i,j=1}^n \xi_i \xi_j a_{ij}$$

is non-negative.

When A is positive semi-definite in the sense above, its entries are absolutely continuous with respect to the positive measure $\mu = \mathrm{Tr}\, A$:

$$a_{ij} = f_{ij}\mu, \qquad f_{ij} \in L^1(\mathrm{d}\mu).$$

The tensor $F = (f_{ij})_{i,j}$ is positive semi-definite too and, since $(\det F)^{\frac1n} \in L^1(\mathrm{d}\mu)$, we may define unambiguously the positive measure:

$$(\det A)^{\frac1n} := (\det F)^{\frac1n}\mu.$$

CI tells us that $(\det A)^{\frac{1}{n}}$ enjoys a higher integrability, which is reminiscent to the Gagliardo–Nirenberg–Sobolev embedding $BV(\mathbb{R}^n) \subset L^{\frac{n}{n-1}}(\mathbb{R}^n)$. The main statement of the theory deals with the case $U = \mathbb{R}^n$:

Theorem 1 (D. S. [10, 11]). *Let A be positive semi-definite and Div-BV over $\mathbb{R}^n$. Then $(\det A)^{\frac{1}{n}}$ is actually a measurable function, belonging to $L^{\frac{n}{n-1}}(\mathbb{R}^n)$. Its norm is controlled by a functional inequality*

$$\int_{\mathbb{R}^n} (\det A)^{\frac{1}{n-1}}\,\mathrm{d}x \le c_n \|\mathrm{Div}\,A\|_{\mathcal{M}}^{\frac{n}{n-1}}. \tag{FIn}$$

The above statement includes the fact that in the Radon–Nikodym decomposition

$$A = A^{\mathrm{ac}}\,\mathrm{d}x + A^s,$$

the singular part satisfies $(\det A^s)^{\frac{1}{n}} \equiv 0$, μ-almost everywhere. In other words, $\mathrm{rk}\,A^s \le n-1$, μ-almost everywhere. This provides an example of a much more general result established in [2].

The best constant c_n in (FIn) is actually known. The equality case occurs for instance if $A(x) = \chi_B(x) I_n$ where χ_B is the characteristic function of a ball B. More generally, if $V \subset \mathbb{R}^n$ is open and $A(x) = \chi_V(x) I_n$, then (FIn) is nothing but the Isoperimetric Inequality:

$$\left(\frac{\mathrm{vol}\,V}{\mathrm{vol}\,B}\right)^{\frac{1}{n}} \le \left(\frac{\mathrm{area}\,\partial V}{\mathrm{area}\,\partial B}\right)^{\frac{1}{n-1}}.$$

We are interested herebelow in the case where the tensor A, defined over $\mathbb{R}^n$, is periodic :

$$\forall \gamma \in \Gamma, \qquad A(\cdot + \gamma) = A.$$

Hereabove, Γ is a discrete subgroup of $\mathbb{R}^n$, of rank $m \le n$. If $m = n$, Γ is a lattice, but we wish to allow the possibility that $m < n$. Thus we shall be able to treat the physically relevant situation of a space-periodic process in the physical space $\mathbb{R}^d$, which evolves as time goes on; then we have $n = 1 + d$ and $m = d$.

The periodicity prevents A from being Div-BV in $\mathbb{R}^n$; the measures a_{ij} and $(\mathrm{Div}\,A)_i$ have infinite mass. Therefore Theorem 1 does not apply to A. Instead we shall assume that their masses per cell,

$$\|a_{ij}\|_{\mathcal{M}(\mathbb{R}^n/\Gamma)}, \quad \|(\mathrm{Div}\,A)_i\|_{\mathcal{M}(\mathbb{R}^n/\Gamma)}$$

are finite. The only known result about periodic tensors so far, which tells us that the function $\det^{\frac{1}{n-1}}$ is Div-quasiconcave over $\mathbf{Sym}_n^+$, is limited to the Div-free case. It reads as follows.

Theorem 2 (D. S. [10, 11]). *Let A be symmetric, positive semi-definite over $\mathbb{R}^n$, periodic with respect to a lattice Λ, and* **Div-free**. *Then*

$$⨍_{\mathbb{R}^n/\Lambda} (\det A)^{\frac{1}{n-1}}\,\mathrm{d}x \le \left(\det ⨍_{\mathbb{R}^n/\Lambda} A\,\mathrm{d}x\right)^{\frac{1}{n-1}}.$$

Notice that the inequality with the exponents $\frac{1}{n}$, instead of $\frac{1}{n-1}$, is true even without Div-freeness; it follows from Jensen Inequality and the concavity of $\det^{\frac{1}{n}}$ over $\mathbf{Sym}_n^+$.

1.2. *Abstract statements*

Our intention below is to unify somehow Theorems 1 and 2 above. In particular, we wish to relax the hypotheses in the latter by allowing A to be Div-BV, instead of Div-free. From a physical point of view, the periodic context has the great advantage of introducing a notion of characteristic length. Whenever the data contains a velocity field, we inherit therefore a characteristic time too.

Our main result in this direction is sharp, except for the fact that the constants in the functional inequality are not. For the sake of simplicity, we begin with the simple case where the compact factor is the canonical torus :

Theorem 3 (Periodic tensors.). *Denote $n = k+m$. Let A be positive semi-definite and Div-BV over $\mathbb{R}^k \times \mathbb{T}_m$. Then $(\det A)^{\frac{1}{n}}$ is actually a measurable function, belonging to $L^{\frac{n}{n-1}}(\mathbb{R}^k \times \mathbb{T}_m)$. Its norm is controlled by the functional inequality*

$$\int_{\mathbb{R}^k\times\mathbb{T}_m} (\det A)^{\frac{1}{n-1}}\,\mathrm{d}x \le_{k,m} \left(\|\mathrm{Div}\,A\|_{\mathcal{M}(\mathbb{R}^k\times\mathbb{T}_m)} + \|\mathrm{Tr}_m A\|_{\mathcal{M}(\mathbb{R}^k\times\mathbb{T}_m)}\right)^{\frac{n}{n-1}}, \tag{FIper}$$

where $\mathrm{Tr}_m A$*, defined as the m-terms sum*

$$\mathrm{Tr}_m A = \sum_{j=k+1}^{n} a_{jj},$$

is the trace of the down-right block in the decomposition

$$A = \begin{pmatrix} A^{u\ell} \in \mathbf{M}_{k\times k} & A^{ur} \in \mathbf{M}_{k\times m} \\ A^{d\ell} \in \mathbf{M}_{m\times k} & A^{dr} \in \mathbf{M}_{m\times m}\end{pmatrix}.$$

Notice that since there are finitely many partitions $n = k+m$, we may write $\le_n$ instead of $\le_{k,m}$.

The case where the compact factor is an arbitrary torus $\mathbb{R}^m/\Gamma$ follows from the observation (see [10, Lemma 1.1] for the Div-free case) that for every $P \in \mathbf{GL}_n(\mathbb{R})$, the congruence

$$A \longmapsto B(z) := PA(P^{-1}z)P^T$$

acts over $\mathrm{DivBV}_{\mathrm{loc}}(\mathbb{R}^n)$. There holds

$$\mathrm{Div}_z\, B(z) = P(\mathrm{Div}_x\, A)(P^{-1}z).$$

This transformation brings us back to the context of Theorem 3, by choosing $Q \in \mathbf{GL}_m(\mathbb{R})$ such that $\Gamma = 2\pi Q^{-1}\mathbb{Z}^m$, and setting $P = \mathrm{diag}(I_k, Q)$. Applying (FIper) to B, we receive easily our most general estimate:

Corollary 4. *If instead A is positive and Div-BV over $\mathcal{T} := \mathbb{R}^k \times (\mathbb{R}^m/\Gamma)$, and $P = \mathrm{diag}(I_k, Q)$ with $\Gamma = 2\pi Q^{-1}\mathbb{Z}^m$, then we have*

$$\int_{\mathcal{T}} (\det A)^{\frac{1}{n-1}}\,\mathrm{d}x \le_{k,m} (\det P)^{-\frac{1}{n-1}} \left(\|P\,\mathrm{Div}\,A\|_{\mathcal{M}(\mathcal{T})} + \|\mathrm{Tr}_m PAP^T\|_{\mathcal{M}(\mathcal{T})}\right)^{\frac{n}{n-1}}, \tag{1}$$

where it may be noticed that

$$\det P = \det Q, \qquad \mathrm{Tr}_m PAP^T = \mathrm{Tr}\, QA^{dr}Q^T.$$

Strategy of the proof. Since the dual of L^n is $L^{\frac{n}{n-1}}$, Theorem 3 amounts to having the following functional inequality: for every positive semi-definite Div-BV tensor A over $\mathbb{R}^k \times \mathbb{T}_m$, and every $\psi \in \mathcal{D}_+(\mathbb{R}^k \times \mathbb{T}_m)$,

$$\left\langle (\det A)^{\frac{1}{n}}, \psi \right\rangle \le_n \|\psi\|_n (\|\mathrm{Div}\,A\|_{\mathcal{M}} + \|\mathrm{Tr}_m A\|_{\mathcal{M}}). \tag{2}$$

It is actually sufficient to have (2) whenever A is smooth (Lemma 7 below). The core of the proof proceeds by induction over the dimension m of the compact factor. The case $m = 0$ is nothing but Theorem 1. When the statement is valid at the level $m-1$ (induction hypothesis), we embed isometrically $\mathbb{T}_m$ as $\mathbb{T}_{m-1} \times \mathbb{S}_1$ in $\mathbb{R}^{m+1}$. Transporting A, we obtain a tensor A' over $\mathbb{R}^{n+1}$, supported by the cylinder $\mathbb{R}^{n-1} \times \mathbb{S}_1$, which is still Div-BV. This new tensor is periodic in $m-1$ coordinates $x_{k+1}, \dots, x_{n-1}$ only. Two important remarks must be made at this stage:

- The isometric embedding results in an extrinsic curvature of the image (the cylinder), although $\mathbb{R}^k \times \mathbb{T}_m$ itself is flat. This curvature causes $\mathrm{Div}\,A'$ to have a normal component of the form $\mathrm{Tr}(A\kappa)$, where κ is the curvature tensor, a phenomenon already observed in [13]. Here this contribution is nothing but a_{nn}.

- The tensor A' is singularly supported and so is $(\det A')^{\frac{1}{n+1}}$. Saying that the latter is a function of class L^p (for $p = 1 + \frac{1}{n}$ here) means that $(\det A')^{\frac{1}{n+1}} \equiv 0$; this can be seen directly by observing that $A'\nu \equiv 0$ where ν is the normal unit vector to the cylindrical support. Therefore applying CI directly to A' is useless.

  This leads us to improve CI, so as to handle singular contributions of the tensor under consideration. Such a step was carried out, in an even more singular situation, in [12]. Doing so, we must apply CI not to A' itself, but to $A' + \phi\nu \otimes \nu$ where $\phi \ge 0$ is a smooth function; this correction restores the full-rank quality of the tensor. Then we optimize the resulting estimate by choosing carefully the auxiliary function ϕ.

Since this strategy combines several difficulties, we offer in Appendix B a proof[1] that CI in dimension $n+1$ implies CI in dimension n. It covers, in a rather simple context, the aspects of i) transporting a tensor, ii) dealing with a now singularly supported tensor, and iii) adding a parametrized corrector. Being familiar with this appendix, the interesting reader will find the proof of Theorem 3 easier to follow.

1.3. *Application to gas dynamics*

The second aspect of this paper deals, as it is often the case, with the Euler equations governing the flow of and inviscid gas. Here $n = 1 + d$ and $x = (t, y)$ where t is the time variable and y stands for the space coordinates. The system reads

$$\begin{aligned}
\partial_t \rho + \operatorname{div}_y(\rho u) = 0,\\
\partial_t(\rho u) + \operatorname{Div}_y(\rho u \otimes u) + \nabla_y p = 0,\\
\partial_t \left(\frac{1}{2}\rho|u|^2 + \rho\varepsilon\right) + \operatorname{div}_y \left(\left(\frac{1}{2}\rho|u|^2 + \rho\varepsilon + p\right) u\right) = 0.
\end{aligned}$$

The mass density ρ, the pressure p and the specific internal energy ε are non-negative quantities, often related to each other by an *equation of state*. The vector field u is the fluid velocity. The two first equations above can be recast as $\operatorname{Div}_x A = 0$, where

$$A = \rho \begin{pmatrix} 1 \\ u \end{pmatrix} \otimes \begin{pmatrix} 1 \\ u \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & pI_d \end{pmatrix} \succ 0_n, \tag{3}$$

the *mass-momentum tensor*, is symmetric.

Let us recall an *a priori* estimate established for the Cauchy problem when the physical domain is $\mathbb{R}^d$, and the total mass and energy are finite, that is

$$M := \int_{\mathbb{R}^d} \rho(0, y)\, \mathrm{d}y < +\infty, \qquad E_0 := \int_{\mathbb{R}^d} \left(\frac{1}{2}\rho|u|^2 + \rho\varepsilon\right)(0, y)\, \mathrm{d}y < +\infty.$$

We say that a flow is admissible if it preserves the total mass, while the total energy is a non-increasing function of time. Using CI, and noticing that $\det A$ equals ρp^d, we proved in [10] that admissible flows satisfy

$$\int_0^{+\infty} \mathrm{d}t \int_{\mathbb{R}^d} \rho^{\frac{1}{d}} p\, \mathrm{d}y \le_d M^{\frac{1}{d}} \sqrt{ME_0}\,.$$

Herebelow we consider the alternate situation where the initial data is space-periodic, say with respect to $L\mathbb{Z}^d$, where L is a characteristic length. We define instead the mass and energy *per cell*:

$$M := \int_{\mathbb{R}^d/L\mathbb{Z}^d} \rho(0, y)\, \mathrm{d}y, \qquad E_0 := \int_{\mathbb{R}^d/L\mathbb{Z}^d} \left(\frac{1}{2}\rho|u|^2 + \rho\varepsilon\right)(0, y)\, \mathrm{d}y.$$

We define also the *mean density* $\bar\rho := L^{-d}M$. Our new estimate applies to gases which obey the equation of state (perfect gas law) $p = (\gamma - 1)\rho\varepsilon$ where $\gamma > 1$ is the *adiabatic constant*.

[1] Of course we do not need it, as CI is proved directly in every dimension. Its role is purely pedagogical.

Theorem 5. *Let a perfect gas flow be $L\mathbb{Z}^d$-periodic in the space variable, and admissible. Then it satisfies the estimate*

$$\forall T>0, \qquad \int_0^T \mathrm{d}t \int_{\mathbb{R}^d/L\mathbb{Z}^d} \rho^{\frac{1}{d}} p \,\mathrm{d}y \leq_d \gamma M^{\frac{1}{d}} \left(\sqrt{ME_0} + \frac{TE_0}{L}\right). \tag{4}$$

Interpretation. When $t \to +\infty$, (4) expresses that in the Cesàro sense ("in time average"),

$$\int_{\mathbb{R}^d/L\mathbb{Z}^d} \rho^{\frac{1}{d}} p \,\mathrm{d}y \overset{\text{Ces.}}{=} O\left(\gamma \bar{\rho}^{\frac{1}{d}} E_0\right).$$

1.4. *Collision estimate for the billiard on a torus*

Our second application concerns the dynamics of hard spheres of equal radii $a>0$ in $\mathbb{R}^d$, which interact through elastic collisions. In absence of a force field, a particle travels with constant speed along a segment, until it collides with another one. The linear momentum and kinetic energy are conserved in every collision. As long as the collisions involve only pairs of particles, and infinitely many collisions do not accumulate, the dynamics can be continued in a unique manner; these are the motions that we consider herebelow. Recall that for finitely many particles, these globally defined motions occur for almost every initial data [1]; we are not aware of such a result for space-periodic configurations.

Since Sinai's seminal paper [14], the major question of the theory has been estimating the number of collisions that happen along the dynamics. So far, the literature seems to focus on finite configurations in the whole space $\mathbb{R}^d$. Vaserstein [15] proved first that the collisions are finitely many. Huge upper bounds were derived in terms of the number N of particles : $O(N^{cN^2})$ by Burago & al. [3], then $O(N^{cN})$ by Burdzy [5]. It is known [4] from explicit constructions that the number of collisions may be as large as $2^{N/2}$. Thus Burdzy's super-exponential bound could not be improved in a better form than exponential. Nevertheless we showed in [12] that it can be converted into a quadratic bound, provided that each collision is weighted by the jump of the velocity $[v]$ of its particles (the exchange of linear momentum between both). Our estimate reads as

$$\sum_{\text{coll.}} |[v]| \leq_d N^2 \bar{v}, \tag{5}$$

where $\bar{v}$ is the standard deviation of the velocity, a constant of the motion; the sum runs over all collisions. Even though (5) does not imply the finiteness of the collision set, it tells us that all but an $O(N^2)$ collisions are negligible, in the sense that $|[v]| \ll \bar{v}$. For instance, most of the collisions of the motion exhibited in [4] must be exponentially weak.

Our method was based upon the construction of a mass-momentum tensor $\mathbb{M}$ encoding the dynamics. This Div-free tensor over $\mathbb{R}^{1+d}$ is supported by a graph formed by the trajectories of the centers of the balls, together with the segments joining the centers of colliding pairs. Compensated Integrability does not apply directly because $(\det\mathbb{M})^{\frac{1}{1+d}} \equiv 0$. To overcome this flaw, we elaborated a version of CI dedicated to singular tensors, which we recall below (Theorem 10).

The situation is rather different for space-periodic configurations, because the particles are infinitely many. In particular, we do not expect the collisions *per cell* to be finitely many over the whole time interval $(0,+\infty)$. However, using the same tensor $\mathbb{M}$, and a version of Theorem 10 adapted to tensors over $\mathbb{R}^k \times \mathbb{T}_m$ (we use actually the pair $(k=1, m=d)$), see Theorem 11, we establish here a weighted estimate of the collisions over finite time intervals. Interestingly, our result involves an assumption about the "linear" density aN/L, where L is the length of the torus.

Theorem 6. *There exists two constants $\kappa_d, C_d \in (0,+\infty)$ such that for every $L, T>0$ and hard sphere dynamics over $(0,T)\times L\mathbb{T}_d$ satisfying*

$$Na < \kappa_d L, \tag{6}$$

one has the estimate

$$\sum_{\text{coll.}}^{(0,T)} |[v]| \le C_d N^2 \bar{v}\left(1+\frac{\bar{v}T}{L}\right). \tag{7}$$

Herabove, the summation runs over all the collisions happening in $(0,T)\times L\mathbb{T}_d$.

Comments.

- Estimate (7) plays the same role as (4) does for perfect gases. It can be interpreted again in the Cesàro sense as $t\to+\infty$, namely

$$\sum_{\text{coll.}}^{(t,t+1)} |[v]| \overset{\text{Ces.}}{=} O\left(\frac{N^2\bar{v}^2}{L}\right), \tag{8}$$

  where $\bar{v}^2$ can be viewed as the temperature of the medium. Since there are $\binom{N}{2}$ pairs of particles, this means that, in average over a unit time interval, each pair (p,q) experiences finitely many non-negligible collisions.
- Curiously, the hypothesis (6) involves a "linear" (that is, a one-dimensional) density. It assumes that if the balls were pearls along a necklace of length $2\pi L$, their density would be less than κ_d/π. Estimate (8) seems a little pessimistic for large L, unless $d=1$. At fixed $(N,a,\bar{v},T)$ and $L\to+\infty$, one should expect an $O(L^{-d})$ instead of an $O(L^{-1})$, at least for generic initial data. But our bound is valid for arbitrary configurations, including the worst ones. Our result says that the latters are those for which all the particles move along a single line, and therefore behave as if $d=1$.
- Let us consider the applicability of the theorem to physical situations, for which we have $d=3$. The rarefied gas must consist of inert atoms like He or Ar because our analysis is limited to spherical molecules. A typical radius is $a\sim 10^{-10}$m, while the number of particles is $N=N_0L^3$, where N_0 is their number per cube meter. Because of (6), an admissible characteristic length must be smaller than

$$10^5\sqrt{\frac{\kappa_3}{N_0}}.$$

  Of course, the gas distribution is never space-periodic, and the gas is made of non-spherical molecules H_2 in general, but our analysis gives a hint of what is expected in more realistic situations, in particular when a characteristic length is given. The following table tells us that our analysis might be useless for a dense gas such as the atmosphere on earth, though it could be meaningful for gases at a much larger scale and lower density.

	particles per m^3	admissible length L (meters)
atmosphere	$3\cdot 10^{25}$	10^{-8}
interstellar cloud	$3\cdot 10^5/3\cdot 10^7$	100/1000
galactic corona	$10^{-3}/10^{-4}$	10^6

Plan of the paper. The proof of Theorem 3 is given in Section 2. That of Theorem 5 is detailed in Section 3. Section 4 is dedicated to the theory of so-called *Determinantal masses*, initiated in [12]. It culminates with Theorem 11, which is adapted to the periodic context. Its application to billiard dynamics is analyzed in Section 5. An appendix gives a short proof of Theorem 3 in the special case $k=0$. Another one shows how the technique developed in Section 2 can be used in a rather simple manner, to prove that CI in dimension $n+1$ implies the same in dimension n.

2. Proof of Theorem 3

We recall that A is a non-negative Div-BV tensor over $\mathbb{R}^k \times \mathbb{T}_m$ $(n = k + m)$.

2.1. *From smooth tensors to general ones*

Lemma 7. *If the functional inequality* (2) *is true for $\mathscr{C}^\infty$-periodic tensors $A \succ 0_n$, then it is so for all periodic Div-BV-tensors $A \succ 0_n$.*

Proof. Let $A \succ 0_n$ be Div-BV over $\mathbb{R}^k \times \mathbb{T}_m$. Let $\rho \in \mathscr{D}_+(\mathbb{R}^n)$ be even and such that $\int \rho(x)\,\mathrm{d}x = 1$, and let us denote $\rho_\epsilon(x) = \epsilon^{-n}\rho(x/\epsilon)$. The tensor $\rho_\epsilon \star A$ is smooth and Div-BV over $\mathbb{R}^k \times \mathbb{T}_m$. From the assumed smooth case of (2), we have

$$\left\langle \left(\det \rho_\epsilon \star A\right)^{\frac{1}{n}}, \psi \right\rangle \le_n \|\psi\|_n \left(\|\mathrm{Div}(\rho_\epsilon \star A)\|_{\mathscr{M}} + \|\mathrm{Tr}_m(\rho_\epsilon \star A)\|_{\mathscr{M}}\right)$$

for every $\psi \in \mathscr{D}_+(\mathbb{R}^k \times \mathbb{T}_m)$. On the right-hand side, we use

$$\|\mathrm{Div}(\rho_\epsilon \star A)\|_{\mathscr{M}} = \|\rho_\epsilon \star \mathrm{Div}\, A\|_{\mathscr{M}} \le \|\mathrm{Div}\, A\|_{\mathscr{M}}, \qquad \|\mathrm{Tr}_m(\rho_\epsilon \star A)\|_{\mathscr{M}} \le \|\mathrm{Tr}_m\, A\|_{\mathscr{M}}.$$

For the left-hand side, we recall that $\det^{\frac{1}{n}}$ is a concave function over $\mathbf{Sym}_n^+$ (see for instance [9, Theorem 6.10]). We thus have

$$\left(\det(\rho_\epsilon \star A)\right)^{\frac{1}{n}} \ge \rho_\epsilon \star (\det A)^{\frac{1}{n}}.$$

We infer

$$\left\langle \rho_\epsilon \star (\det A)^{\frac{1}{n}}, \psi \right\rangle \le_n \|\psi\|_n \left(\|\mathrm{Div}\, A\|_{\mathscr{M}} + \|\mathrm{Tr}_m\, A\|_{\mathscr{M}}\right).$$

The left-hand side in the inequality above can be recast as

$$\left\langle (\det A)^{\frac{1}{n}}, \rho_\epsilon \star \psi \right\rangle,$$

which converges to $\left\langle (\det A)^{\frac{1}{n}}, \psi \right\rangle$ as $\epsilon \to 0$. This proves (2) in full generality. □

2.2. *Transporting the tensor*

Our proof proceeds by induction over m. The case $m = 0$ is covered by Theorem 1. We therefore assume $m \ge 1$ and suppose that, for every $\ell \in \mathbb{N}$, the statement of Theorem 3 is valid for every non-negative Div-BV tensors over $\mathbb{R}^\ell \times \mathbb{T}_{m-1}$.

We start by defining a "transported tensor" A' over $\mathbb{R}^k \times \mathbb{T}_{m-1} \times \mathbb{R}^2$. Supported by the cylinder $C := \mathbb{R}^k \times \mathbb{T}_{m-1} \times \mathbb{S}_1$, it will be shown positive semi-definite and Div-BV. The factor $\mathbb{T}_{m-1}$ means that A' is $(2\pi\mathbb{Z}^{m-1})$-periodic in the variable $z := (y_{k+1}, \dots, y_{n-1}) \in \mathbb{R}^{m-1}$. We denote as well $w := (y_1, \dots, y_k)$ and $\theta := y_n$, so that $y = (w, z, \theta)$.

To define A', we use the canonical isometry $\mathscr{J} : \mathbb{R}^k \times \mathbb{T}_m \to \mathbb{R}^k \times \mathbb{T}_{m-1} \times \mathbb{S}_1$,

$$\mathscr{J}(w, z, \theta) = (w, z, \cos\theta, \sin\theta).$$

The differential $\mathrm{d}\mathscr{J}$ maps $\mathbb{R}^k \times \mathbb{T}_m$ onto the tangent bundle:

$$d\mathscr{J}_{(w,z,\theta)}(a, b, \xi) = (a, b, -\xi\sin\theta, \xi\cos\theta).$$

The entries of A' being distributions (actually measures), they are defined by their action over test functions $\phi_{ij} \in \mathscr{D}(\mathbb{R}^{n+1})$ $(1 \le i, j \le n+1)$. The action of A' over $\Phi = (\phi_{ij})_{i,j}$ is given by

$$\langle A', \Phi \rangle := \sum_{\alpha,\beta=1}^{n-1} \langle a_{\alpha\beta}, \phi_{\alpha\beta} \circ \mathscr{J} \rangle + \sum_{\alpha=1}^{n-1} \left\langle a_{\alpha n}, (\phi_{\alpha,n+1} + \phi_{n+1,\alpha}) \circ \mathscr{J} \cos\theta - (\phi_{\alpha n} + \phi_{n\alpha}) \circ \mathscr{J} \sin\theta \right\rangle$$
$$+ \left\langle a_{nn}, \phi_{nn} \circ \mathscr{J} \sin^2\theta - (\phi_{n,n+1} + \phi_{n+1,n}) \circ \mathscr{J} \sin\theta\cos\theta + \phi_{n+1,n+1} \circ \mathscr{J} \cos^2\theta \right\rangle.$$

In other words, with $\mathrm{d}\mathscr{J}_y \in \mathbf{M}_{n+1,n}(\mathbb{R})$,

$$\langle A', \Phi\rangle = \langle \mathrm{d}\mathscr{J}\, A(\mathrm{d}\mathscr{J})^T, \Phi\circ\mathscr{J}\rangle. \tag{9}$$

That A' is symmetric and its entries are finite measures are obvious. By construction, A' vanishes in the normal direction to the cylinder:

$$\left\langle A'\begin{pmatrix}0\\ \cos\theta\\ \sin\theta\end{pmatrix}, \vec{\psi}\right\rangle = \left\langle A', \begin{pmatrix}0\\ \cos\theta\\ \sin\theta\end{pmatrix}\otimes\vec{\psi}\right\rangle \equiv 0.$$

Formula (9) shows that A' is positive semi-definite as well; for $\eta\in\mathbb{R}^{n+1}$ and $\Gamma\in\mathscr{D}(\mathbb{R}^{n+1})$, one has

$$\langle \eta^T A'\eta, \Gamma\rangle = \langle \widehat{\eta}^T A\widehat{\eta}, \Gamma\circ\mathscr{J}\rangle,$$

for $\widehat{\eta}^T\,\mathrm{d}\mathscr{J} = \eta^T$. When $\Gamma\ge 0$, the above quantity is non-negative.

2.2.1. *The Divergence of the transported tensor*

Let $\vec{\psi}$ be a test vector field over $\mathbb{R}^{n+1}$. Forming $\Phi = \nabla\vec{\psi}$, we express the Divergence of A' by the following calculation, in which we use Einstein's convention of summation over repeated indices:

$$\begin{aligned}\langle \mathrm{Div}\,A', \vec{\psi}\rangle &= -\langle A', \nabla\vec{\psi}\rangle = -\left\langle \mathrm{d}_\alpha\mathscr{J}_i a_{\alpha\beta}\mathrm{d}_\beta\mathscr{J}_k, (\mathrm{d}_i\psi_k)\circ\mathscr{J}\right\rangle\\ &= -\left\langle a_{\alpha\beta}\mathrm{d}_\beta\mathscr{J}_k, \mathrm{d}_\alpha(\psi_k\circ\mathscr{J})\right\rangle = \left\langle \mathrm{d}_\alpha(a_{\alpha\beta}\mathrm{d}_\beta\mathscr{J}_k), \psi_k\circ\mathscr{J}\right\rangle\\ &= \left\langle \mathrm{d}\mathscr{J}_k\cdot\mathrm{Div}\,A + \mathrm{Tr}(A\nabla^2\mathscr{J}_k), \psi_k\circ\mathscr{J}\right\rangle.\end{aligned}$$

We infer the estimate

$$\|\mathrm{Div}\,A'\|_{\mathscr{M}(\mathbb{R}^k\times\mathbb{T}_{m-1}\times\mathbb{R}^2)} \le \left\| |\mathrm{d}\mathscr{J}|_{\mathrm{op}}\,\mathrm{Div}\,A\right\|_{\mathscr{M}(\mathbb{R}^k\times\mathbb{T}_m)} + \left\|\mathrm{Tr}(A\nabla^2\mathscr{J})\right\|_{\mathscr{M}(\mathbb{R}^k\times\mathbb{T}_m)},$$

from which we know that A' is Div-BV.

On the one hand, $\mathscr{J}$ being an isometry, we have $|\mathrm{d}\mathscr{J}|_{\mathrm{op}}\equiv 1$. On the other hand $\mathrm{d}_\alpha\mathrm{d}_\beta\mathscr{J}_k$ vanishes, unless $\alpha=\beta=n$ and $k=n$ or $n+1$, in which cases it equals either $-\cos\theta$ or $-\sin\theta$. Thus $\mathrm{Tr}(A\nabla^2\mathscr{J})$ reduces to the vector-valued measure $(0,\dots,0,-a_{nn}\cos\theta,-a_{nn}\sin\theta)$. We obtain therefore

$$\|\mathrm{Div}\,A'\|_{\mathscr{M}(\mathbb{R}^k\times\mathbb{T}_{m-1}\times\mathbb{R}^2)} \le \|\mathrm{Div}\,A\|_{\mathscr{M}(\mathbb{R}^k\times\mathbb{T}_m)} + \|a_{nn}\|_{\mathscr{M}(\mathbb{R}^k\times\mathbb{T}_m)}. \tag{10}$$

2.3. *Compensated Integrability* vs *singular support*

Thanks to Lemma 7, we assume from now on that the tensors $A:\mathbb{R}^k\times\mathbb{T}_m\to\mathbf{Sym}_n^+$ under consideration are $\mathscr{C}^\infty$. We split the coordinates in $\mathbb{R}^k\times\mathbb{T}_{m-1}\times\mathbb{R}^2$ as $x=(\widehat{x},x')$, where $x'\in\mathbb{R}^2$. When using cylindrical coordinates $(\widehat{x},r,\theta)$ with $x'=re^{i\theta}$, we denote $\vec{e}_r=(\widehat{0},e^{i\theta})$. The support of the tensor A' is the cylinder C defined by $r=1$, whose unit normal is $\nu=\vec{e}_r$. The density of A' with respect to the Lebesgue measure over C is a $\mathscr{C}^\infty$ function denoted $S(\widehat{x},e^{i\theta})\in\mathbf{Sym}_{n+1}^+$.

We now define a few auxiliary positive symmetric tensors. On the one hand

$$\Sigma(\widehat{x},x') := \begin{pmatrix} I_{n-1} & 0\\ 0 & rI_2\end{pmatrix} S\left(\widehat{x},\frac{x'}{r}\right)\begin{pmatrix} I_{n-1} & 0\\ 0 & rI_2\end{pmatrix}$$

is smooth away from the origin. If μ is a positive measure, compactly supported over $(0,+\infty)$, we set

$$B_\mu(\widehat{x},x') = \mu(r)\Sigma(\widehat{x},x').$$

For instance $B_\mu = A'$ when μ is the Dirac mass at $r=1$.

Lemma 8. *We have*

$$\|\mathrm{Div}\,B_\mu\|_{\mathscr{M}} \le c(\mu)\|\mathrm{Div}\,A'\|_{\mathscr{M}}, \qquad c(\mu) = \max(\langle\mu,r\rangle,\langle\mu,r^2\rangle).$$

Proof. The general formula $\mathrm{Div}(\mu\Sigma)=\mu\,\mathrm{Div}\,\Sigma+\Sigma\nabla\mu$ gives, because of $S\vec{e}_r\equiv 0$,

$$\mathrm{Div}\,B_\mu=\mu\,\mathrm{Div}\,\Sigma.$$

Choosing $\mu=\delta_{r=1}$, this gives in particular

$$\|\mathrm{Div}\,A'\|_{\mathcal{M}}=\iint|\mathrm{Div}\,\Sigma|\left(\widehat{x},\mathrm{e}^{\mathrm{i}\theta}\right)\mathrm{d}\widehat{x}\,\mathrm{d}\theta.$$

We notice that the entries σ_{ij} of Σ are homogeneous in r, of respective degrees 0, 1 or 2, depending on whether i,j are $\le n-1$ or $\ge n$. Consequently, the coordinates $(\mathrm{Div}\,\Sigma)_j$ are homogeneous in r, of respective degrees 0 or 1, depending on whether j is $\le n-1$ or $\ge n$. This allows us to express $\|(\mathrm{Div}\,B_\mu)_j\|_{\mathcal{M}}$ in terms of $\|(\mathrm{Div}\,A')_j\|_{\mathcal{M}}$. For instance if $j\le n-1$,

$$\begin{aligned}\left\|(\mathrm{Div}\,B_\mu)_j\right\|_{\mathcal{M}}&=\iint\mu(r)|(\mathrm{Div}\,\Sigma)_j(\widehat{x},x')|\,\mathrm{d}\widehat{x}\,\mathrm{d}x'=\iiint\mu(r)\left|(\mathrm{Div}\,\Sigma)_j\left(\widehat{x},r\,\mathrm{e}^{\mathrm{i}\theta}\right)\right|\mathrm{d}\widehat{x}\,r\,\mathrm{d}r\,\mathrm{d}\theta\\&=\left\|(\mathrm{Div}\,A)'_j\right\|_{\mathcal{M}}\int r\mu(r)\,\mathrm{d}r.\end{aligned}$$

Likewise, when $j=n$ or $n+1$, one has

$$\left\|(\mathrm{Div}\,B_\mu)_j\right\|_{\mathcal{M}}=\left\|(\mathrm{Div}\,A)'_j\right\|_{\mathcal{M}}\int r^2\mu(r)\,\mathrm{d}r.\qquad\square$$

We select from now on some function $\eta\in\mathcal{D}_+(0,+\infty)$, with $\eta\not\equiv 0$. Since this function will not vary any more, we denote B for B_η for simplicity. For every test function $\phi\in\mathcal{D}_+(\mathbb{R}^k\times\mathbb{T}_{m-1}\times\mathbb{R}^2)$, the tensor $\phi\vec{e}_r\otimes\vec{e}_r+B$ is positive and Div-BV. Since the tensor $r^{-1}\vec{e}_r\otimes\vec{e}_r$ is Div-free (see [11]), we actually have

$$\mathrm{Div}(\phi\vec{e}_r\otimes\vec{e}_r)=\frac{1}{r}\partial_r(r\phi)\,\vec{e}_r.\tag{11}$$

Lemma 9. *We have*

$$\det(\phi\vec{e}_r\otimes\vec{e}_r+B)(\widehat{x},x')=\eta(r)^n\phi(\widehat{x},x')\det A(\widehat{x},\theta).$$

Proof. Writing blockwise

$$A=\begin{pmatrix}\widehat{A}&Z\\Z^T&a_{nn}\end{pmatrix},$$

we have

$$\phi\vec{e}_r\otimes\vec{e}_r+B=\begin{pmatrix}\eta\widehat{A}&\eta Z\otimes\binom{\sin\theta}{-\cos\theta}\\\eta(Z\otimes\binom{\sin\theta}{-\cos\theta})^T&\eta a_{nn}\begin{pmatrix}\sin^2\theta&-\sin\theta\cos\theta\\-\sin\theta\cos\theta&\cos^2\theta\end{pmatrix}+\phi\begin{pmatrix}\cos^2\theta&\sin\theta\cos\theta\\\sin\theta\cos\theta&\sin^2\theta\end{pmatrix}\end{pmatrix},$$

this matrix being unitarily similar to

$$\begin{pmatrix}\eta A&0\\0&\phi\end{pmatrix}.\qquad\square$$

End of the proof

The tensor $\phi\vec{e}_r\otimes\vec{e}_r+B$ being periodic in $m-1$ coordinates $(x_{k+1},\dots,x_{n-1})$, we may apply to it the induction assumption, namely (FIper) with $(n+1,m-1)$ instead of (n,m). Mind that we identify $\mathbb{R}^{k+2}\times\mathbb{T}_{m-1}\sim\mathbb{R}^k\times\mathbb{T}_{m-1}\times\mathbb{R}^2$, so that $\mathrm{Tr}_{m-1}A'$ equals $\sum_{k+1}^{n-1}a'_{jj}$. Together with Lemma 8, Lemma 9 and (11), this gives

$$\begin{aligned}&\iiint\phi\left(\widehat{x},r\,\mathrm{e}^{\mathrm{i}\theta}\right)^{\frac{1}{n}}(\det A(\widehat{x},\theta))^{\frac{1}{n}}\,\eta(r)\,r\,\mathrm{d}\widehat{x}\,\mathrm{d}r\,\mathrm{d}\theta\\&\qquad\le_n\left(\iiint|\partial_r(r\phi)|\,\mathrm{d}\widehat{x}\,\mathrm{d}r\,\mathrm{d}\theta+\|\mathrm{Div}\,A'\|_{\mathcal{M}}+\sum_{k+1}^{n-1}\|a'_{jj}\|_{\mathcal{M}}\right)^{1+\frac{1}{n}}\end{aligned}$$

Since a'_{jj} is nothing but a_{jj}, transported isometrically, we have $\|a'_{jj}\|_{\mathscr{M}} = \|a_{jj}\|_{\mathscr{M}}$. With (10), we obtain therefore

$$\iiint \phi\left(\widehat{x}, r\,\mathrm{e}^{\mathrm{i}\theta}\right)^{\frac{1}{n}} (\det A(\widehat{x},\theta))^{\frac{1}{n}}\, \eta(r)\, r\, \mathrm{d}\widehat{x}\, \mathrm{d}r\, \mathrm{d}\theta \\ \leq_n \left(\iiint |\partial_r(r\phi)|\, \mathrm{d}\widehat{x}\, \mathrm{d}r\, \mathrm{d}\theta + \|\mathrm{Div}\, A\|_{\mathscr{M}} + \sum_{k+1}^{n} \|a_{jj}\|_{\mathscr{M}} \right)^{1+\frac{1}{n}}. \quad (12)$$

Our last step is a classical scaling argument. Given any test function $\psi \in \mathscr{D}_+(\mathbb{R}^k \times \mathbb{T}_m)$, and scalar parameter $\lambda > 0$, we apply (12) to the function $\phi = \lambda \eta(r) \psi(\widehat{x},\theta)^n$:

$$\lambda^{\frac{1}{n}} \int \eta(r)^{1+\frac{1}{n}}\, r\, \mathrm{d}r \int \psi (\det A)^{\frac{1}{n}}\, \mathrm{d}y \leq_n \left(\lambda \int \psi^n\, \mathrm{d}y + \|\mathrm{Div}\, A\|_{\mathscr{M}} + \sum_{k+1}^{n} \|a_{jj}\|_{\mathscr{M}} \right)^{1+\frac{1}{n}}.$$

We choose

$$\lambda = \frac{\|\mathrm{Div}\, A\|_{\mathscr{M}} + \sum_{k+1}^{n} \|a_{jj}\|_{\mathscr{M}}}{\int \psi^n\, \mathrm{d}y}$$

and we conclude that

$$\int \psi (\det A)^{\frac{1}{n}}\, \mathrm{d}y \leq_n \|\psi\|_n \left(\|\mathrm{Div}\, A\|_{\mathscr{M}} + \sum_{k+1}^{n} \|a_{jj}\|_{\mathscr{M}} \right)$$

This ends the proof of the validity of (2) for smooth tensors. Notice that the a_{jj} are positive measures, so that

$$\sum_{k+1}^{n} \|a_{jj}\|_{\mathscr{M}} = \|\mathrm{Tr}_m A\|_{\mathscr{M}}.$$

3. The estimate for space-periodic perfect gas flows

Let $(\rho, u, p, \varepsilon)$ be a space-periodic solution of the Euler system, over $(0,T) \times \mathbb{R}^d$, with lattice of periods $2\pi L\mathbb{Z}_d$. We assume that the total mass M and initial energy E_0 (per cell) are finite and that the flow is admissible.

We apply Corollary 4 to the tensor A' obtained by extending the mass-momentum tensor (see (3)) by 0_n (we recall $n = 1+d$ and $m = d$) away from $(0,T) \times \mathbb{R}^d$. Therefore $\mathrm{Div}\, A'$ consists in the vector-valued measure

$$\begin{pmatrix} \rho \\ \rho u \end{pmatrix} \mathscr{H}_d \Big|_{t=T} - \begin{pmatrix} \rho \\ \rho u \end{pmatrix} \mathscr{H}_d \Big|_{t=0},$$

where $\mathscr{H}_d$ is the d-dimensional Hausdorff measure, or equivalently here the Lebesgue measure over subspaces of dimension d. The matrix Q being $\frac{1}{L} I_d$, we infer

$$\|P\, \mathrm{Div}\, A'\|_{\mathscr{M}} \leq \left(\int_{\{T\}\times L\mathbb{T}_d} + \int_{\{0\}\times L\mathbb{T}_d} \right) \left(\rho + \frac{1}{L} \rho |u| \right) \mathrm{d}y \leq 2 \left(M + \frac{1}{L} \sqrt{2ME_0} \right).$$

On the other hand, $\mathrm{Tr}_d A = \rho |u|^2 + dp$ yields

$$\|\mathrm{Tr}_d P A' P^T\|_{\mathscr{M}} = \frac{1}{L^2} \int_0^T \mathrm{d}t \int_{\mathbb{R}^d} (\rho |u|^2 + \mathrm{d}p) \leq \max\{2, (\gamma-1)d\}\, \frac{T}{L^2} E_0.$$

The functional inequality (1) therefore writes

$$\int_0^T \mathrm{d}t \int_{L\mathbb{T}_d} \rho^{\frac{1}{d}}\, p\, \mathrm{d}y \leq_d \gamma L \left(M + \frac{\sqrt{ME_0}}{L} + \frac{TE_0}{L^2} \right)^{1+\frac{1}{d}}. \quad (13)$$

We notice the uncomfortable fact that the parenthesis above is not homogeneous from a phyisical point of view, with only the last two terms being of the same dimension (a mass per time). Meanwhile, the left-hand side has the same dimension as $M^{1+\frac{1}{d}} L T^{-1}$. We have explained in our seminal paper [10] how to cure this flaw, with the help of a scaling argument.

The appropriate scaling concerns both dependent and independent variables. If $\mu > 0$ is a constant parameter, we form

$$(\tau, z, r, v, q, \epsilon) := \left(\mu t, y, \rho, \frac{u}{\mu}, \frac{p}{\mu^2}, \frac{\varepsilon}{\mu^2}\right),$$

which still satisfies the Euler system. It is space-periodic according to the same lattice, with same mass M, but with initial energy (per cell) $\mu^{-2}E_0$. Let us apply the estimate (13) to this flow, on the time interval $(0, \mu T)$:

$$\int_0^{\mu T} d\tau \int_{L\mathbb{T}_d} r^{\frac{1}{d}} q \, dz \leq_d \gamma L \left(M + \frac{\sqrt{ME_0}}{\mu L} + \frac{TE_0}{\mu L^2}\right)^{1+\frac{1}{d}}.$$

Expressing the integral in terms of the original (t, y) variables, we obtain

$$\int_0^{T} dt \int_{L\mathbb{T}_d} \rho^{\frac{1}{d}} p \, dy \leq_d \gamma \mu L \left(M + \frac{\sqrt{ME_0}}{\mu L} + \frac{TE_0}{\mu L^2}\right)^{1+\frac{1}{d}}.$$

To balance the terms of different physical dimensions, we choose

$$\mu = \frac{1}{M}\left(\frac{\sqrt{ME_0}}{L} + \frac{TE_0}{L^2}\right),$$

which yields the desired estimate

$$\int_0^{T} dt \int_{L\mathbb{T}_d} \rho^{\frac{1}{d}} p \, dy \leq_d \gamma M^{\frac{1}{d}} \left(\sqrt{ME_0} + \frac{TE_0}{L}\right).$$

4. CI with determinantal masses, for periodic tensors

We adapt here the Section 4 of [12], to the periodic context. We recall that the Theorem 4.1 of [12] dealt with positive Div-BV tensors S over $\mathbb{R}^n$ which are, in the neighborhood of finitely many points, Div-free and positively homogeneous of degree $1-n$. This situation is extreme in the realm of Div-free tensors (see [11, 12]), in that such tensors are *special* : they derive locally from a convex potential θ, meaning that $S = \widehat{D^2\theta}$ is the cofactor matrix of the Hessian. In addition, the potential is positively homogeneous of degree 1. Its existence follows from Pogorelov's solution to Minkowski's Problem [8]; uniqueness occurs up to the addition of an affine function.

Given such a singularity at $X^* \in \mathbb{R}^n$, the *Determinantal mass* $\mathrm{Dm}(S; X^*)$ was defined as the volume of the convex body enclosed in the image $\nabla\theta(\mathbb{R}^n \setminus \{X^*\})$. This body is nothing but the subgradient of $\partial\theta(X^*)$. We proved in [12] the following improvement of Theorem 1,

Theorem 10. *Let $S > 0_n$ be symmetric and Div-BV over $\mathbb{R}^n$. Let $X^1, \dots, X^r \in \mathbb{R}^n$ be points at which S is locally Div-free and positively homogeneous of degree $1-n$. Then we have the estimate*

$$\int_{\mathbb{R}^n} (\det S)^{\frac{1}{n-1}} dx + \sum_1^r \mathrm{Dm}(S; X^j) \leq_n \|\mathrm{Div}\, S\|_{\mathcal{M}}^{\frac{n}{n-1}}. \tag{FI:Dm}$$

In other words, $(\det S)^{\frac{1}{n-1}}$ behaves as if it contained Dirac masses $\mathrm{Dm}(S; X^j)\delta_{x=X^j}$. Let us emphasize that the constant understated by $\leq_n$ does not depend upon the number r of singularities.

We shall prove below the following adaptation to the periodic context.

Theorem 11. *Let $S > 0_n$ be symmetric and Div-BV over $\mathbb{R}^k \times \mathbb{T}_m$, with $n = k + m$. Let $X^1, \dots, X^r \in \mathbb{R}^k \times \mathbb{T}_m$ be points at which S is locally Div-free and positively homogeneous of degree $1-n$. Then we have the estimate*

$$\int_{\mathbb{R}^k \times \mathbb{T}_m} (\det S)^{\frac{1}{n-1}} dx + \sum_1^r \mathrm{Dm}(S; X^j) \leq_n \left(\|\mathrm{Div}\, S\|_{\mathcal{M}} + \|\mathrm{Tr}_m S\|_{\mathcal{M}}\right)^{\frac{n}{n-1}}. \tag{14}$$

Proof. We follow the strategy developed in [12], approximating S by tensors S^ϵ which are smooth at the singularities X^j. Then we apply (FIper) to S^ϵ and pass to the limit as $\epsilon \to 0+$. Since the smoothing is done locally around each X^j separately, it suffices to consider only one singularity, say at $X^* = 0$. Thus $S = \widehat{D^2\theta}$ in some ball $B(0;\rho)$, and θ is convex, positively homogeneous of degree 1.

We begin by smoothing out θ in a neighborhood of the origin. We may always assume $\mathrm{Dm}(S;0) > 0$, which means that the bounded convex subset $K = \partial\theta(0)$ has a non-empty interior. Up to the addition of an affine function to θ, we may assume that 0 is interior to K.

We recall (John's Theorem [6]) that there exists a smallest ellipsoid $\mathscr{E}$ containing K, and that the homothetic $\mathscr{F} = \frac{1}{n}\mathscr{E}$ is contained in K (by convention, 0 is the center of $\mathscr{E}$). Let $x^T\Sigma^{-1}x \le 1$ be the equation of $\mathscr{F}$, with $\Sigma \in \mathbf{SPD}_n$, and denote $g(x) = \sqrt{x^T\Sigma x}$. We have

$$\partial g(0) = \mathscr{F} \subset \partial\theta(0) \subset \mathscr{E} = \partial(ng)(0).$$

Since these functions are positively homogeneous of degree one, this means $g \le \theta \le ng$.

Choosing

$$J(s) = \begin{cases} \frac{1}{2}|s| & \text{if } |s| \ge 1, \\ \frac{1+s^2}{4} & \text{if } |s| \le 1, \end{cases}$$

a convex increasing function, we set

$$\bar\theta = \max\{\theta, J\circ g\},$$

which is a convex function. For $g(x) < 2n - \sqrt{4n^2-1} =: a_n$, we have $J\circ g > ng \ge \theta$, hence $\bar\theta = J\circ g$, and actually $\bar\theta = \frac{1+g^2}{4}$. This happens in the domain $a_n\mathscr{F}^0$. Likewise, for $g > 2-\sqrt{3} =: b$, we have $J\circ g < g \le \theta$, hence $\bar\theta = \theta$. This happens away from $b\mathscr{F}^0$ (we denote $\mathscr{F}^0$ the polar set of $\mathscr{F}$).

Eventually we set $\theta_\epsilon(x) = \epsilon\bar\theta(\frac{x}{\epsilon})$, which coincides with θ away from $\epsilon b\mathscr{F}^0$. Whenever $\epsilon > 0$ is small enough, so that $\epsilon b\mathscr{F}^0 \subset B(0;\rho)$, we may define (both formulæ below agree in the coronna $B(0;\rho)\setminus\epsilon b\mathscr{F}^0$)

$$S_\epsilon = \begin{cases} S & \text{away from } \epsilon b\mathscr{F}^0, \\ \widehat{D^2\theta_\epsilon} & \text{in } B(0;\rho). \end{cases}$$

We observe that $\mathrm{Div}\,S_\epsilon \equiv \mathrm{Div}\,S$, because they both vanish in $B(0;\rho)$. Likewise, $(\det S_\epsilon)^{\frac{1}{n-1}}$ differs from $(\det S)^{\frac{1}{n-1}}$ only within $\epsilon b\mathscr{F}^0$, where the former is $\det D^2\theta_\epsilon$, while the latter vanishes identically. Again $\mathrm{Tr}_m S$ and $\mathrm{Tr}_m S_\epsilon$ differ only within $\epsilon b\mathscr{F}^0$. On the one hand

$$\int_{\epsilon b\mathscr{F}^0} \mathrm{Tr}_m S \xrightarrow{\epsilon\to 0} 0$$

because $\mathrm{Tr}_m S$ does not charge the origin. On the other hand

$$\int_{\epsilon b\mathscr{F}^0} \mathrm{Tr}_m S_\epsilon = \epsilon\int_{b\mathscr{F}^0} \mathrm{Tr}_m \widehat{D^2\bar\theta}(x)\,dx \longrightarrow 0.$$

Applying (FIper) to S_ϵ, we therefore have

$$\int_{\mathbb{R}^k\times\mathbb{T}_m} (\det S)^{\frac{1}{n-1}}\,dx + \int_{\epsilon b\mathscr{F}^0} \det D^2\theta_\epsilon\,dx \le_n \left(\|\mathrm{Div}\,S\|_{\mathscr{M}} + \|\mathrm{Tr}_m S\|_{\mathscr{M}} + O(\epsilon)\right)^{\frac{n}{n-1}}$$

Observing that

$$\int_{\epsilon b\mathscr{F}^0} \det D^2\theta_\epsilon\,dx \ge \int_{\epsilon a_n\mathscr{F}^0} \det D^2\theta_\epsilon\,dx = \mathrm{vol}\left(\nabla\theta_\epsilon(\epsilon a_n\mathscr{F}^0)\right)$$

and

$$\nabla\theta_\epsilon(x) = (\nabla\bar\theta)\left(\frac{x}{\epsilon}\right) = \frac{1}{2}\Sigma\frac{x}{\epsilon},$$

we find (invoquing John's Theorem for the last inequality)

$$\int_{\epsilon b\mathscr{F}^0} \det D^2\theta_\epsilon\,dx \ge \mathrm{vol}\left(\frac{a_n}{2}\Sigma\mathscr{F}^0\right) = \left(\frac{a_n}{2}\right)^n \mathrm{vol}(\mathscr{F}) \ge \left(\frac{a_n}{2n}\right)^n \mathrm{vol}(K).$$

Letting $\epsilon \to 0$, we infer

$$\int_{\mathbb{R}^k \times \mathbb{T}_m} (\det S)^{\frac{1}{n-1}}\, dx + \mathrm{vol}(K) \le_n (\|\mathrm{Div}\, S\|_{\mathcal{M}} + \|\mathrm{Tr}_m S\|_{\mathcal{M}})^{\frac{n}{n-1}}$$

Recalling that $\mathrm{vol}(K) = \mathrm{Dm}(S;0)$, we have proved Formula (14). □

5. Collisions estimate for the billiard in a torus

In [12], we constructed the mass-momentum tensor $\mathbb{M}$ as the sum of terms that describe on the one hand the kinematics of each particle, and on the other hand the collisions (the latter terms being called *collitons*). A given particle p is associated with its center $p(t)$, whose graph $\gamma(p)$ is a broken line. The kinks correspond to the collisions experienced by p. In terms of the velocity $v = \dot{p}$, the tensor associated with this particle is the tensor-valued measure

$$\begin{pmatrix} 1 \\ v \end{pmatrix} \otimes \begin{pmatrix} 1 \\ v \end{pmatrix} \, dt|_{\gamma(p)}.$$

When a pair (p, q) collides a time τ, with incoming/outgoing velocities $v_\pm(p, q)$, the conservation of momentum implies $v_+(p) - v_-(p) = v_-(q) - v_+(q)$ (denoted herebelow $[v]$). The corresponding colliton is defined as

$$\frac{1}{|[v]|} \begin{pmatrix} 0 \\ [v] \end{pmatrix} \otimes \begin{pmatrix} 0 \\ [v] \end{pmatrix} \, d\ell|_{[(\tau, p(\tau)), (\tau, q(\tau))]},$$

where $d\ell$ is the element of length along the segment. Notice that $|q(\tau) - p(\tau)| = 2a$. It was shown that summing all these constributions results in a (locally) Div-free tensor $\mathbb{M}$ over $\mathbb{R}^n$, with $n = 1 + d$. Remark that $\mathbb{M}$ is rank-one on its support (a graph), so that $(\det \mathbb{M})^{\frac{1}{n}} \equiv 0$. However $\mathbb{M}$ is homogeneous of degree $1 - n = -d$ about each node of this graph. This suggests that we apply Theorem 11, though to a suitable modification of $\mathbb{M}$.

We denote N the finite number of particles *per cell.* Another meaningful quantity is the kinetic energy

$$E = \sum_p \frac{1}{2} |v|^2,$$

a constant of the motion. Notice that we may always assume, after the choice of a suitable Galilean frame, that the mean velocity $\sum_p v$ vanishes. Then $E = \frac{N}{2} \bar{v}^2$, where $\bar{v}$ is again the root mean square velocity, a constant of the motion.

We begin by considering $\mathbb{T}_d$-periodic configurations. We proceed as in [12], by adding correctors at each nodes of the graph suporting $\mathbb{M}$. If $X = (\tau, p(\tau))$ is such a node, we choose a unitary basis $\{z_1, \ldots, z_{d-1}\}$ of the subspace orthogonal to $\binom{1}{v_\pm(p)}$ and form the tensor

$$S_X := \sum_1^{d-1} z_j \otimes z_j \, d\ell|_{[X - \epsilon z_j, X + \epsilon z_j]}.$$

Given a time interval $(0, T)$, we now consider the augmented tensor

$$S = \begin{cases} \mathbb{M} + \sum_{\text{nodes}} b_X S_X & \text{in } (0, T) \times \mathbb{T}_d, \\ 0_n & \text{otherwise.} \end{cases}$$

The positive parameters b_X will be chosen later on. We may always assume that T is not a collision time, and choose $\epsilon > 0$ small enough so that the support of the corrector does not meet the initial/final times 0 and T.

Let us apply (14) to S. Since $(\det S)^{\frac{1}{n}} \equiv 0$, the left-hand side consists only in the sum of the determinantal masses at nodes. These have been calculated in [12] :

$$\mathrm{Dm}(S : X) = 2^{d-3} b_X^{1 - \frac{1}{d}} \left| \begin{pmatrix} 1 \\ v_-(p) \end{pmatrix} \wedge \begin{pmatrix} 1 \\ v_+(p) \end{pmatrix} \right|^{\frac{1}{d}}.$$

In particular

$$\mathrm{Dm}(S:X) \geq 2^{d-3} b_X^{1-\frac{1}{d}} \max(|[v]|, |v_-(p) \wedge v_+(p)|)^{\frac{1}{d}}. \tag{15}$$

We now evaluate the right-hand side of (14). As noticed in [12], Div S consists in two parts: – the restriction of its first column to the initial and final hyperplanes, $t = 0$ or $t = T$, – the Divergence of the correctors, which concentrate at the ends of the segments $[X - \epsilon z_j, X + \epsilon z_j]$. Overall, we have

$$\|\mathrm{Div}\, S\|_{\mathcal{M}} \leq N + \sqrt{2NE} + 2(d-1) \sum_{\text{nodes}} b_X. \tag{16}$$

The tracial contribution is

$$\mathrm{Tr}_d \mathbb{M} = \sum_p |v|^2 \, \mathrm{d}t|_{\gamma(p)} + \sum_{\text{coll.}} |[v]| \, \mathrm{d}\ell|_{[(\tau,p(\tau)),(\tau,q(\tau))]} ,$$

so that

$$\|\mathrm{Tr}_d \mathbb{M}\|_{\mathcal{M}} = 2ET + 2a \sum_{\text{coll.}} |[v]|. \tag{17}$$

On the other hand $\|\mathrm{Tr}_d S_X\|_{\mathcal{M}} \leq 2\epsilon$. Assembling (16) and (17) in (14), we therefore have

$$\sum_{\text{nodes}} b_X^{1-\frac{1}{d}} |[v]|^{\frac{1}{d}} \leq_d \left(N + \sqrt{NE} + ET + a \sum_{\text{nodes}} |[v]| + \sum_{\text{nodes}} b_X + O(\epsilon) \right)^{1+\frac{1}{d}},$$

in which we may let $\epsilon \to 0+$. We thus have

$$\sum_{\text{nodes}} b_X^{1-\frac{1}{d}} |[v]|^{\frac{1}{d}} \leq_d \left(N + \sqrt{NE} + ET + a \sum_{\text{nodes}} |[v]| + \sum_{\text{nodes}} b_X \right)^{1+\frac{1}{d}}.$$

Let us relax the parameters b_X by setting $b_X = \kappa \beta_X^{\frac{d}{d-1}}$, where β_X, to be chosen later, are positive. Taking

$$\kappa = \frac{N + \sqrt{NE} + ET + a \sum_{\text{nodes}} |[v]|}{\sum \beta_X^{\frac{d}{d-1}}},$$

we obtain

$$\sum_{\text{nodes}} \beta_X |[v]|^{\frac{1}{d}} \leq_d \|\vec{\beta}\|_{\frac{d}{d-1}} \left(N + \sqrt{NE} + ET + a \sum_{\text{nodes}} |[v]| \right)^{\frac{2}{d}}.$$

The latter inequality, being valid for every positive $\vec{\beta}$, yields and estimate in $(\ell^{\frac{d}{d-1}})' = \ell^d$ of the vector whose coordinates are the quantities $|[v]|^{\frac{1}{d}}$. This reads

$$\sum_{\text{coll}} |[v]| \leq_d \left(N + \sqrt{NE} + ET + a \sum_{\text{coll}} |[v]| \right)^2. \tag{18}$$

So far, (18) is not homogeneous in terms of physical units. We remedy to this flaw by the same scaling argument as in Paragraph 3. If $\mu > 0$ is a constant parameter, the change of time $\tau = \mu t$ yields another motion, though with particle velocities $\frac{v}{\mu}$. The corresponding energy per cell is $\mu^{-2} E$. Appplying (18) to this new motion, over the times interval $(0, \mu T)$, we obtain

$$\frac{1}{\mu} \sum_{\text{coll}} |[v]| \leq_d \left(N + \frac{1}{\mu} \left(\sqrt{NE} + ET + a \sum_{\text{coll}} |[v]| \right) \right)^2.$$

Taking

$$\mu = \frac{1}{N} \left(\sqrt{NE} + ET + a \sum_{\text{coll}} |[v]| \right),$$

we conclude

$$\sum_{\text{coll}} |[v]| \leq_d N \left(\sqrt{NE} + ET + a \sum_{\text{coll}} |[v]| \right).$$

There exists therefore a number $\theta(d) > 0$ such that, whenever $Na < \theta(d)$, we have

$$\sum_{\text{coll}} |[v]| \leq_d N(\sqrt{NE} + ET). \tag{19}$$

We close our analysis by considering $L\mathbb{T}_d$-periodic configurations, where $L > 0$ is a characteristic length. Such a motion can be reduced to a $\mathbb{T}_d$-configuration, *via* the transformation

$$(t, y, v, E, a) \longmapsto \left(t, \frac{y}{L}, \frac{v}{L}, \frac{E}{L^2}, \frac{a}{L}\right).$$

Applying (19) to the latter, we obtain our final estimate

$$\sum_{\text{coll}} |[v]| \leq_d N\left(\sqrt{NE} + \frac{ET}{L}\right), \tag{20}$$

valid whenever $Na < \theta(d)L$.

Further estimate. Because of (15), the estimate (18) implies also

$$\sum_{\text{nodes}} |v_-(p) \wedge v_+(p)| \leq_d \left(N + \sqrt{NE} + ET + a \sum_{\text{nodes}} |[v]|\right)^2.$$

After the time-scaling argument, we thus have (mind that we estimate a quadratic quantity in terms of velocities)

$$\sum_{\text{nodes}} |v_-(p) \wedge v_+(p)| \leq_d \left(\sqrt{NE} + ET + a \sum_{\text{nodes}} |[v]|\right)^2$$

for $\mathbb{T}_d$-periodic configurations. This becomes

$$\sum_{\text{nodes}} |v_-(p) \wedge v_+(p)| \leq_d \left(\sqrt{NE} + \frac{ET}{L} + \frac{a}{L} \sum_{\text{nodes}} |[v]|\right)^2$$

for $L\mathbb{T}_d$-periodic configurations.

Combined with (20), this yields

$$\sum_{\text{nodes}} |v_-(p) \wedge v_+(p)| \leq_d \left(\sqrt{NE} + \frac{ET}{L}\right)^2$$

whenever the initial configuration satisfies $Na < \theta(d)L$.

Appendix A. Short proof of Theorem 3 when $k = 0$

If the tensor A is fully $\mathbb{T}_n$-periodic, it suffices to revisit the proof of Theorem 2, see [10], and consider Div-BV tensors instead of Div-free tensors.

Let $f > 0$ be a periodic smooth test function. For every $S \in \mathbf{SPD}_n$ satisfying

$$\det S = \fint f(x)\,dx, \tag{21}$$

there exists (see [7]) a unique (up to and additive constant) smooth periodic solution θ of the elliptic Monge–Ampère equation

$$\det(S + D^2\theta) = f, \tag{22}$$

$$S + D^2\theta(x) \in \mathbf{SPD}_n, \qquad \forall\, x \in \mathbb{R}^d. \tag{23}$$

The constraint (23) means that $x \mapsto \frac{1}{2} x^T S x + \theta(x)$ is a strongly convex function.

Because both $S+\mathrm{D}^2\theta$ and A are symmetric positive, the spectrum of their product is real, non negative. Applying the AM-GM Inequality, we have

$$(f\det A)^{\frac{1}{n}} = \left(\det((S+\mathrm{D}^2\theta)A)\right)^{\frac{1}{n}} \le \frac{1}{n}\,\mathrm{Tr}((S+\mathrm{D}^2\theta)A) \le \frac{1}{n}(\mathrm{Tr}(SA)+\mathrm{div}(A\nabla\theta)-(\mathrm{Div}\,A)\cdot\nabla\theta). \tag{24}$$

Integrating (24) over $\mathbb{T}_n$, we have

$$⨍(f\det A)^{\frac{1}{n}}\,\mathrm{d}x \le \frac{1}{n}\left(\mathrm{Tr}\left(S⨍ A(x)\,\mathrm{d}x\right)-⨍(\mathrm{Div}\,A)\cdot\nabla\theta\,\mathrm{d}x\right).$$

We actually choose

$$S=\mu I_n, \qquad \mu=\left(⨍ f(x)\,\mathrm{d}x\right)^{\frac{1}{n}},$$

so that $\det(\mu I_n+\mathrm{D}^2\theta)=f$ and

$$⨍(f\det A)^{\frac{1}{n}}\,\mathrm{d}x \le \frac{1}{n}\left(\mu⨍\mathrm{Tr}\,A(x)\,\mathrm{d}x-⨍(\mathrm{Div}\,A)\cdot\nabla\theta\,\mathrm{d}x\right). \tag{25}$$

To estimate the second term in the right-hand side, we use the

Lemma 12. *Let $\theta\in\mathscr{C}^2(\mathbb{T}_n)$ be such that $x\mapsto\frac{\mu}{2}|x|^2+\theta(x)$ is convex. One has*

$$\sup_x|\nabla\theta(x)|\le 2\mu\sqrt{n}\pi.$$

We infer

$$⨍(f\det A)^{\frac{1}{n}}\,\mathrm{d}x \le \frac{\mu}{n}\left(⨍\mathrm{Tr}\,A(x)\,\mathrm{d}x+2\sqrt{n}\pi⨍|\mathrm{Div}\,A|\,\mathrm{d}x\right).$$

In terms of $g=f^{\frac{1}{n}}$, this reads

$$⨍ g(\det A)^{\frac{1}{n}}\,\mathrm{d}x \le \frac{1}{n}\left(⨍\mathrm{Tr}\,A(x)\,\mathrm{d}x+2\sqrt{n}\pi⨍|\mathrm{Div}\,A|\,\mathrm{d}x\right)\|g\|_n.$$

This inequality, being valid whenever g is smooth and positive, extends by Fatou Lemma to uniformly positive L^n-functions, then to non-negative L^n-functions. This implies $(\det A)^{\frac{1}{n}}\in(L^n(\mathbb{T}_n))'=L^{\frac{n}{n-1}}(\mathbb{T}_n)$ and the inequality

$$\left(⨍(\det A)^{\frac{n}{n-1}}\,\mathrm{d}x\right)^{1-\frac{1}{n}} \le \frac{1}{n}\left(⨍\mathrm{Tr}\,A(x)\,\mathrm{d}x+2\sqrt{n}\pi⨍|\mathrm{Div}\,A|\,\mathrm{d}x\right).$$

To prove Lemma 12, we use the monotonicity of the gradient of a convex function:

$$\left\langle \mu x+\nabla\theta(x)-\mu y-\nabla\theta(y),x-y\right\rangle\ge 0.$$

Replacing x by $x+\omega$ where $\omega\in 2\pi\mathbb{Z}^n$, and using the periodicity of $\nabla\theta$, we infer

$$\left\langle \mu\omega+\mu x+\nabla\theta(x)-\mu y-\nabla\theta(y),\omega+x-y\right\rangle\ge 0, \qquad \forall\,\omega\in 2\pi\mathbb{Z}^n.$$

To minimize the left-hand side with respect to ω, we choose ω such that

$$\alpha:=\omega+x-y+\frac{1}{2}\mu^{-1}\left(\nabla\theta(x)-\nabla\theta(y)\right)\in[-\pi,\pi]^n.$$

Then the inequality rewrites

$$\|\nabla\theta(x)-\nabla\theta(y)\|\le 2\mu\|\alpha\|\le 2\mu\sqrt{n}\pi.$$

Choosing a point y at which θ reaches its maximum, we obtain the announced inequality.

Remark 13. An alternate proof consists in applying Theorem 1 to the tensor ϕA, where $\phi\in\mathscr{D}_+(\mathbb{R}^n)$ is $\equiv 1$ on a fundamental domain F, and vanishes on every cell $F+\omega$ that is not a neighbour of F.

Appendix B. Proving CI in $\mathbb{R}^n$, from CI in $\mathbb{R}^{n+1}$

Given $n \geq 2$, let us pretend that we know CI in $\mathbb{R}^{n+1}$ only. We shall prove that it is valid in $\mathbb{R}^n$ too. In other words, CI can be established by backward induction. We warn the reader that we are not interested in sharp constants. Embedding $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$ by the isometry $y \mapsto (0, y)$, we denote the general coordinate in $\mathbb{R}^{n+1}$ by $x = (x_0, y)$.

Let $A \succ 0_n$ be a Div-BV tensor in $\mathbb{R}^n$. The first step is to transport A to the subspace $\{0\} \times \mathbb{R}^n$. This allows us to define a tensor over $\mathbb{R}^{n+1}$,

$$A' := \begin{pmatrix} 0 & 0 \\ 0 & \delta_{x_0=0} \otimes A \end{pmatrix} \succ 0_{n+1}.$$

Since

$$\operatorname{Div}_x A' = \begin{pmatrix} 0 \\ \delta_{x_0=0} \otimes (\operatorname{Div}_y A) \end{pmatrix},$$

the tensor A' is Div-BV, with $\|\operatorname{Div}_x A'\|_{\mathcal{M}} = \|\operatorname{Div}_y A\|_{\mathcal{M}}$.

Let us choose an $\eta \in \mathcal{D}_+(\mathbb{R})$ so that $\eta(0) = 1$. Likewise, let an even function $\rho \in \mathcal{D}_+(\mathbb{R}^{n+1})$ be such that

$$\int_{\mathbb{R}^{n+1}} \rho(x)\,\mathrm{d}x = 1.$$

We denote as usual $\rho_\epsilon(x) = \epsilon^{-n-1}\rho(\frac{x}{\epsilon})$ the approximate Dirac mass.

For every $f \in \mathcal{D}_+(\mathbb{R}^n)$ and $\epsilon > 0$, we consider the non-negative tensor

$$B_f + \rho_\epsilon \star A' = \begin{pmatrix} \eta(x_0) f(y) & 0 \\ 0 & \rho_\epsilon \star (\delta_{x_0=0} \otimes A) \end{pmatrix}.$$

From

$$\operatorname{Div}_x (B_f + \rho_\epsilon \star A') = \begin{pmatrix} \eta' f \\ \rho_\epsilon \star (\delta_{x_0=0} \otimes (\operatorname{Div}_y A)) \end{pmatrix},$$

we infer $\|\operatorname{Div}_x (B_f + \rho_\epsilon \star A')\|_{\mathcal{M}} \leq 2\|f\|_1 + \|\operatorname{Div}_y A\|_{\mathcal{M}}$. The hypothesis (CI in $\mathbb{R}^{n+1}$) tells us therefore

$$\iint (\eta f)^{\frac{1}{n}} \left(\det\left(\rho_\epsilon \star (\delta_{x_0=0} \otimes A)\right)\right)^{\frac{1}{n}} \mathrm{d}y\,\mathrm{d}x_0 \leq_n \left(\|f\|_1 + \|\operatorname{Div}_y A\|_{\mathcal{M}}\right)^{1+\frac{1}{n}}.$$

Since $\det^{\frac{1}{n}}$ is a concave function over $\mathbf{Sym}_n^+$ (see for instance [9, Theorem 6.10]), we have by Jensen Inequality

$$\left(\det\left(\rho_\epsilon \star (\delta_{x_0=0} \otimes A)\right)\right)^{\frac{1}{n}} \geq \rho_\epsilon \star \left(\left(\det(\delta_{x_0=0} \otimes A)\right)^{\frac{1}{n}}\right) = \rho_\epsilon \star \left(\delta_{x_0=0} \otimes \left((\det A)^{\frac{1}{n}}\right)\right).$$

The above inequality therefore implies

$$\iint (\eta f)^{\frac{1}{n}} \rho_\epsilon \star \left(\delta_{x_0=0} \otimes \left((\det A)^{\frac{1}{n}}\right)\right) \mathrm{d}y\,\mathrm{d}x_0 \leq_n \left(\|f\|_1 + \|\operatorname{Div}_y A\|_{\mathcal{M}}\right)^{1+\frac{1}{n}}.$$

The left-hand side can be recast as

$$\left\langle \delta_{x_0=0} \otimes \left((\det A)^{\frac{1}{n}}\right), \rho_\epsilon \star \left((\eta f)^{\frac{1}{n}}\right)\right\rangle.$$

Passing to the limit as $\epsilon \to 0$, there remains

$$\left\langle \delta_{x_0=0} \otimes \left((\det A)^{\frac{1}{n}}\right), (\eta f)^{\frac{1}{n}} \right\rangle \leq_n \left(\|f\|_1 + \|\operatorname{Div}_y A\|_{\mathcal{M}}\right)^{1+\frac{1}{n}},$$

where the left-hand side is nothing but $\langle (\det A)^{\frac{1}{n}}, f^{\frac{1}{n}}\rangle$.

The end of the proof uses a classical argument of scaling. For every $\phi \in \mathcal{D}_+(\mathbb{R}^n)$ and parameter $\lambda \geq 0$, we apply the above inequality to $f = (\lambda\phi)^n$, so that

$$\left\langle (\det A)^{\frac{1}{n}}, \lambda\phi \right\rangle \leq_n \left(\lambda^n \|\phi\|_n^n + \|\operatorname{Div}_y A\|_{\mathcal{M}}\right)^{1+\frac{1}{n}},$$

Choosing

$$\lambda := \frac{\|\mathrm{Div}_y A\|_{\mathscr{M}}^{\frac{1}{n}}}{\|\phi\|_n},$$

we receive

$$\left\langle (\det A)^{\frac{1}{n}}, \phi \right\rangle \leq_n \|\phi\|_n \|\mathrm{Div}_y A\|_{\mathscr{M}}.$$

We conclude with Radon–Nikodym.

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