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Comptes Rendus
Mathématique

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Volume 362 (2024), p. 1449-1453

Online since: 14 November 2024

https://doi.org/10.5802/crmath.655

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www.centre-mersenne.org — e-ISSN : 1778-3569

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Comptes Rendus. Mathématique
2024, Vol. 362, p. 1449-1453
https://doi.org/10.5802/crmath.655

Research article / *Article de recherche*
Representation theory / *Théorie des représentations*

On a theorem of B. Keller on Yoneda algebras of simple modules

Sur un théorème de B. Keller sur les algèbres de Yoneda de modules simples

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Abstract. A theorem of Keller states that the Yoneda algebra of the simple modules over a finite-dimensional algebra is generated in cohomological degrees 0 and 1 as a minimal A_∞-algebra. We provide a proof of an extension of Keller's theorem to abelian length categories by reducing the problem to a particular class of Nakayama algebras, where the claim can be shown by direct computation.

Résumé. Un théorème de Keller stipule que l'algèbre de Yoneda des modules simples sur une algèbre de dimension finie est générée en degrés cohomologiques 0 et 1 comme une A_∞-algèbre minimale. Nous prouvons une extension du théorème de Keller aux catégories de longueur abélienne en réduisant le problème à une classe particulière d'algèbres de Nakayama, où l'affirmation peut être démontrée par un calcul direct.

Keywords. Yoneda algebras, simple modules, Nakayama algebras, A_∞-algebras.

Mots-clés. Algèbres de Yoneda, modules simples, algèbres de Nakayama, A_∞-algèbres.

2020 Mathematics Subject Classification. 18G70, 16G20, 16G70.

Funding. The author's research was supported by the Swedish Research Council (Vetenskapsrådet) Research Project Grant 2022-03748 'Higher structures in higher-dimensional homological algebra.'

Manuscript received 23 February 2024, revised 15 May 2024, accepted 3 June 2024.

We work over an arbitrary field $\mathbf{k}$. Let A be a finite-dimensional algebra and S the direct sum of a complete set of representatives of the simple (right) A-modules. The Yoneda algebra $\operatorname{Ext}_A^*(S,S)$, as a graded algebra, does not determine the algebra A up to Morita equivalence, as the following example shows (see [8, Section 2.1] or [13, Example B.2.2]): Let $A=\mathbf{k}[x]/(x^\ell)$, $\ell\geq 3$; then

$$\operatorname{Ext}_A^*(S,S)\cong\mathbf{k}[u,v]/(u^2),\qquad |u|=1,\ |v|=2,\tag{1}$$

does not depend on ℓ. On the other hand, the Yoneda algebra of an arbitrary finite-dimensional algebra inherits, via Kadeishvili's Homotopy Transfer Theorem [5, 15], the structure of a minimal[1] A_∞-algebra since the Yoneda algebra is the cohomology of the differential graded algebra $\operatorname{RHom}_A(S,S)$. Endowed with this additional A_∞-structure the Yoneda algebra does determine the algebra A up to Morita equivalence [7, Section 7.8]. The purpose of this short article is to provide a proof of a minor extension of a theorem of Keller [8, Section 2.2, Proposition 1(b)] that is stated

[1] Recall that an A_∞-algebra is minimal if its underlying complex has vanishing differential.

below. A proof of Keller's theorem that utilises the calculus of Massey products was announced by Minamoto in [16]; our proof should have some similarities with his.

Theorem (Keller). *Let $\mathscr{A}$ be a $\mathbf{k}$-linear abelian length category [10] with only finitely many pairwise non-isomorphic simple objects, for example the category of finite-dimensional modules over a finite-dimensional algebra. Let $S_1, \ldots, S_n$ be a complete set of representatives of the simple objects in $\mathscr{A}$ and set $S := S_1 \oplus \cdots \oplus S_n$. Then, the Yoneda algebra $\operatorname{Ext}^*_{\mathscr{A}}(S,S)$ is generated by its homogeneous components of cohomological degrees 0 and 1 as an A_∞-algebra.*

Remark. Since the abelian category $\mathscr{A}$ is not assumed to have any non-zero projective or injective objects, we interpret the Yoneda algebra $\operatorname{Ext}^*_{\mathscr{A}}(S,S)$ in terms of Yoneda equivalence classes of exact sequences [19]. The results in [18, Section III.3] (see in particular paragraph III.3.3.2 therein) readily imply that we can identify the Yoneda algebra with the graded algebra $\bigoplus_{k\geq 0} \operatorname{Hom}(S, S[k])$ of endomorphisms of S in the derived category of $\mathscr{A}$. Thus, we may and we will identify minimal A_∞-algebra structures on the Yoneda algebra with those of the latter graded algebra.

Remark. The isomorphism (1) shows that the conclusion of the theorem may fail if the Yoneda algebra $\operatorname{Ext}^*_{\mathscr{A}}(S,S)$ is considered as a graded algebra only.

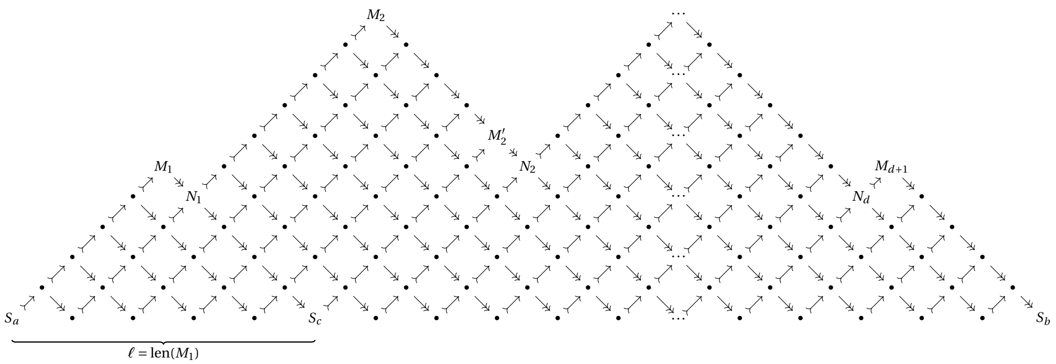
Proof of the theorem. We use freely the theory of A_∞-categories [7, 12], as well as the theory of differential graded (=DG) categories and their derived categories [6, 9]. We also assume familiarity with the Auslander–Reiten (=AR) theory of Nakayama algebras, see for example [1, Chapter V].

Let $d \geq 1$ and $\delta \in \operatorname{Ext}^{d+1}_{\mathscr{A}}(S,S)$ a Yoneda class represented by an exact sequence

$$0 \longrightarrow S_a \longrightarrow M_1 \longrightarrow M_2 \longrightarrow \cdots \longrightarrow M_{d+1} \longrightarrow S_b \longrightarrow 0 \tag{2}$$

between some simple objects; notice that we identify $\operatorname{Ext}^k_{\mathscr{A}}(S,S) = \bigoplus_{i,j=1}^n \operatorname{Ext}^k_{\mathscr{A}}\big(S_i, S_j\big)$. Below we prove that δ is generated by Yoneda classes of degrees 1 and d under the operations $m_k^{\mathscr{A}}$ with $k \leq \operatorname{len}(M_1)$. The theorem then follows by induction on $d \geq 1$.

A straightforward inductive argument shows that the exact sequence (2) can be extended, by first choosing composition series of $N_i := \operatorname{img}(M_i \to M_{i+1})$, to a commutative diagram of the following form in which all squares are bicartesian, the apparent sequences at the bottom are short exact, and all of the objects along the bottom of the diagram are simple:



Consider now the Nakayama algebra $B = \mathbf{k}Q_B/I$ with Gabriel quiver

$$Q_B: \quad 1 \to 2 \to \ldots \to p, \qquad p = \textstyle\sum_{i=1}^{d+1} \operatorname{len}(M_i) - \sum_{j=1}^{d} \operatorname{len}\big(N_j\big),$$

and Kupisch series

$$(1,2,\dots,\operatorname{len}(M_1),\operatorname{len}(N_1)+1,\operatorname{len}(N_1)+2,\dots,\operatorname{len}(M_2),\\ \dots,\operatorname{len}(N_{d-1})+1,\operatorname{len}(N_{d-1})+2,\dots,\operatorname{len}(M_d),\operatorname{len}(M_{d+1})).$$

Observe that the above commutative diagram is indexed by the AR quiver of the category $\operatorname{mod} B$ of finite-dimensional (right) B-modules. In other words, we can extend the exact sequence (2) to an exact functor $F\colon \operatorname{mod} B \to \mathscr{A}$ that sends the simple B-modules to simple objects of $\mathscr{A}$. The exact functor F lifts to an A_∞-functor

$$\widetilde{\mathscr{F}}\colon \mathrm{D^b}(\operatorname{mod} B)_{\mathrm{dg}} \longrightarrow \mathrm{D^b}(\mathscr{A})_{\mathrm{dg}}$$

between the corresponding bounded derived DG categories, for example because $\mathrm{D^b}(\operatorname{mod} B)_{\mathrm{dg}}$ is the pre-triangulated hull of $\operatorname{mod} B$ [3, Theorem 6.1 and Example 6.2]. The many-objects version of the Homotopy Transfer Theorem yields minimal models of $\mathrm{D^b}(\operatorname{mod} B)_{\mathrm{dg}}$ and $\mathrm{D^b}(\mathscr{A})_{\mathrm{dg}}$ so that we may consider the induced A_∞-functor

$$\begin{array}{ccc} H^*(\mathrm{D^b}(\operatorname{mod} B)_{\mathrm{dg}}) & \overset{\mathscr{F}}{\dashrightarrow} & H^*(\mathrm{D^b}(\mathscr{A})_{\mathrm{dg}}) \\ \wr\downarrow & & \uparrow\wr \\ \mathrm{D^b}(\operatorname{mod} B)_{\mathrm{dg}} & \xrightarrow{\tilde{\mathscr{F}}} & \mathrm{D^b}(\mathscr{A})_{\mathrm{dg}} \end{array} \qquad \begin{array}{ccc} H^*(\mathrm{D^b}(\operatorname{mod} B)_{\mathrm{dg}}) & \overset{\mathscr{F}_1}{\dashrightarrow} & H^*(\mathrm{D^b}(\mathscr{A})_{\mathrm{dg}}) \\ \mathrm{id}\downarrow & & \uparrow\mathrm{id} \\ H^*(\mathrm{D^b}(\operatorname{mod} B)_{\mathrm{dg}}) & \xrightarrow{H^*(\tilde{\mathscr{F}})} & H^*(\mathrm{D^b}(\mathscr{A})_{\mathrm{dg}}) \end{array}$$

We obtain, in particular, an A_∞-morphism

$$\mathscr{F}\colon \operatorname{Ext}^*_B(S,S) \longrightarrow \operatorname{Ext}^*_{\mathscr{A}}(S,S),$$

where the Yoneda algebras are now endowed with minimal A_∞-algebra structures.

We claim that the underlying morphism of graded algebras

$$\mathscr{F}_1\colon \operatorname{Ext}^*_B(S,S) \longrightarrow \operatorname{Ext}^*_{\mathscr{A}}(S,S), \qquad \gamma \longmapsto \delta, \tag{3}$$

maps the Yoneda class γ of an augmented minimal projective resolution of the simple B-module concentrated at the vertex p of the quiver Q_B to the class δ. Indeed, $\gamma = \alpha_{d+1}\alpha_2\cdots\alpha_1$ is the product of Yoneda classes of short exact sequences between certain indecomposable B-modules and, therefore,

$$\mathscr{F}_1(\gamma) = \mathscr{F}_1(\alpha_{d+1}\alpha_d\cdots\alpha_1) = \mathscr{F}_1(\alpha_{d+1})\mathscr{F}_1(\alpha_d)\cdots\mathscr{F}_1(\alpha_1) = \delta.$$

Here, the classes $\mathscr{F}_1(\alpha_i)$, $i = 1,\dots,d+1$, satisfy the equality on the right-hand side by construction since the restriction of the graded functor $\mathscr{F}_1\colon H^*(\mathrm{D^b}(\operatorname{mod} B)_{\mathrm{dg}}) \to H^*(\mathrm{D^b}(\mathscr{A})_{\mathrm{dg}})$ to the graded subcategories spanned by $\operatorname{mod} B$ and $\mathscr{A}$, respectively, is induced by the exact functor $F\colon \operatorname{mod} B \to \mathscr{A}$ above.

Since Nakayama algebras are monomial algebras, an explicit description of a minimal model of the Yoneda algebra $\operatorname{Ext}^*_B(S,S)$ is available due to independent work of Chuang and King (unpublished) and of Tamaroff [17, Theorem 4.9].[2] In particular, we can assume that[3]

$$m^B_\ell(\eta_1 \otimes \eta_2 \otimes \cdots \otimes \eta_{\ell-1} \otimes \eta) = \gamma, \qquad \ell = \operatorname{len}(M_1), \tag{4}$$

where $\eta_i \in \operatorname{Ext}^1_B(S_{i+1},S_i) \cong \mathbf{k}$ is the Yoneda class of an AR sequence and $\eta \in \operatorname{Ext}^d_B(S_p,S_\ell) \cong \mathbf{k}$ is the Yoneda class of an exact sequence of the form

$$0 \longrightarrow S_\ell \longrightarrow M'_2 \longrightarrow \cdots \longrightarrow M_{d+1} \longrightarrow S_p \longrightarrow 0,$$

[2] The computation of the explicit product that we need is straightforward using the HTT; see also [4].

[3] Compare with [16, Theorem 1] where the equivalent of the Massey product $\langle \eta_1,\dots,\eta_{\ell-1},\eta\rangle \ni \pm\gamma$ appears.

with M_2' the injective hull of the simple B-module S_ℓ and M_{d+1} the projective cover of S_p, as indicated in the above diagram that we now interpret as the AR quiver of $\operatorname{mod} B$. Moreover, an elementary computation shows that

$$m^B_{j-i+1}(\eta_i \otimes \eta_{i+1} \otimes \cdots \otimes \eta_j) \in \operatorname{Ext}^2_B\big(S_{j+1}, S_i\big) = 0 \tag{5}$$

whenever $1 \leq i < j < \ell$, and also

$$m^B_{\ell-i+1}(\eta_i \otimes \eta_{i+1} \otimes \cdots \otimes \eta_{\ell-1} \otimes \eta) \in \operatorname{Ext}^*_B\big(S_p, S_i\big) = 0 \tag{6}$$

whenever $1 < i < \ell$. Here we use that the operation m^B_k has degree $2-k$.

Finally, the A_∞-morphism $\mathscr{F}\colon \operatorname{Ext}^*_B(S,S) \to \operatorname{Ext}^*_{\mathscr{A}}(S,S)$ satisfies in particular an equation of the form

$$0 = \partial(\mathscr{F}_\ell) = \sum_{\substack{r+1+t=k\\ r+s+t=\ell}} \pm \mathscr{F}_k \circ (1^{\otimes r} \otimes m^B_s \otimes 1^{\otimes t}) - \sum_{\substack{2\leq k\leq \ell\\ i_1+\cdots+i_k=\ell}} \pm m^{\mathscr{A}}_k \circ (\mathscr{F}_{i_1} \otimes \cdots \otimes \mathscr{F}_{i_k}),$$

where $\partial = 0$ since the A_∞-algebras involved are minimal. Therefore, using equations (3) and (4),[4]

$$\begin{aligned}\delta = \mathscr{F}_1(\gamma) &= \mathscr{F}_1(m^B_\ell(\eta_1 \otimes \cdots \otimes \eta_{\ell-1} \otimes \eta))\\ &= m^{\mathscr{A}}_\ell\underbrace{(\mathscr{F}_1(\eta_1) \otimes \cdots \otimes \mathscr{F}_1(\eta_{\ell-1}) \otimes \mathscr{F}_1(\eta))}_{\in \operatorname{Ext}^{1,d}_{\mathscr{A}}(S,S)} + \omega,\end{aligned}$$

where

$$\omega = \sum_{\substack{r+1+t=k>1\\ r+s+t=\ell}} \pm \mathscr{F}_k(\underbrace{\;\cdots\;}_{=0}) + \sum_{\substack{2\leq k<\ell\\ i_1+\cdots+i_k=\ell}} \pm m^{\mathscr{A}}_k(\underbrace{\quad\cdots\quad}_{\in \operatorname{Ext}^{1,d}_{\mathscr{A}}(S,S)}).$$

To argue the vanishing of the first term we use that the inputs involve one the higher products (5)-(6) that we know vanish, and for the condition on the degrees of the inputs in the second term we note that

$$\begin{aligned}\mathscr{F}_{j-i+1}(\eta_i \otimes \cdots \otimes \eta_j) &\in \operatorname{Ext}^1_{\mathscr{A}}(S,S) && 1 \leq i < j < \ell,\\ \mathscr{F}_{\ell-i+1}(\eta_i \otimes \cdots \otimes \eta_{\ell-1} \otimes \eta) &\in \operatorname{Ext}^d_{\mathscr{A}}(S,S) && 1 < i < \ell,\end{aligned}$$

since the component $\mathscr{F}_k$ is a morphism of degree $1-k$. This finishes the proof. □

Remark. In a MathOverflow post [14], Madsen gives an outline of Keller's original proof of the theorem, which is unpublished.[5] Keller's proof uses the description of the category of finite-dimensional modules over a finite-dimensional algebra A as the category of twisted stalks on the Yoneda algebra $\operatorname{Ext}^*_A(S,S)$ with its minimal A_∞-algebra structure. In fact, one may replace the ambient abelian category by the exact subcategory of objects filtered by a finite collection of objects, with Ext^* now understood in this exact subcategory. Our proof, which can be adapted to this more general setting with the appropriate modifications, avoids making explicit use of the category of twisted stalks by instead reducing the problem to a computation with Nakayama algebras.

Acknowledgements

The author thanks Julian Külshammer for informing him of Madsen's MathOverflow post [14] and for interesting discussions on possible extensions of Keller's theorem. The author also thanks Bernhard Keller for clarifications on his original proof of the theorem.

[4] Compare with [2, Theorem A(ii)] and see also Section 3 therein.
[5] But see [11, Theorem 3.22].

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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