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Intersection of parabolic subgroups in Euclidean braid groups: a short proof

Intersection de sous-groupes paraboliques dans les groupes de tresses euclidiens : une démonstration courte

María Cumplido [*,a], **Federica Gavazzi** [b] **and Luis Paris** [b]

[a] Departamento de Álgebra, Facultad de Matemáticas, Universidad de Sevilla, Calle Tarfia s/n 41012, Seville, Spain

[b] IMB, UMR5584, CNRS, Université de Bourgogne, 21000 Dijon, France

E-mails: cumplido@us.es, federica.gavazzi@u-bourgogne.fr, luis.paris@u-bourgogne.fr

Abstract. We give a short proof for the fact, already proven by Thomas Haettel, that the arbitrary intersection of parabolic subgroups in Euclidean Braid groups $A[\widetilde{A}_n]$ is again a parabolic subgroup. To that end, we use that the spherical-type Artin group $A[B_{n+1}]$ is isomorphic to $A[\widetilde{A}_n] \rtimes \mathbb{Z}$.

Résumé. Nous donnons une démonstration courte du fait, déjà démontré par Thomas Haettel, que l'intersection arbitraire de sous-groupes paraboliques dans les groupes de tresses euclidiens $A[\widetilde{A}_n]$ est à nouveau un sous-groupe parabolique. À cette fin, nous utilisons le fait que le groupe d'Artin de type sphérique $A[B_{n+1}]$ est isomorphe à $A[\widetilde{A}_n] \rtimes \mathbb{Z}$.

Keywords. Group theory, Artin groups, Euclidean braid groups, parabolic subgroups, group isomorphism.

Mots-clés. Théorie des groupes, groupes d'Artin, groupes de tresses euclidiens, sous-groupes paraboliques, isomorphisme de groupes.

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An Artin (or Artin-Tits) group A_S is any group with a presentation

$$\langle S | \underbrace{sts\cdots}_{m_{st}} = \underbrace{tst\cdots}_{m_{st}},\ s,t \in S,\ m_{s,t} \neq \infty,\ s \neq t \rangle$$

where S is a finite set of generators and $(m_{s,t})_{s,t\in S}$ is a symmetric matrix with 1's in the diagonal and the other entrances in $\{2,3,\ldots,\infty\}$. The number of known global results for these groups is very limited, and for some decades now, classic problems such as the word problem, the conjugacy problem, or the $K(\pi,1)$ conjecture have been the subject of study by group theorists. Specifically, it has become necessary to better study the properties of certain specific subgroups: the parabolic subgroups. A standard parabolic subgroup A_X of A_S is a subgroup generated by a subset $X \subset S$, and thanks to [8], we also know that it coincides with the Artin group

* Corresponding author

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on X with the same relations that these generators have in A_S. A parabolic subgroup is any conjugate of a standard parabolic subgroup. These subgroups play a principal role in the construction of simplicial complexes associated with Artin groups [1–4, 7], either as stabilizers of simplices or as the building blocks of the complexes. In the case of braid groups, the parabolic subgroups coincide with the isotopy classes of non-degenerate multicurves in the n-punctured disc (containing the vertices of the same curve complex). However, the basic question of whether the arbitrary intersection of parabolic subgroups is a parabolic subgroup remains open in most cases.

To establish notations, we will use Coxeter graphs. A Coxeter graph Γ_S encodes the information of an Artin group A_S as follows: each generator corresponds to a vertex, and two vertices s, t are connected by an edge if the vertices do not commute. This edge is labeled by $m_{s,t}$ if $m_{s,t} > 3$ and by ∞ if there is no relation between s and t. In this way, we can also refer to A_S as $A[\Gamma_S]$. The Euclidean braid group (also known as affine braid group) with $n+1$ generators is the group $A[\widetilde{A}_n]$, where $\widetilde{A}_n$ is the graph in Figure 1:

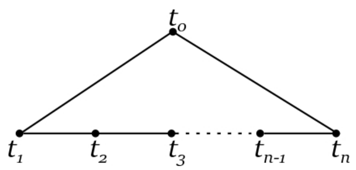

Figure 1. The Coxeter graph $\widetilde{A}_n$.

In this article, we will give an alternative and concise proof of the following theorem:

Theorem 1 ([5, Corollary N]). *Any intersection of parabolic subgroups in $A[\widetilde{A}_n]$ is a parabolic subgroup.*

We know, according to [2], that the intersection of parabolic subgroups of a spherical-type Artin group remains a parabolic subgroup. In particular, this is true in the group $A[B_{n+1}]$, where B_{n+1} is the graph illustrated in Figure 2.

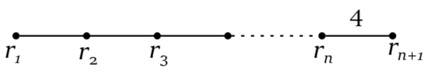

Figure 2. The Coxeter graph B_{n+1}.

Using the notation of the figure, let $\rho = r_1 \dots r_n r_{n+1}$. For $1 \le i \le n-1$, it holds that $\rho r_i \rho^{-1} = r_{i+1}$. If we additionally define $r_0 := \rho r_n \rho^{-1}$, we can state that the previous equality is true modulo $n+1$ (observing that $\rho^2 r_n \rho^{-2} = r_1$ is sufficient). Furthermore, we consider the following exterior automorphism f of $A[\widetilde{A}_n]$:

$$f : A[\widetilde{A}_n] \longrightarrow A[\widetilde{A}_n]$$
$$t_i \longmapsto t_{i+1},$$

again with the indexes taken modulo $n+1$. Given this, we can define an action of the infinite cyclic group $\mathbb{Z} \cong \langle u \rangle$ on $A[\widetilde{A}_n]$ by setting $u \cdot g = ugu^{-1} := f(g)$, for all $g \in A[\widetilde{A}_n]$. With this action, we can define the semidirect product $A[\widetilde{A}_n] \rtimes \langle u \rangle$, that has a presentation with generators $\{t_1, \dots, t_n, u\}$ and relations given by the union of the Artin relations of $A[\widetilde{A}_n]$ and the set $\{ut_i u^{-1} = t_{i+1} \mid 0 \le i \le n\}$ modulo $n+1$.

Theorem 2 ([6]). *The map $\varphi : A[\widetilde{A}_n] \rtimes \langle u \rangle \longrightarrow A[B_{n+1}]$ that sends t_i to r_i and u to ρ is an isomorphism.*

Remark 3. The restriction of φ to $A[\widetilde{A}_n]$ gives an embedding of this Artin group in $A[B_{n+1}]$.

The proof of the next result is a straightforward consequence of the definition of φ.

Lemma 4. *Let ξ be the group homomorphism defined by*

$$\begin{aligned} \xi : A[B_{n+1}] &\longrightarrow \mathbb{Z} \\ r_i &\longmapsto 0 \qquad \text{for } 1 \le i \le n, \\ r_{n+1} &\longmapsto 1. \end{aligned}$$

Then ρ is mapped to 1, and the kernel of ξ is $\varphi(A[\widetilde{A}_n])$.

Proof of Theorem 1. Firstly, notice that if P is a proper parabolic subgroup of $A[\widetilde{A}_n]$, then $\varphi(P)$ is a parabolic subgroup of $A[B_{n+1}]$. The only case in which this is not clear is when $P = gA[\widetilde{A}_n]_X g^{-1}$ and X contains t_0, which is sent to $\rho r_n \rho^{-1}$ by φ. However, since P is proper, X does not contain all the generators of $A[\widetilde{A}_n]$, thus, using ρ, we can always conjugate $\varphi(X)$ to a subset of $\{r_1, \dots, r_n\}$. Now suppose that P_1 and P_2 are two parabolic subgroups of $A[\widetilde{A}_n]$. Since in $A[B_{n+1}]$ the intersection of parabolic subgroups is a parabolic subgroup, we have, in particular, that $\varphi(P_1 \cap P_2) = \varphi(P_1) \cap \varphi(P_2)$ is a parabolic subgroup of $A[B_{n+1}]$.

To complete the proof, it remains to show that if Q is a parabolic subgroup of $A[B_{n+1}]$ such that $P := \varphi^{-1}(Q) \subset A[\widetilde{A}_n]$, then P is a parabolic subgroup of $A[\widetilde{A}_n]$. We can write $Q = hA[B_{n+1}]_Y h^{-1}$, with $Y \subset \{r_1, \dots, r_{n+1}\}$ and $h \in A[B_{n+1}]$. First, we show that $r_{n+1} \notin Y$. If we suppose otherwise, then $hr_{n+1}h^{-1} \in Q$ and $\varphi^{-1}(hr_{n+1}h^{-1}) \in P \subset A[\widetilde{A}_n]$. In particular, by Lemma 4, the element $hr_{n+1}h^{-1}$ must belong to the kernel of ξ, that coincides with $\varphi(A[\widetilde{A}_n])$. But computing $\xi(hr_{n+1}h^{-1}) = 1$, we get a contradiction. Therefore, $Y \subset \{r_1, \dots, r_n\} = \varphi(\{t_1, \dots, t_n\})$. Furthermore, since $A[\widetilde{A}_n] \rtimes \langle u \rangle \cong A[B_{n+1}]$, we can write $h = h_1 \rho^m$, with $\varphi^{-1}(h_1) \in A[\widetilde{A}_n]$ and $m \in \mathbb{Z}$. Thus, $Q = h_1 \rho^m A[B_{n+1}]_Y \rho^{-m} h_1^{-1}$, and exploiting the fact that $u^m A[\widetilde{A}_n]_{\varphi^{-1}(Y)} u^{-m} = f^m(A[\widetilde{A}_n]_{\varphi^{-1}(Y)})$, we get

$$P = \varphi^{-1}(Q) = \varphi^{-1}(h_1) f^m(A[\widetilde{A}_n]_{\varphi^{-1}(Y)})(\varphi^{-1}(h_1))^{-1} = \varphi^{-1}(h_1) A[\widetilde{A}_n]_{f^m(\varphi^{-1}(Y))}(\varphi^{-1}(h_1))^{-1},$$

which has the form of a parabolic subgroup in $A[\widetilde{A}_n]$. Finally (see the complete argument in [3, Corollary 16]), in any Artin group, any descending chain of inclusions of parabolic subgroups has to stabilize, and then the finite intersection of parabolic subgroups, being a parabolic subgroup, implies that the same is true for an arbitrary intersection. □

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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