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A note on the exact formulas for certain 2-color partitions

Note sur les formules exactes de certaines partitions à 2 couleurs

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Abstract. Let $p \leq 23$ be a prime and $a_p(n)$ count the number of partitions of n where parts that are multiple of p come up with 2 colors. Using a result of Sussman, we derive the exact formula for $a_p(n)$ and obtain an asymptotic formula for $\log a_p(n)$. Our results partially extend the work of Mauth, who proved the asymptotic formula for $\log a_2(n)$ conjectured by Banerjee et al.

Résumé. Soit $p \leq 23$ un nombre premier et $a_p(n)$ compte le nombre de partitions de n où les parties qui sont multiples de p donnent 2 couleurs. En utilisant un résultat de Sussman, nous dérivons la formule exacte pour $a_p(n)$ et obtenons une formule asymptotique pour $\log a_p(n)$. Nos résultats étendent partiellement le travail de Mauth, qui a prouvé la formule asymptotique pour $\log a_2(n)$ conjecturée par Banerjee et al.

Keywords. Circle method, η-quotients, partitions, asymptotic formula.

Mots-clés. Méthode des cercles, η-quotients, partitions, formule asymptotique.

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1. Introduction

Throughout this paper, we denote $(a;q)_\infty = \prod_{n\geq 0}(1-aq^n)$ for $a \in \mathbb{C}$. Recall that a partition of a positive integer n is a nonincreasing finite sequence of positive integers, known as its parts, whose sum is n. We define $p(n)$ as the number of partitions of n, which can be seen as the coefficients of its generating function given by

$$\frac{1}{(q;q)_\infty} = \sum_{n=0}^{\infty} p(n)q^n.$$

Hardy and Ramanujan [9] proved the following asymptotic formula for $p(n)$ given by

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right), \quad n \longrightarrow \infty,$$

using the celebrated Circle Method. Rademacher [14] refined the Hardy–Ramanujan Circle Method and derived the exact formula

$$p(n) = \frac{1}{\pi\sqrt{2}}\sum_{k=1}^{\infty} A_k(n)\sqrt{k}\frac{\mathrm{d}}{\mathrm{d}n}\left(\frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right)}{\sqrt{n-\frac{1}{24}}}\right)$$

where

$$A_k(n) := \sum_{\substack{0\le h<k\\ \gcd(h,k)=1}} \exp\left[\pi\mathrm{i}\left(s(h,k)-\frac{2nh}{k}\right)\right]$$

and

$$s(h,k) := \sum_{j=1}^{k-1}\frac{j}{k}\left(\left\{\frac{hj}{k}\right\}-\frac{1}{2}\right)$$

is the Dedekind sum, where $\{t\}$ denotes the fractional part of t. Recently, Banerjee et al. [3] proved the following refined asymptotic formula for $p(n)$ given by

$$\log p(n) \sim \pi\sqrt{\frac{2n}{3}} - \log n - \log 4\sqrt{3} - \frac{0.44\cdots}{\sqrt{n}}, \quad n\longrightarrow\infty,$$

from a family of inequalities for $p(n)$, which was used to prove a refinement of the inequality of DeSalvo and Pak [8], and Chen, Wang and Xie [6] that reads

$$\left(1+\frac{\pi}{24n^{3/2}}-\frac{1}{n^2}\right)p(n-1)p(n+1) < p(n)^2 < \left(1+\frac{\pi}{24n^{3/2}}\right)p(n-1)p(n+1)$$

for $n \ge 120$.

Let $a_k(n)$ be the number of partitions of n where parts that are multiple of k come up with 2 colors. For $k=1$, we have $a_1(n) = p_2(n)$ where $p_2(n)$ is the total number of partitions of n where parts come up with 2 colors and for $k=2$, $a_2(n)$ also counts the number of cubic partitions (see [10]). The generating function for $a_k(n)$ is given by [1]

$$\frac{1}{(q;q)_\infty(q^k;q^k)_\infty} = \sum_{n=0}^{\infty} a_k(n)q^n.$$

Kotesovec [11] found the following asymptotic formula for $a_2(n)$ given by

$$a_2(n) \sim \frac{1}{8n^{5/4}}\exp(\pi\sqrt{n}), \quad n\longrightarrow\infty.$$

Banerjee et al. [3] conjectured that $a_2(n)$ satisfies the following asymptotic formula

$$\log a_2(n) \sim \pi\sqrt{n} - \frac{5}{4}\log n - \log 8 - \frac{0.79\cdots}{\sqrt{n}}, \quad n\longrightarrow\infty. \tag{1}$$

Recently, Mauth [12] proved (1) by finding the exact formula of Rademacher type for $a_2(n)$ using a result of Zuckerman [16].

In this paper, we use a result of Sussman [15] to derive Rademacher-type formulas for $a_p(n)$ when $p \le 23$ is a prime, and deduce their asymptotic formulas in the spirit of (1), which can be considered as a partial extension of Mauth's work. We give our main results as follows.

Theorem 1. *Let $p \le 23$ be a prime. Then for $n \ge 1$ we have*

$$a_p(n) = 2\pi\sqrt{p}\left(\frac{1+p^{-1}}{24n-p-1}\right)\sum_{j=1}^{p-1}\sum_{\substack{m=1\\ m\equiv j \bmod p}}^{\infty} I_2\left(\frac{\pi}{6m}\sqrt{(1+p^{-1})(24n-p-1)}\right)\frac{b_m(n)}{m}$$
$$+ 2\pi\left(\frac{1+p}{24n-p-1}\right)\sum_{\substack{m=1\\ p\mid m}}^{\infty} I_2\left(\frac{\pi}{6m}\sqrt{(1+p)(24n-p-1)}\right)\frac{b_m(n)}{m},$$

where

$$b_k(n) := \sum_{\substack{0\le h<k\\ \gcd(h,k)=1}} \exp\left[-\frac{2\pi nh\mathrm{i}}{k} + \pi s(h,k)\mathrm{i} + \pi s\left(\frac{ph}{\gcd(p,k)}, \frac{k}{\gcd(p,k)}\right)\mathrm{i}\right]$$

and $I_2(s)$ is the second modified Bessel function of the first kind (see Section 2).

Corollary 2. *For $p \le 23$ prime, we have*

$$\log a_p(n) \sim \pi\sqrt{\frac{2n(1+p^{-1})}{3}} - \frac{5}{4}\log n + \log\frac{2\sqrt{3p}(1+p^{-1})^{3/4}}{24^{5/4}} - \frac{c_p}{24\sqrt{6n}}, \quad n \longrightarrow \infty$$

where

$$c_p := \frac{135}{\pi\sqrt{1+p^{-1}}} + \pi\sqrt{\frac{(1+p)^3}{p}}.$$

Setting $p = 2$ in Corollary 2, we deduce the asymptotic formula due to Mauth [12] given by

$$\log a_2(n) \sim \pi\sqrt{n} - \frac{5}{4}\log n - \log 8 - \left(\frac{15}{8\pi} + \frac{\pi}{16}\right)\frac{1}{\sqrt{n}}, \quad n \longrightarrow \infty,$$

which immediately yields (1).

The paper is organized as follows. In Section 2, we state the Sussman's result on the exact formulas for the Fourier coefficients of a class of η-quotients. In Section 3, we apply this result to prove Theorem 1, and using the asymptotic expansion of $I_2(s)$ due to Banerjee [2], we deduce Corollary 2.

2. Fourier coefficients of a class of η-quotients

We consider in this section the exact formulas for the Fourier coefficients of the following class of holomorphic functions on the open disk given by

$$G(q) := \prod_{r=1}^{R} (q^{m_r}; q^{m_r})_\infty^{\delta_r} = \sum_{n=0}^{\infty} g(n) q^n,$$

where $\mathbf{m} = (m_1, \ldots, m_R)$ is a sequence of R distinct positive integers and $\delta = (\delta_1, \ldots, \delta_R)$ is a sequence of R nonzero integers. The functions $G(q)$ can be seen as η-quotients since $(q;q)_\infty = q^{-1/24}\eta(\tau)$, where Dedekind's eta function, denoted by $\eta(\tau)$, is defined by the infinite product $\eta(\tau) := q^{1/24}\prod_{n\ge1}(1-q^n)$ with $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$ and $\tau \in \mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. For useful properties of the Dedekind's eta function and the definition of η-quotients, we refer the interested reader to the book [13]. Given a particular $G(q)$ with $\sum_{r=1}^{R}\delta_r < 0$, Sussman [15] gave a Rademacher-type exact formula for $g(n)$, which is a special case of the work of Bringmann and Ono [4] on the coefficients of harmonic Maass forms. Sussman's proof follows the original approach of Rademacher [14] on the Hardy–Ramanujan Circle Method. In the case where $\sum_{r=1}^{R}\delta_r \ge 0$, Chern [7] obtained an analogous formula with an error term using the method of O-Y. Chan [5].

Before we state Sussman's result, we need some definitions. For $(h,k) \in \mathbb{N}^2$ with $\gcd(h,k) = 1$, we set

$$\Delta_1 = -\frac{1}{2}\sum_{r=1}^{R}\delta_r, \quad \Delta_2 = \sum_{r=1}^{R} m_r\delta_r,$$
$$\Delta_3(k) = -\sum_{r=1}^{R}\frac{\gcd(m_r,k)^2}{m_r}\delta_r, \quad \Delta_4(k) = \prod_{r=1}^{R}\left(\frac{\gcd(m_r,k)}{m_r}\right)^{\delta_r/2},$$

and

$$\widehat{A}_k(n) = \sum_{\substack{0\le h<k\\ \gcd(h,k)=1}} \exp\left(-\frac{2\pi nh\mathrm{i}}{k} - \pi\mathrm{i}\sum_{r=1}^{R}\delta_r s\left(\frac{m_r h}{\gcd(m_r,k)}, \frac{k}{\gcd(m_r,k)}\right)\right),$$

where $s(h,k)$ is the Dedekind sum defined in Section 1. We also set L as the least common multiple of $m_1,\dots,m_R$, and partition the set $\{1,\dots,L\}$ into two disjoint subsets

$$\mathscr{L}_{>0} := \{1\le l\le L : \Delta_3(l) > 0\},$$
$$\mathscr{L}_{\le 0} := \{1\le l\le L : \Delta_3(l) \le 0\}.$$

Theorem 3 ([15]). *If $\Delta_1 > 0$ and the inequality*

$$\min_{1\le r\le R} \frac{\gcd(m_r,l)^2}{m_r} \ge \frac{\Delta_3(l)}{24} \tag{2}$$

holds for $1\le l\le L$, then for positive integers $n > -\Delta_2/24$, we have

$$g(n) = 2\pi \sum_{l\in\mathscr{L}_{>0}} \Delta_4(l)\left(\frac{\Delta_3(l)}{24n+\Delta_2}\right)^{(\Delta_1+1)/2} \sum_{\substack{k=1\\ k\equiv l \bmod L}}^{\infty} I_{\Delta_1+1}\left(\frac{\pi}{6k}\sqrt{\Delta_3(l)(24n+\Delta_2)}\right)\frac{\widehat{A}_k(n)}{k},$$

where

$$I_\nu(s) := \sum_{m=0}^{\infty} \frac{(\frac{s}{2})^{\nu+2m}}{m!\Gamma(\nu+m+1)}$$

is the νth modified Bessel function of the first kind, and $\Gamma(s) = \int_0^\infty \mathrm{e}^{-t}t^{s-1}\,\mathrm{d}t$ is the gamma function.

3. Proofs of Theorem 1 and Corollary 2

In this section, we use Theorem 3 to prove Theorem 1. As a consequence, we obtain Corollary 2 using the asymptotic expansion of the modified Bessel function of the first kind due to Banerjee [2].

Proof of Theorem 1. Writing

$$\frac{1}{(q;q)_\infty(q^p;q^p)_\infty} = \sum_{n=0}^{\infty} a_p(n)q^n,$$

we have $\mathbf{m} = (1,p), \delta = (-1,-1)$ and $L = p$. We compute $\Delta_1 = 1$ and $\Delta_2 = -p-1$. Recalling that p is a prime, we have that for $l\in\{1,\dots,p\}$,

$$\Delta_3(l) = \begin{cases} 1+\frac{1}{p}, & l\ne p\\ 1+p, & l = p\end{cases}, \qquad \Delta_4(l) = \begin{cases}\sqrt{p}, & l\ne p\\ 1, & l=p\end{cases},$$

so that $\mathscr{L}_{>0} = \{1,\dots,p\}$. Since $p\le 23$, we see that condition (2) holds for $l\in\{1,\dots,p\}$. Applying Theorem 3 with $\widehat{A}_k(n) = b_k(n)$, we obtain

$$a_p(n) = 2\pi\sum_{l\in\mathscr{L}_{>0}}\Delta_4(l)\left(\frac{\Delta_3(l)}{24n-p-1}\right)\sum_{\substack{m=1\\ m\equiv l \bmod p}}^{\infty} I_2\left(\frac{\pi}{6m}\sqrt{\Delta_3(l)(24n-p-1)}\right)\frac{b_m(n)}{m}$$

for $n > (p+1)/24$. Plugging in the values of $\Delta_3(l)$ and $\Delta_4(l)$ yields the desired result. □

We see that the series for $a_p(n)$ in Theorem 1 converges rapidly and that the term $m = 1$ contributes signficantly to the series. We thus obtain the asymptotic formula

$$a_p(n) \sim 2\pi\sqrt{p}\left(\frac{1+p^{-1}}{24n-p-1}\right) I_2\left(\frac{\pi}{6}\sqrt{(1+p^{-1})(24n-p-1)}\right), \quad n\longrightarrow\infty. \tag{3}$$

To prove Corollary 2, we use the following asymptotic formula [2, Equation (2.11)]

$$I_\nu(s) \sim \frac{e^s}{\sqrt{2\pi s}} \sum_{m=0}^{\infty} \frac{(-1)^m d_m(\nu)}{x^m}, \quad s \longrightarrow \infty, \tag{4}$$

where

$$d_m(\nu) = \frac{\binom{\nu-1/2}{m}(\nu+\frac{1}{2})_m}{2^m}$$

with

$$\binom{a}{m} = \begin{cases} \frac{a(a-1)\cdots(a-m+1)}{m!}, & m \in \mathbb{N} \\ 1, & m = 0 \end{cases},$$

$$(a)_m = \begin{cases} a(a+1)\cdots(a+m-1), & m \in \mathbb{N} \\ 1, & m = 0 \end{cases}$$

for $a \in \mathbb{R}$.

Proof of Corollary 2. From (4) we get

$$I_2(s) = \frac{e^s}{\sqrt{2\pi s}}\left(1 - \frac{15}{8s} + O\left(\frac{1}{s^2}\right)\right).$$

In view of (3), we have

$$a_p(n) \sim 2\sqrt{3p} \cdot \frac{(1+p^{-1})^{3/4}}{(24n-p-1)^{5/4}} \cdot \exp\left(\frac{\pi}{6}\sqrt{(1+p^{-1})(24n-p-1)}\right)\left(1 - \frac{45}{8\pi\sqrt{6n(1+p^{-1})}} + O\left(\frac{1}{n}\right)\right). \tag{5}$$

Since

$$\frac{1}{(24n-p-1)^{5/4}} = \frac{1}{(24n)^{5/4}}\left(1 + O\left(\frac{1}{n}\right)\right),$$

$$\exp\left(\frac{\pi}{6}\sqrt{(1+p^{-1})(24n-p-1)}\right) = \exp\left(\pi\sqrt{\frac{2n(1+p^{-1})}{3}}\right)\left(1 - \frac{p^{-1/2}(1+p)^{3/2}\pi}{24\sqrt{6n}} + O\left(\frac{1}{n}\right)\right),$$

we infer from (5) that

$$a_p(n) \sim 2\sqrt{3p} \cdot \frac{(1+p^{-1})^{3/4}}{(24n)^{5/4}}\left(1 + O\left(\frac{1}{n}\right)\right) \cdot \exp\left(\pi\sqrt{\frac{2n(1+p^{-1})}{3}}\right)$$
$$\cdot \left(1 - \frac{p^{-1/2}(1+p)^{3/2}\pi}{24\sqrt{6n}} + O\left(\frac{1}{n}\right)\right)\left(1 - \frac{45}{8\pi\sqrt{6n(1+p^{-1})}} + O\left(\frac{1}{n}\right)\right),$$

and taking logarithms yields the desired conclusion. □

We remark that when $p > 23$ is a prime, condition (2) of Theorem 3 fails. Nonetheless, by numerical experiments via *Mathematica*, we propose the following conjecture.

Conjecture 4. *Corollary 2 also holds for primes $p > 23$.*

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