ACADÉMIE
DES SCIENCES
INSTITUT DE FRANCE

Comptes Rendus
Mathématique

Julien Bichon

Bialgebra cohomology and exact sequences

Volume 362 (2024), p. 1475-1483

Online since: 14 November 2024

https://doi.org/10.5802/crmath.659

CENTRE MERSENNE

The Comptes Rendus. Mathématique *are a member of the Mersenne Center for open scientific publishing*
www.centre-mersenne.org — e-ISSN : 1778-3569

ACADÉMIE DES SCIENCES
INSTITUT DE FRANCE

Comptes Rendus. Mathématique
2024, Vol. 362, p. 1475-1483
https://doi.org/10.5802/crmath.659

Research article / *Article de recherche*
Algebra / *Algèbre*

Bialgebra cohomology and exact sequences

Cohomologie de bigèbres et suites exactes

Julien Bichon [a]

[a] Université Clermont Auvergne, CNRS, LMBP, F-63000 Clermont-Ferrand, France
E-mail: julien.bichon@uca.fr

Abstract. We show how the bialgebra cohomologies of two Hopf algebras involved in an exact sequence are related, when the third factor is finite-dimensional cosemisimple. As an application, we provide a short proof of the computation of the bialgebra cohomology of the universal cosovereign Hopf algebras in the generic (cosemisimple) case, done recently by Baraquin, Franz, Gerhold, Kula and Tobolski.

Résumé. Nous montrons comment les cohomologies de Gertstenhaber–Schack de deux algèbres de Hopf imbriquées dans une suite exacte courte sont reliées, quand le troisième facteur est cosemisimple de dimension finie. Nous en déduisons une preuve rapide du calcul de la cohomologie de bigèbre des algèbres de Hopf cosouveraines universelles dans le cas générique, établi récemment par Baraquin, Franz, Gerhold, Kula et Tobolski.

Keywords. Hopf algebras, bialgebra cohomology, Yetter–Drinfeld modules.

Mots-clés. Algèbres de Hopf, cohomologie des bigèbres, modules de Yetter–Drinfeld.

2020 Mathematics Subject Classification. 16T05, 16E40.

Manuscript received 9 October 2023, revised 29 May 2024 and 10 June 2024, accepted 17 June 2024.

1. introduction

Gerstenhaber–Schack cohomology, which includes bialgebra cohomology as a special instance, is a cohomology theory adapted to Hopf algebras. It was introduced in [17, 18] by means of an explicit bicomplex modeled on the Hochschild complex of the underlying algebra and the Cartier complex of the underlying coalgebra, with deformation theory as a motivation. See [24] for an exposition, with the original coefficients being Hopf bimodules, but in view of the equivalence between Hopf bimodules and Yetter–Drinfeld modules [23], one can work in the simpler framework of Yetter–Drinfeld modules.

Gerstenhaber–Schack cohomology has been useful in proving some fundamental results in Hopf algebra theory [16, 25], but few concrete computations were known (see [22, 24]) until it was shown by Taillefer [26] that Gerstenhaber–Schack cohomology can be identified with the Ext functor on the category of Yetter–Drinfeld modules: if A is a Hopf algebra, V is a Yetter–Drinfeld module over A and k is the trivial Yetter–Drinfeld module, one has

$$H^*_{\mathrm{GS}}(A,V) \simeq \mathrm{Ext}^*_{\mathscr{YD}^A_A}(k,V).$$

The bialgebra cohomology of A is then defined by $H^*_b(A) = H^*_{\mathrm{GS}}(A,k)$. We will use this Ext description, which opens the way to use classical tools of homological algebra, as a definition.

Note that the category $\mathcal{YD}_A^A$ has enough injective objects [12, 26], so the above Ext spaces can be studied using injective resolutions of V, and when $\mathcal{YD}_A^A$ has enough projective objects (for example if A is cosemisimple, or more generally if A is co-Frobenius), they can also be computed by using projective resolutions of the trivial module.

This note is a contribution to the study of Gerstenhaber–Schack cohomology: we show how the bialgebra (and Gerstenhaber–Schack) cohomologies of two Hopf algebras involved in an exact sequence of Hopf algebras are related when the third factor is a finite-dimensional cosemisimple Hopf algebra, see Theorem 6. When the third factor is the semisimple group algebra of a finite abelian group, the result even takes a nicer form, see Corollary 8.

We apply our result to provide a computation of the bialgebra cohomology of the universal cosovereign Hopf algebras [9] in the generic (cosemisimple) case, a class of Hopf algebras that we believe to be of particular interest in view of their universal property, see [5]. Such a computation has just been done by Baraquin, Franz, Gerhold, Kula and Tobolski [3], but the present proof is shorter.

2. Preliminaries

We work over an algebraically closed field k, and use standard notation from Hopf algebra theory, for which a standard reference is [21].

2.1. *Exact sequences of Hopf algebras*

Recall that a sequence of Hopf algebra maps

$$k \longrightarrow B \xrightarrow{i} A \xrightarrow{p} L \longrightarrow k$$

is said to be exact [1] if the following conditions hold:

(1) i is injective and p is surjective,
(2) $\mathrm{Ker}(p) = Ai(B)^+ = i(B)^+A$, where $i(B)^+ = i(B) \cap \mathrm{Ker}(\varepsilon)$,
(3) $i(B) = A^{\mathrm{co}\,L} = \{a \in A : (\mathrm{id} \otimes p)\Delta(a) = a \otimes 1\} = {}^{\mathrm{co}\,L}A = \{a \in A : (p \otimes \mathrm{id})\Delta(a) = 1 \otimes a\}$.

Note that condition (2) implies $pi = \varepsilon 1$.

In an exact sequence as above, we can assume, without loss of generality, that B is Hopf subalgebra and i is the inclusion map.

A Hopf algebra exact sequence $k \to B \xrightarrow{i} A \xrightarrow{p} L \to k$ is said to be cocentral if the Hopf algebra map p is cocentral, that is for any $a \in A$, we have $p(a_{(1)}) \otimes a_{(2)} = p(a_{(2)}) \otimes a_{(1)}$.

2.2. *Yetter–Drinfeld modules*

Recall that a (right-right) Yetter–Drinfeld module over a Hopf algebra A is a right A-comodule and right A-module V satisfying the condition, $\forall v \in V$, $\forall a \in A$,

$$(v \cdot a)_{(0)} \otimes (v \cdot a)_{(1)} = v_{(0)} \cdot a_{(2)} \otimes S(a_{(1)})v_{(1)}a_{(3)}.$$

The category of Yetter–Drinfeld modules over A is denoted $\mathcal{YD}_A^A$: the morphisms are the A-linear and A-colinear maps. The category $\mathcal{YD}_A^A$ is obviously abelian, and, endowed with the usual tensor product of modules and comodules, is a monoidal category, with unit the trivial Yetter–Drinfeld module, denoted k. If A has bijective antipode, for a finite-dimensional Yetter–Drinfeld module V over A, the dual vector space V^* has a Yetter–Drinfeld module structure given by, for $f \in V^*$ and $v \in V$

$$f \cdot a(v) = f(v \cdot S^{-1}(a)), \quad f_{(0)}(v)f_{(1)} = f(v_{(0)})S(v_{(1)}).$$

See for example [19, Lemma 4.2.2] in the left-left case. In this way the usual evaluation and coevaluation maps $V^* \otimes V \to k$ and $k \to V \otimes V^*$ are morphisms of Yetter–Drinfeld modules, and the Yetter–Drinfeld module V^* becomes a left dual of V in the monoidal category sense, and in particular the functors $- \otimes V$ and $- \otimes V^*$ form a pair of adjoint functors: we have for all Yetter–Drinfeld modules X, Y over A, natural isomorphisms

$$\mathrm{Hom}_{\mathscr{YD}_A^A}(X \otimes V, Y) \simeq \mathrm{Hom}_{\mathscr{YD}_A^A}(X, Y \otimes V^*).$$

Since the functors $- \otimes V$ and $- \otimes V^*$ are exact, the above isomorphisms extend to isomorphisms

$$\mathrm{Ext}^*_{\mathscr{YD}_A^A}(X \otimes V, Y) \simeq \mathrm{Ext}^*_{\mathscr{YD}_A^A}(X, Y \otimes V^*).$$

This is similar to [20, IV.12], using injective resolutions (as said in the introduction, the category of Yetter–Drinfeld modules has enough injective objects).

Example 1. Let $B \subset A$ be a Hopf subalgebra, and consider the quotient coalgebra $L = A/B^+A$. Endow L with the right A-module structure induced by the quotient map $p\colon A \to L$, i.e $p(a)\cdot b = p(ab)$ and with the coadjoint A-comodule structure given $p(a) \mapsto p(a_{(2)}) \otimes S(a_{(1)})a_{(3)}$ (this is well-defined since for $b \in B$ and $a \in A$, one has $p(ba) = \varepsilon(b)a$). Then L, endowed with these two structures, is a Yetter–Drinfeld module over A. In particular if $k \to B \xrightarrow{i} A \xrightarrow{p} L \to k$ is an exact sequence of Hopf algebras, then L inherits a Yetter–Drinfeld module structure over A.

Example 2. Let $\psi\colon A \to k$ be an algebra map satisfying $\psi(a_{(1)})a_{(2)} = \psi(a_{(2)})a_{(1)}$ for any $a \in A$. Endow k with the trivial A-comodule structure and with the A-module structure induced by ψ. Then k, endowed with these two structures, is a Yetter–Drinfeld module over A, that we denote k_ψ.

Examples 1 and 2 are related by the following lemma.

Lemma 3. *Let $p\colon A \to k\Gamma$ be a surjective cocentral Hopf algebra map, where Γ is a group. For $\psi \in \widehat{\Gamma} = \mathrm{Hom}(\Gamma, k^*)$, we still denote by $\psi\colon A \to k$ the composition of the linear extension of ψ to $k\Gamma$ with p. If Γ is finite abelian and $|\Gamma| \neq 0$ in k, the Fourier transform is an isomorphism*

$$k\Gamma \simeq \bigoplus_{\psi \in \widehat{\Gamma}} k_\psi$$

in the category $\mathscr{YD}_A^A$, where $k\Gamma$ has the coadjoint Yetter–Drinfeld structure given in Example 1, and the right-handed term has the Yetter–Drinfeld structure from Example 2.

Proof. The Fourier transform is defined by

$$\mathscr{F}\colon k\Gamma \longrightarrow \bigoplus_{\psi \in \widehat{\Gamma}} k_\psi, \quad \Gamma \ni g \longmapsto \sum_{\psi \in \widehat{\Gamma}} \psi(g) e_\psi$$

where e_ψ denotes the basis element in k_ψ, and since k is algebraically closed, the assumption $|\Gamma| \neq 0$ in k ensures that $\mathscr{F}$ is a linear isomorphism. The cocentrality assumption on p ensures that the A-comodule structure on $k\Gamma$ from Example 1 is trivial, so $\mathscr{F}$ is a comodule map as well. To prove the A-linearity of $\mathscr{F}$, recall first that $p\colon A \to k\Gamma$ induces a $k\Gamma$-comodule algebra structure $(\mathrm{id} \otimes p) \circ \Delta\colon A \to A \otimes k\Gamma$ on A and hence an algebra grading

$$A = \bigoplus_{g \in \Gamma} A_g$$

where $A_g = \{a \in A \mid a_{(1)} \otimes p(a_{(2)}) = a \otimes g\}$, with $a \in A_g \Rightarrow p(a) = \varepsilon(a)g$. For $g \in \Gamma$, pick $a \in A_g$ such that $\varepsilon(a) = 1$. For $h \in \Gamma$ and $a' \in A_h$, we have $aa' \in A_{gh}$ and hence

$$\begin{aligned}\mathscr{F}(g \cdot a') = \mathscr{F}(p(aa')) = \mathscr{F}(\varepsilon(aa')gh) = \varepsilon(a') \sum_{\psi \in \widehat{\Gamma}} \psi(gh) e_\psi = \varepsilon(a') \sum_{\psi \in \widehat{\Gamma}} \psi(g)\psi(h) e_\psi \\ = \sum_{\psi \in \widehat{\Gamma}} \psi(g)\varepsilon(a')\psi(h) e_\psi = \sum_{\psi \in \widehat{\Gamma}} \psi(g)\psi(p(a')) e_\psi = \sum_{\psi \in \widehat{\Gamma}} \psi(g) e_\psi \cdot a' = \mathscr{F}(g) \cdot a'\end{aligned}$$

and this concludes the proof. □

3. Main results

The main tool to prove our main results will be induction and restriction of Yetter–Drinfeld modules, that we first recall.

Let $B \subset A$ be a Hopf subalgebra. Recall [9, 13] that we have a pair of adjoint functors

$$\begin{array}{ccc} \mathscr{YD}_A^A \longrightarrow \mathscr{YD}_B^B & \qquad & \mathscr{YD}_B^B \longrightarrow \mathscr{YD}_A^A \\ X \longmapsto X^{(B)} & & V \longmapsto V \otimes_B A \end{array}$$

constructed as follows:

(1) For an object X in $\mathscr{YD}_A^A$, $X^{(B)} = \{x \in X \mid x_{(0)} \otimes x_{(1)} \in X \otimes B\}$ is equipped with the obvious B-comodule structure, and is a B-submodule of X. We have $X^{(B)} \simeq X \square_A B$, where the right term is the cotensor product, and we say that $B \subset A$ is (left) coflat when the above functor $X \mapsto X \square_A B$, $\mathscr{M}^A \to \mathscr{M}^B$ is exact.

(2) For an object $V \in \mathscr{YD}_B^B$, the induced A-module $V \otimes_B A$ has the A-comodule structure given by the map

$$v \otimes_B a \longmapsto v_{(0)} \otimes_B a_{(2)} \otimes S(a_{(1)}) v_{(1)} a_{(3)}.$$

We then have the following result [9, Proposition 3.3], which follows from the general machinery of pairs of adjoint functors.

Proposition 4. *Let $B \subset A$ be a Hopf subalgebra. If $B \subset A$ is coflat and A is flat as a left B-module, we have, for any object X in $\mathscr{YD}_A^A$ and any object V in $\mathscr{YD}_B^B$, natural isomorphisms*

$$\mathrm{Ext}^*_{\mathscr{YD}_A^A}(V \otimes_B A, X) \simeq \mathrm{Ext}^*_{\mathscr{YD}_B^B}(V, X^{(B)}).$$

Remark 5. Let $B \subset A$ be a Hopf subalgebra, and consider the quotient coalgebra $L = A/B^+A$. Recall from Example 1 that L has a natural Yetter–Drinfeld module structure over A. The induced Yetter–Drinfeld module $k \otimes_B A$ is isomorphic to L in $\mathscr{YD}_A^A$.

Theorem 6. *Let $k \to B \to A \to L \to k$ be an exact sequence of Hopf algebras, with L finite-dimensional and cosemisimple, and A having bijective antipode. We have, for any $X \in \mathscr{YD}_A^A$,*

$$H^*_{\mathrm{GS}}(B, X^{(B)}) \simeq H^*_{\mathrm{GS}}(A, X \otimes L^*)$$

and hence in particular

$$H^*_b(B) \simeq H^*_{\mathrm{GS}}(A, L^*)$$

where L^* is the dual Yetter–Drinfeld module of L.

Proof. Since $L = A/B^+A$ is cosemisimple, $B \subset A$ is coflat [9, Proposition 3.4]. Moreover, still because L is cosemisimple, the quotient map $A \to L$ is faithfully coflat, and hence A is (faithfully) flat as a B-module by the left version of [27, Theorem 2]. Hence we can use Proposition 4 applied to $V = k$ to get

$$\mathrm{Ext}^*_{\mathscr{YD}_A^A}(k \otimes_B A, X) \simeq \mathrm{Ext}^*_{\mathscr{YD}_B^B}(k, X^{(B)})$$

and hence, by Remark 5,

$$\mathrm{Ext}^*_{\mathcal{YD}^A_A}(L,X) \simeq \mathrm{Ext}^*_{\mathcal{YD}^B_B}(k, X^{(B)}).$$

Since L is assumed to be finite-dimensional, the adjunction between the exact functors $-\otimes L$ and $-\otimes L^*$ provides the announced isomorphism, see Subsection 2.2. □

Remark 7. Let $k \to B \to A \to L \to k$ be an exact sequence of Hopf algebras, with A cosemisimple. Then by L is cosemisimple by the discussion before Theorem 2.5 in [14]. Hence if L is finite-dimensional, the conclusion of Theorem 6 holds.

Corollary 8. *Let $k \to B \to A \to k\Gamma \to k$ be a cocentral exact sequence of Hopf algebras, with A having bijective antipode and Γ a finite abelian group with $|\Gamma| \neq 0$ in k. We have, for any $X \in \mathcal{YD}^A_A$,*

$$H^*_{\mathrm{GS}}(B, X^{(B)}) \simeq \bigoplus_{\psi\in\widehat{\Gamma}} H^*_{\mathrm{GS}}(A, X \otimes k_\psi)$$

and hence in particular

$$H^*_b(B) \simeq \bigoplus_{\psi\in\widehat{\Gamma}} H^*_{\mathrm{GS}}(A, k_\psi).$$

Proof. We are in the situation of Theorem 6, hence

$$H^*_{\mathrm{GS}}(B, X^{(B)}) \simeq H^*_{\mathrm{GS}}(A, X \otimes L^*)$$

for $L = k\Gamma$. The assumption on Γ, ensures, by Lemma 3, that $L \simeq \oplus_{\psi\in\widehat{\Gamma}} k_\psi$ as Yetter–Drinfeld modules over A, and hence in particular $L \simeq L^*$. The statement follows. □

Remark 9. Recall that the Gerstenhaber–Schack cohomological dimension of a Hopf algebra A is defined by

$$\mathrm{cd}_{\mathrm{GS}}(A) = \sup\left\{n : \ H^n_{\mathrm{GS}}(A, V) \neq 0 \text{ for some } V \in \mathcal{YD}^A_A\right\} \in \mathbb{N} \cup \{\infty\}.$$

Let $k \to B \to A \to k\Gamma \to k$ be a cocentral exact sequence with Γ a finite abelian group such $|\Gamma| \neq 0$. Then it follows from Corollary 8 that $\mathrm{cd}_{\mathrm{GS}}(B) \geq \mathrm{cd}_{\mathrm{GS}}(A)$. If A is cosemisimple, then $\mathrm{cd}_{\mathrm{GS}}(B) = \mathrm{cd}_{\mathrm{GS}}(A)$ by [9, Theorem 4.8]. We expect that equality holds in general.

4. Application to the bialgebra cohomology of universal cosovereign Hopf algebras

4.1. *Universal cosovereign Hopf algebras*

Recall that for $n \geq 2$ and $F \in \mathrm{GL}_n(k)$, the universal cosovereign Hopf algebra $H(F)$ is the algebra presented by generators $(u_{ij})_{1\leq i,j\leq n}$ and $(v_{ij})_{1\leq i,j\leq n}$, and relations:

$$uv^t = v^t u = I_n; \quad vFu^tF^{-1} = Fu^tF^{-1}v = I_n,$$

where $u = (u_{ij})$, $v = (v_{ij})$ and I_n is the identity $n \times n$ matrix. The algebra $H(F)$ has a Hopf algebra structure defined by

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}, \quad \Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj},$$

$$\varepsilon(u_{ij}) = \varepsilon(v_{ij}) = \delta_{ij}, \quad S(u) = v^t, \quad S(v) = Fu^tF^{-1}.$$

When $k = \mathbb{C}$ and F is a positive matrix, so that $F = K^*K$ for some $K \in \mathrm{GL}_n(\mathbb{C})$, then $H(F)$ carries a Hopf $*$-algebra structure so that it is the coordinate algebra of the universal unitary compact quantum group U^+_K of Van Daele and Wang [29]. We refer the reader to [5, 9] for more information and background on the Hopf algebras $H(F)$. Recall from [5] that a matrix $F \in \mathrm{GL}_n(k)$ is said to be

- normalizable if $\mathrm{tr}(F) \neq 0$ and $\mathrm{tr}(F^{-1}) \neq 0$ or $\mathrm{tr}(F) = 0 = \mathrm{tr}(F^{-1})$;
- generic if it is normalizable and the solutions of the equation $q^2 - \sqrt{\mathrm{tr}(F)\,\mathrm{tr}(F^{-1})}\,q + 1 = 0$ are generic, i.e. are not roots of unity of order ≥ 3 (this property does not depend on the choice of the above square root);
- an asymmetry if there exists $E \in \mathrm{GL}_n(k)$ such that $F = E^tE^{-1}$.

4.2. *Hopf algebras of bilinear forms*

Let $E \in \mathrm{GL}_n(k)$. The Hopf algebra $\mathscr{B}(E)$ defined by Dubois–Violette and Launer [15] is presented by generators a_{ij}, $1 \le i, j \le n$, and relations $E^{-1}a^tEa = I_n = aE^{-1}a^tE$, where a is the matrix (a_{ij}). The Hopf algebra structure is given by

$$\Delta(a_{ij}) = \sum_k a_{ik} \otimes a_{kj}, \quad \varepsilon(a_{ij}) = \delta_{ij}, \quad S(a) = E^{-1}a^tE.$$

For an appropriate matrix E_q, one has $\mathscr{B}(E_q) = \mathscr{O}_q(\mathrm{SL}_2(k))$, the coordinate algebra on quantum SL_2. The Hopf algebra $\mathscr{B}(E)$ is cosemisimple if and only if $F = E^tE^{-1}$ is generic in the sense of the previous subsection: this follows from [4] and the classical result for $\mathscr{O}_q(\mathrm{SL}_2(k))$.

When $k = \mathbb{C}$ and $E\overline{E} = rI_n$ for $r \in \mathbb{R}^*$, then $\mathscr{B}(E)$ has a Hopf $*$-algebra structure making it the coordinate algebra of the universal orthogonal compact quantum group O_E^+ of Van Daele and Wang [29].

Denote by $\mathscr{B}_+(E)$ the subalgebra of $\mathscr{B}(E)$ generated by the products $a_{ij}a_{kl}$, $1 \le i, j, k, l \le n$. This is a Hopf subalgebra of $\mathscr{B}(E)$, that fits into a cocentral exact sequence

$$k \longrightarrow \mathscr{B}_+(E) \longrightarrow \mathscr{B}(E) \longrightarrow k\mathbb{Z}_2 \longrightarrow k$$

where the projection on the right is given by $p(a_{ij}) = \delta_{ij}g$, with g being the generator of the cyclic group $\mathbb{Z}_2$. By Example 1, $k\mathbb{Z}_2$ inherits a Yetter–Drinfeld module structure over $\mathscr{B}(E)$, whose module structure is induced by p, and comodule structure is trivial.

The bialgebra cohomology of $\mathscr{B}(E)$ was computed in the cosemisimple case in [6, Theorem 6.5] with $\mathbb{C}$ as a base field. We record and supplement the result here, taking care of the characteristic of the base field, together with another computation of Gerstenhaber–Schack cohomology, with coefficients in $k\mathbb{Z}_2$.

Theorem 10. *Let $E \in \mathrm{GL}_n(k)$, $n \ge 2$, and assume that E^tE^{-1} is generic.*

(1) *We have*

$$H^p_{\mathrm{GS}}(\mathscr{B}(E), k\mathbb{Z}_2) \simeq \begin{cases} k & \text{if } p = 0,3 \\ \{0\} & \text{otherwise.} \end{cases}$$

(2) *If* $\mathrm{char}(k) \neq 2$, *then*

$$H^p_b(\mathscr{B}(E)) \simeq \begin{cases} k & \text{if } p = 0,3 \\ \{0\} & \text{otherwise.} \end{cases}$$

(3) *If* $\mathrm{char}(k) = 2$, *then*

$$H^p_b(\mathscr{B}(E)) \simeq \begin{cases} k & \text{if } p = 0,1,2,3 \\ \{0\} & \text{otherwise.} \end{cases}$$

Proof. The resolution given in [6, Theorem 5.1] is valid over any field, and can be used to compute the above cohomologies, since the involved Yetter–Drinfeld modules are free, and hence projective by the cosemisimplicity assumption on $\mathscr{B}(E)$. The result is then obtained by direct computations, which depend on whether k has, or not, characteristic 2. □

As a first application of the results of Section 3, we recover in a shorter way the bialgebra cohomology computation of $\mathscr{B}_+(E)$ in the cosemisimple case [8, Theorem 6.4], that we supplement in the characteristic 2 case.

Corollary 11. *Let $E \in \mathrm{GL}_n(k)$, $n \ge 2$, and assume that E^tE^{-1} is generic. We have*

$$H^p_b(\mathscr{B}_+(E)) \simeq \begin{cases} k & \text{if } p = 0,3 \\ \{0\} & \text{otherwise.} \end{cases}$$

Proof. The Yetter–Drinfeld module $k\mathbb{Z}_2$ is self dual, hence the result is the combination of the first part of Theorem 10 and of Theorem 6. □

4.3. *Relation between $H(F)$ and $\mathscr{B}(E)$*

The first relation between $H(F)$ and $\mathscr{B}(E)$ was observed by Banica in [2], when $F = E^t E^{-1} \in \mathrm{GL}_n(\mathbb{C})$ is positive matrix, and a key result from [2] in that case is the existence of a Hopf algebra embedding

$$H(F) \hookrightarrow \mathscr{B}(E) * \mathbb{C}\mathbb{Z} \tag{4.1}$$

which, according to [28, Proposition 6.20], can be refined to an embedding

$$H(F) \hookrightarrow \mathscr{B}(E) * \mathbb{C}\mathbb{Z}_2. \tag{4.2}$$

This is strengthened in [3, Theorem 4.11], where it is shown that the embedding is still valid for any generic asymmetry F.

In fact, there is a simple proof of this result, valid over any field k and any asymmetry $F = E^t E^{-1}$.

Proposition 12. *Let $E \in \mathrm{GL}_n(k)$ and let $F = E^t E^{-1}$. There exists a $\mathbb{Z}_2$-action on $H(F)$ such one gets a Hopf algebra isomorphism*

$$H(F) \rtimes k\mathbb{Z}_2 \simeq \mathscr{B}(E) * k\mathbb{Z}_2.$$

Proof. The announced $\mathbb{Z}_2$-action, from [11, Example 2.18], is provided by the order 2 Hopf algebra automorphism of $H(F)$ given in matrix form as follows

$$\tau(u) = (E^t)^{-1} v E^t, \quad \tau(v) = E^t u (E^t)^{-1}.$$

We therefore form the usual crossed product Hopf algebra $H(F) \rtimes k\mathbb{Z}_2$. Denoting by g the generator of $\mathbb{Z}_2$, it is a straightforward verification to check the existence of a Hopf algebra map, written in matrix form

$$\begin{aligned} H(F) \rtimes k\mathbb{Z}_2 &\longrightarrow \mathscr{B}(E) * k\mathbb{Z}_2 \\ u,\ v,\ g &\longmapsto ag,\ E^t g a (E^t)^{-1},\ g. \end{aligned}$$

Similarly, it is straightforward to construct an inverse isomorphism

$$\begin{aligned} \mathscr{B}(E) * k\mathbb{Z}_2 &\longrightarrow H(F) \rtimes k\mathbb{Z}_2 \\ a,\ g &\longmapsto ug,\ g. \end{aligned}$$

We leave the detailed verification to the reader. □

4.4. *Bialgebra cohomology of $H(F)$ in the generic case*

Theorem 13. *Let $F \in \mathrm{GL}_n(k)$, $n \geq 2$, with F generic. The bialgebra cohomology of $H(F)$ is*

$$H_b^p(H(F)) \simeq \begin{cases} k & \text{if } p = 0,1,3 \\ \{0\} & \text{otherwise.} \end{cases}$$

Proof. First notice that one always has $H_b^0(A) = k$ for any Hopf algebra, while the computation of $H_b^1(H(F))$ is extremely easy (see the complex in [8, Proposition 5.3]), so we concentrate on degree $p \geq 2$. First consider the asymmetry case: $F = E^t E^{-1}$. Consider the $\mathbb{Z}_2$-action of Proposition 12 and the Hopf algebra map

$$\varepsilon \otimes \mathrm{id}\colon H(F) \rtimes k\mathbb{Z}_2 \longrightarrow k\mathbb{Z}_2.$$

This is cocentral, and the associated Hopf subalgebra B is clearly the image of the natural embedding $H(F) \hookrightarrow \mathrm{H}(F) \rtimes k\mathbb{Z}_2$. Theorem 6 gives an isomorphism

$$H_b^*(H(F)) \simeq H_{\mathrm{GS}}^*(H(F) \rtimes k\mathbb{Z}_2, k\mathbb{Z}_2).$$

Considering now the isomorphism of Proposition 12, we obtain the isomorphism

$$H_b^*(H(F)) \simeq H_{\mathrm{GS}}^p(\mathscr{B}(E) * k\mathbb{Z}_2, k\mathbb{Z}_2).$$

Since $\mathscr{B}(E)$ is cosemisimple as well, [9, Theorem 5.9] yields, for $p \geq 2$,

$$H_b^p(H(F)) \simeq H_{\mathrm{GS}}^p(\mathscr{B}(E), k\mathbb{Z}_2) \oplus H_{GS}^p(k\mathbb{Z}_2, k\mathbb{Z}_2).$$

Since $k\mathbb{Z}_2$ is cosemisimple and cocommutative, we have $H_{\mathrm{GS}}^p(k\mathbb{Z}_2, k\mathbb{Z}_2) \simeq \mathrm{Ext}_{k\mathbb{Z}_2}^p(k, k\mathbb{Z}_2)$, and the latter Ext-space is easily seen to vanish if $p \geq 1$. We conclude by the first part of Theorem 10.

For a general matrix F, by [5, Theorem 1.1] there always exists an asymmetry $F(q) \in \mathrm{GL}_2(k)$ such that the tensor categories of comodules $H(F)$ and $H(q)$ are equivalent, hence the monoidal invariance of bialgebra cohomology (see e.g. [7, Theorem 7.10]) gives the result. □

Remark 14. One can also compute the usual Hochschild cohomology for $H(F)$ in the asymmetry case, for particular choices of coefficients, by combining Proposition 12 and the usual adjunction relation for Ext (see e.g. [20, IV.12]). The computation is done in greater generality in [3, Theorem B], and is valid for any normalizable F over any field, since Proposition 12 is. Notice also that it follows from [3] that $\mathrm{cd}(H(F)) = 3$ for any normalizable F, which was only known for F an asymmetry [9] or F generic [10]. Here cd is the cohomological dimension, i.e. the global dimension, which, for Hopf algebras, coincides as well with the Hochschild cohomological dimension.

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

References

[1] N. Andruskiewitsch and J. Devoto, "Extensions of Hopf algebras", *St. Petersbg. Math. J.* **7** (1995), no. 1, pp. 22–61.

[2] T. Banica, "Le groupe quantique compact libre U(n)", *Commun. Math. Phys.* **190** (1997), no. 1, pp. 143–172.

[3] I. Baraquin, U. Franz, M. Gerhold, A. Kula and M. Tobolski, "Free resolutions for free unitary quantum groups and universal cosovereign Hopf algebras", *J. Lond. Math. Soc.* **109** (2024), no. 4, article no. e12898 (50 pages).

[4] J. Bichon, "The representation category of the quantum group of a non-degenerate bilinear form", *Commun. Algebra* **31** (2003), no. 10, pp. 4831–4851.

[5] J. Bichon, "Co-representation theory of universal co-sovereign Hopf algebras", *J. Lond. Math. Soc.* **75** (2007), no. 1, pp. 83–98.

[6] J. Bichon, "Hochschild homology of Hopf algebras and free Yetter–Drinfeld resolutions of the counit", *Compos. Math.* **149** (2013), no. 4, pp. 658–678.

[7] J. Bichon, "Hopf–Galois objects and cogroupoids", *Rev. Unión Mat. Argent.* **55** (2014), no. 2, pp. 11–69.

[8] J. Bichon, "Gerstenhaber–Schack and Hochschild cohomologiesof Hopf algebras", *Doc. Math.* **21** (2016), pp. 955–986.

[9] J. Bichon, "Cohomological dimensions of universal cosovereign Hopfalgebras", *Publ. Mat.* **62** (2018), no. 2, pp. 301–330.

[10] J. Bichon, "On the monoidal invariance of the cohomological dimensionof Hopf algebras", *C. R. Math. Acad. Sci. Paris* **360** (2022), pp. 561–582.

[11] J. Bichon, S. Neshveyev and M. Yamashita, "Graded twisting of categories and quantum groups by group actions", *Ann. Inst. Fourier* **66** (2016), no. 6, pp. 2299–2338.

[12] S. Caenepeel, G. Militaru and S. Zhu, "Crossed modules and Doi–Hopf modules", *Isr. J. Math.* **100** (1997), pp. 221–247.

[13] S. Caenepeel, G. Militaru and S. Zhu, *Frobenius and separable functors for generalized module categories and nonlinear equations*, Springer, 2002, pp. xiv+354.

[14] A. Chirvasitu, "Cosemisimple Hopf algebras are faithfully flat over Hopf subalgebras", *Algebra Number Theory* **8** (2014), no. 5, pp. 1179–1199.

[15] M. Dubois-Violette and G. Launer, "The quantum group of a nondegenerate bilinear form", *Phys. Lett. B* **245** (1990), no. 2, pp. 175–177.

[16] P. Etingof and S. Gelaki, "On finite-dimensional semisimple and cosemisimple Hopf algebras in positive characteristic", *Int. Math. Res. Not.* **1998** (1998), no. 16, pp. 851–864.

[17] M. Gerstenhaber and S. D. Schack, "Bialgebra cohomology, deformations, and quantum groups", *Proc. Natl. Acad. Sci. USA* **87** (1990), no. 1, pp. 478–481.

[18] M. Gerstenhaber and S. D. Schack, "Algebras, bialgebras, quantum groups, and algebraic deformations", in *Deformation theory and quantum groups with applications to mathematical physics (Amherst, MA, 1990)*, American Mathematical Society, 1992, pp. 51–92.

[19] I. Heckenberger and H.-J. Schneider, *Hopf algebras and root systems*, American Mathematical Society, 2020, pp. xix+582.

[20] P. J. Hilton and U. Stammbach, *A course in homological algebra*, Springer, 1971, pp. ix+338.

[21] S. Montgomery, *Hopf algebras and their actions on rings*, American Mathematical Society, 1993, pp. xiv+238.

[22] B. Parshall and J. P. Wang, "On bialgebra cohomology", *Bull. Soc. Math. Belg. Sér. A* **42** (1990), no. 3, pp. 607–642.

[23] P. Schauenburg, "Hopf modules and Yetter–Drinfelʹd modules", *J. Algebra* **169** (1994), no. 3, pp. 874–890.

[24] S. Shnider and S. Sternberg, *Quantum groups. From coalgebras to Drinfelʹd algebras. A guided tour*, International Press, 1993, pp. xxii+496.

[25] D. Ştefan, "The set of types of n-dimensional semisimple and cosemisimple Hopf algebras is finite", *J. Algebra* **193** (1997), no. 2, pp. 571–580.

[26] R. Taillefer, "Injective Hopf bimodules, cohomologies of infinite dimensional Hopf algebras and graded-commutativity of the Yoneda product", *J. Algebra* **276** (2004), no. 1, pp. 259–279.

[27] M. Takeuchi, "Relative Hopf modules – equivalences and freeness criteria", *J. Algebra* **60** (1979), no. 2, pp. 452–471.

[28] P. Tarrago and M. Weber, "Unitary easy quantum groups: the free case and the group case", *Int. Math. Res. Not.* **2017** (2017), no. 18, pp. 5710–5750.

[29] A. Van Daele and S. Wang, "Universal quantum groups", *Int. J. Math.* **7** (1996), no. 2, pp. 255–263.