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A note on flatness of non separable tangent cone at a barycenter

Une note sur la platitude du cône tangeant à un barycentre

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Abstract. Given a probability measure **P** on an Alexandrov space S with curvature bounded below, we prove that the support of the pushforward of **P** on the tangent cone $T_{b^\star}S$ at its (exponential) barycenter $b^\star$ is a subset of a Hilbert space, without separability of the tangent cone.

Résumé. Étant donné une mesure de probabilité **P** sur un espace d'Alexandrov S avec courbure minorée, nous prouvons que le support de la mesure poussée de **P** sur le cône tangent $T_{b^\star}S$ à son barycentre (exponentiel) $b^\star$ est un sous-ensemble d'un espace de Hilbert, sans condition de séparabilité du cône tangent.

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1. Introduction

Barycenter of a probability measure **P** (a.k.a. Fréchet means) provides an extension of expectation on Euclidean space to arbitrary metric spaces. We present here a useful tool for the study of barycenters on Alexandrov spaces with curvature bounded below: the support of $\log_{b^\star}\#\mathbf{P}$ in the tangent cone at the barycenter is included in a Hilbert space. This rigidity result has been stated in [9] as Theorem 45, however the proof is not written. Moreover, there is an extra assumption of support of $\log_{b^\star}\#\mathbf{P}$ being separable, which does not even seem to be a consequence of the support of **P** being separable. As pointed out by [7], it is not clear if even S being proper ensures that the tangent cone is separable. This paper presents a proof of this rigidity result, without this extra separable assumption on the tangent cone. For measurability purposes (see Lemma 6), we suppose however that S is separable. The proof is essentially the one of Theorem 45 of [9], with needed approximations dealt with a bit differently.

2. Setting and main result

We use a classical notion of curvature bounded below for geodesic spaces, referred to as Alexandrov curvature. We recall several notions whose formal definitions can be found for instance in [3] or in the work in progress [2].

For a metric space (S,d), we denote by $\mathscr{P}_1(S)$ the set of probability measures on S with finite moment of order 1 (i.e. such that there exists $x \in S$ such that $\int d(x,y)\mathrm{d}\mathbf{P}(y) < \infty$). The support of a measure $\mathbf{P}$ will be denoted by $\operatorname{supp}\mathbf{P}$. We use both notation $\int f \mathrm{d}\mathbf{P}$ and $\mathbf{P}f$ for the integral of f w.r.t. $\mathbf{P}$.

A *geodesic space* is a metric space (S,d) such that every two points $x,y \in S$ at distance is connected by a curve of length $d(x,y)$. Such shortest curves are called *geodesics*. For $\kappa \in \mathbf{R}$, the *model space* $(\mathbf{M}_\kappa, d_\kappa)$ denotes the 2-dimensional simply connected complete surface of constant Gauss curvature κ. A geodesic space (S,d) is an *Alexandrov space with curvature bounded below by* $\kappa \in \mathbf{R}$ if for every triangle (3-uple) $(x_0,x_1,y) \in S$, and a constant speed geodesic $(x_t)_{t\in[0;1]}$ there exists an isometric triangle $(\tilde{x}_0,\tilde{x}_1,\tilde{y}) \in \mathbf{M}_\kappa$, such that the geodesic $(\tilde{x}_t)_{t\in[0;1]}$ satisfies for all $t \in [0;1]$,

$$d(y,x_t) \geq d_\kappa(\tilde{y},\tilde{x}_t).$$

For such spaces, angles between two unit-speed geodesics γ_1,γ_2 starting at the same point $p \in S$ can be defined as follows:

$$\cos\angle_p(\gamma_1,\gamma_2) = \lim_{t\to 0} \frac{d^2(\gamma_1(t),p) + d^2(\gamma_2(t),p) - d^2(\gamma_1(t),\gamma_2(t))}{2d(p,\gamma_1(t))d(p,\gamma_2(t))},$$

where angle $\angle_p(\gamma_1,\gamma_2) \in [0;\pi]$. Denote by Γ_p the set of all unit-speed geodesics emanating from p. Using angles, we can define the *tangent cone* T_pS *at* $p \in S$ as follows. First define T'_pS as the (quotient) set $\Gamma_p \times \mathbf{R}^+$, equipped with the (pseudo-)metric defined by

$$\|(\gamma_1,t) - (\gamma_2,s)\|_p^2 := s^2 + t^2 - 2s.t\cos\angle_p(\gamma_1,\gamma_2).$$

Then, the *tangent cone* T_pS is defined as the completion of T'_pS equipped with the metric $\|\cdot\|_p$. We will use the notation for $u,v \in T_pS$,

$$\langle u,v\rangle_p := \frac{1}{2}(\|u\|_p^2 + \|v\|_p^2 - \|u-v\|_p^2),$$

We will often identify a point $\gamma(t) \in S$ with $(\gamma,t) \in T_pS$. Although such γ might not be unique, we will assume a choice of a map $\log_p : S \to T_pS$, called *logarithmic map*, such that for all $x \in S$, there exists a unit-speed geodesic γ emanating from p such that, for some $t > 0$, $\gamma(t) = x$ and

$$\log_p(x) = (\gamma,t).$$

This map can be chosen to be $\mathfrak{S}_B$-measurable, where $\mathfrak{S}_B$ denotes the σ-algebra generated by open balls on the tangent cone T_pS (see Lemma 6) and this weak measurability is enough for our results to hold and will be assumed for the rest of the paper. Then the *pushforward* of $\mathbf{P}$ by $\log_p$ will be denoted by $\log_p\#\mathbf{P}$.

The tangent cone is not necessarily a geodesic space (see [4]), however, it is included in a geodesic space, namely the ultratangent space (see for instance Theorem 14.4.2 and 14.4.1 in [2]) that is an Alexandrov space with curvature bounded below by 0.

The tangent cone T_pS contains the subspace Lin_p of all points with an *opposite*, formally defined as follows. A point u belongs to $\mathrm{Lin}_p \subset T_pS$ if and only if there exists $v \in T_pS$ such that $\|u\|_p = \|v\|_p$ and

$$\langle u,v\rangle_p = -\|u\|_p^2.$$

Our main result is based on the following Theorem.

Theorem (Theorem 14.5.4 in [2]). *The set* Lin_p *equipped with the induced metric of* T_pS *is a Hilbert space.*

A point $b^\star \in S$ is a *barycenter* of the probability measure $\mathbf{P} \in \mathscr{P}_1(S)$ if for all $b \in S$

$$0 \le \int d^2(x,b) - d^2(x,b^\star)\,\mathrm{d}\mathbf{P}(x).$$

Such barycenter might not be unique, neither exists. However, when they exist, they satisfy

$$\int \langle x, y\rangle_{b^\star}\,\mathrm{d}\mathbf{P}\otimes\mathbf{P}(x,y) = 0. \tag{1}$$

A point $b^\star \in S$ satisfying (1) is called an *exponential barycenter* of $\mathbf{P}$.

We can now state our main result.

Theorem 1. *Let (S,d) be an Alexandrov space with curvature bounded below by some $\kappa \in \mathbf{R}$ and $\mathbf{P} \in \mathscr{P}_1(S)$. If $b^\star \in S$ is an exponential barycenter of $\mathbf{P}$, then* $\operatorname{supp}\log_{b^\star}\#\mathbf{P} \subset \mathrm{Lin}_{b^\star} S$. *In particular,* $\operatorname{supp}\log_{b^\star}\#\mathbf{P}$ *is included in a Hilbert space.*

This result allows to prove the following Corollary, that has been implicitly used in [1].

Corollary 2 (Linearity). *Let $b \in T_{b^\star} S$. Then, the map $\langle\cdot, b\rangle_{b^\star} : \mathrm{Lin}_{b^\star} \to \mathbf{R}$ is continuous and linear. In particular, if $b^\star$ is an exponential barycenter of $\mathbf{P}$, then*

$$\int \langle x, b\rangle_{b^\star}\,\mathrm{d}\mathbf{P}(x) = 0.$$

3. Proofs

Recall that we always identify a point in S and its image in the tangent cone T_pS by the $\log_p$ map.

Proof of Corollary 2. Linearity is obvious from the definition of $\langle\cdot, b\rangle_{b^\star}$. We check that $x \mapsto \langle x, b\rangle_{b^\star}$ is a convex and concave function in $\mathrm{Lin}_{b^\star} S$. Let $t \in (0,1)$, x_0, x_1 in $\mathrm{Lin}_{b^\star} S$, and set $x_t = (1-t)x_0 + tx_1$. Since the tangent cone is included in an Alexandrov space with curvature bounded below by 0 on one hand, and $\mathrm{Lin}_{b^\star}$ is a Hilbert space on the other hand,

$$\begin{aligned}
\langle x_t, b\rangle_{b^\star} &= \frac{1}{2}\left(\|x_t\|_{b^\star}^2 + \|b\|_{b^\star}^2 - \|x_t - b\|^2\right)\\
&\le \frac{1}{2}\left((1-t)(\|x_0\|_{b^\star}^2 - \|x_0 - b\|_{b^\star}^2) + t(\|x_1\|_{b^\star}^2 - \|x_1 - b\|^2) + \|b\|_{b^\star}^2\right)\\
&= (1-t)\langle x_0, b\rangle_{b^\star} + t\langle x_1, b\rangle_{b^\star}.
\end{aligned}$$

The same lines applied to $-x_0$ and $-x_1$ gives the converse inequality

$$\langle -x_t, b\rangle_{b^\star} \le (1-t)\langle -x_0, b\rangle_{b^\star} + t\langle -x_1, b\rangle_{b^\star}.$$

The second statement follows from the fact that $b^\star$ is a Pettis integral of the pushforward of $\mathbf{P}$ onto $\mathrm{Lin}_{b^\star} \subset T_{b^\star} S$, as a direct consequence of Theorem 1. □

Proof of Theorem 1. Let $L \subset \{x \in S \mid \int \langle x, \cdot\rangle_{b^\star}\,\mathrm{d}\mathbf{P} = 0\}$ be a measurable set such that $\mathbf{P}(L) = 1$ given by Lemma 3. Let $x \in L$. For $U = \{x\}$, use Lemma 5 with $\mathbf{Q} = \mathbf{P}$ and B_δ a ball of radius δ around x in $T_{b^\star} S$, to get a sequence $(y_\delta^n)_n \subset T_{b^\star} S$ such that,

$$\begin{aligned}
\limsup_n \cos\angle(\uparrow_{b^\star}^{x}, \uparrow_{b^\star}^{y_\delta^n}) &= \limsup_n \frac{\langle x, y_\delta^n\rangle_{b^\star}}{d(b^\star, x)\,d(b^\star, y_\delta^n)}\\
&= \frac{1}{d(b^\star, x)} \limsup_n \langle x, y_\delta^n\rangle_{b^\star} \frac{1}{\lim_n d(b^\star, y_\delta^n)}\\
&\le \frac{1}{d(b^\star, x)} \frac{\int_{B_\delta^c} \langle x, y\rangle_{b^\star}\,\mathrm{d}\mathbf{P}(x)}{\mathbf{P}(B_\delta)} \frac{\mathbf{P}(B_\delta)}{\left(\int_{B_\delta}\int_{B_\delta} \langle x, y\rangle_{b^\star}\,\mathrm{d}\mathbf{P}\otimes\mathbf{P}(x,y)\right)^{1/2}}.
\end{aligned}$$

Then, since $\int \langle x, y\rangle_{b^\star} \mathrm{d}\mathbf{P}(y) = 0$, letting $\delta \to 0$, one gets

$$\frac{1}{\mathbf{P}(B_\delta)}\int_{B_\delta^c} \langle x, y\rangle_{b^\star} \mathrm{d}\mathbf{P}(y) = -\frac{1}{\mathbf{P}(B_\delta)}\int_{B_\delta} \langle x, y\rangle_{b^\star} \mathrm{d}\mathbf{P}(y) \to -d^2(b^\star, x).$$

and

$$\frac{\left(\int_{B_\delta}\int_{B_\delta} \langle x, y\rangle_{b^\star} \mathrm{d}\mathbf{P}\otimes\mathbf{P}(x,y)\right)^{1/2}}{\mathbf{P}(B_\delta)} \to d(b^\star, x)$$

Thus,

$$\lim_{\delta\to 0^+} \limsup_n \cos\angle(\uparrow_{b^\star}^{x}, \uparrow_{b^\star}^{y_\delta^n}) = -1$$

One can thus choose $(\bar{y}^n)_n$ a sequence in $(y_\delta^n)_{n,\delta}$ such that $\cos\angle(\uparrow_{b^\star}^{x}, \uparrow_{b^\star}^{\bar{y}^n}) \to -1$. Since $T_{b^\star}S$ is a subspace of an Alexandrov space of curvature bounded below by 0, we also have

$$\begin{aligned}\angle(\uparrow_{b^\star}^{\bar{y}^n}, \uparrow_{b^\star}^{\bar{y}^k}) &\le 2\pi - \angle(\uparrow_{b^\star}^{\bar{y}^n}, \uparrow_{b^\star}^{x}) - \angle(\uparrow_{b^\star}^{x}, \uparrow_{b^\star}^{\bar{y}^k})\\ &\to 0,\end{aligned}$$

as $n, k \to \infty$. Thus $(\bar{y}^n)_n$ corresponds to a Cauchy sequence in the space of direction, and thus admits a limit in $T_{b^\star}S$, since its "norm" also admits a limit $d(b^\star, x)$. Its limit $\bar{y}$ satisfies $\cos\angle(\uparrow_{b^\star}^{x}, \uparrow_{b^\star}^{\bar{y}}) = -1$, and therefore, it is the opposite $\bar{y} = -x$.

Finally, by definition of the support, for $x \in \operatorname{supp}(\log_{b^\star}\mathbf{P})$, every ball centered at x have a positive probability, and thus there exists a sequence $(x_n)_{n\ge 1} \subset L$ such that $x_n \to x$. We conclude with the completeness of $\mathrm{Lin}_{b^\star}$. □

Lemma 3 (Proposition 1.7 of [8] for non separable metric space). *Suppose (S,d) is an Alexandrov space with curvature bounded below. Then, for any probability measure $\mathbf{Q} \in \mathscr{P}_1(S)$, and any $b^\star \in S$,*

$$\int \langle x, y\rangle_{b^\star} \mathrm{d}\mathbf{Q}\otimes\mathbf{Q}(x,y) \ge 0.$$

Moreover, if $b^\star$ is an exponential barycenter of $\mathbf{Q}$, then for $\mathbf{Q}$-almost all $x \in S$,

$$\int \langle x, y\rangle_{b^\star} \mathrm{d}\mathbf{Q}(y) = 0.$$

Proof. For brevity, we will adopt the notation $\mathbf{Q}g$ for $\int g\mathrm{d}\mathbf{Q}$.

The result for $\mathbf{Q}$ finitely supported is the Lang–Schroeder inequality (Proposition 3.2 in [5]). Thus, we just need to approximate $\mathbf{Q}\otimes\mathbf{Q}\langle\cdot,\cdot\rangle_{b^\star}$ by some $\mathbf{Q}_n\otimes\mathbf{Q}_n\langle\cdot,\cdot\rangle_{b^\star}$ for some finitely supported $\mathbf{Q}_n$.

To approximate $\mathbf{Q}\otimes\mathbf{Q}\langle\cdot,\cdot\rangle_{b^\star}$, draw two independent sequences of i.i.d. random variables $(X_i^1)_i$ and $(X_i^2)_i$ of common law $\mathbf{Q}$, and denote $\mathbf{Q}_n^1$ and $\mathbf{Q}_n^2$ the corresponding empirical measures. In particular, $\mathbf{Q}_n^1\otimes\mathbf{Q}_n^2$ and $\mathbf{Q}_n^2\otimes\mathbf{Q}_n^1$ are both empirical measures of $\mathbf{Q}\otimes\mathbf{Q}$. Since S is not separable, we can not apply the fundamental theorem of statistics that ensures almost sure weak convergence of $\mathbf{Q}_n^1\otimes\mathbf{Q}_n^1$ to $\mathbf{Q}\otimes\mathbf{Q}$. However, for a measurable function $f : S\times S \to \mathbf{R}$, such that $\mathbf{Q}\otimes\mathbf{Q}f < \infty$, the law of large number ensures that almost surely

$$\mathbf{Q}_n^1\otimes\mathbf{Q}_n^2 f \to \mathbf{Q}\otimes\mathbf{Q}f$$

and

$$\mathbf{Q}_n^2\otimes\mathbf{Q}_n^1 f \to \mathbf{Q}\otimes\mathbf{Q}f.$$

Since the sequence $(X_1^1, X_1^2, X_2^1, X_2^2, X_3^1, X_3^2, \ldots)$ is also an i.i.d. sequence of random variables of common law $\mathbf{Q}$, the subsequence of the associated empirical measures $(\mathbf{Q}_n^3)_n$ defined by

$$\mathbf{Q}_n^3 := \frac{1}{2}(\mathbf{Q}_n^1 + \mathbf{Q}_n^2)$$

also satisfies the almost sure convergence

$$\mathbf{Q}_n^3\otimes\mathbf{Q}_n^3 f \to \mathbf{Q}\otimes\mathbf{Q}f.$$

Now, since

$$\mathbf{Q}_n^3 \otimes \mathbf{Q}_n^3 = \frac{1}{4}(\mathbf{Q}_n^1 \otimes \mathbf{Q}_n^2 + \mathbf{Q}_n^1 \otimes \mathbf{Q}_n^1 + \mathbf{Q}_n^2 \otimes \mathbf{Q}_n^1 + \mathbf{Q}_n^2 \otimes \mathbf{Q}_n^2),$$

we proved that almost surely

$$\mathbf{Q}_n^1 \otimes \mathbf{Q}_n^1 f + \mathbf{Q}_n^2 \otimes \mathbf{Q}_n^2 f \to 2\mathbf{Q} \otimes \mathbf{Q} f.$$

And since $(\mathbf{Q}_n^1)_n$ and $(\mathbf{Q}_n^2)_n$ are independent and with same law, it implies that both $\mathbf{Q}_n^1 \otimes \mathbf{Q}_n^1 f$ and $\mathbf{Q}_n^2 \otimes \mathbf{Q}_n^2 f$ converge to $\mathbf{Q} \otimes \mathbf{Q} f$ almost surely. In particular, since $\mathbf{Q}_n^1$ is supported on n points, there exists a sequence of finitely supported measures (that we rename $(\mathbf{Q}_n)_n$) such that $\mathbf{Q}_n \otimes \mathbf{Q}_n f \to \mathbf{Q} \otimes \mathbf{Q} f$. We thus proved the first result applying $f = \langle \cdot, \cdot \rangle_{b^\star}$.

Now, for any $x \in S$, applying this first result to the measure $\mathbf{Q}_\varepsilon := \frac{1}{1+\varepsilon}\mathbf{Q} + \frac{\varepsilon}{1+\varepsilon}\delta_x$, we get

$$\begin{aligned} 0 &\le (1+\varepsilon)\mathbf{Q}_\varepsilon \otimes \mathbf{Q}_\varepsilon \langle \cdot, \cdot \rangle_{b^\star} \\ &= \mathbf{Q} \otimes \mathbf{Q}\langle \cdot, \cdot \rangle_{b^\star} + 2\varepsilon \mathbf{Q}\langle x, . \rangle_{b^\star} + \varepsilon^2 \|x\|_{b^\star}^2. \end{aligned}$$

Letting $\varepsilon \to 0^+$, we get

$$\mathbf{Q}\langle x, \cdot \rangle_{b^\star} \ge 0.$$

Then equality follows from the hypothesis $\mathbf{Q} \otimes \mathbf{Q}\langle \cdot, \cdot \rangle_{b^\star} = 0$ meaning that $b^\star$ is an exponential barycenter. □

Lemma 4 (Subadditivity, Lemma A.6 of [5]). *Let (S,d) be an Alexandrov space with curvature bounded below. Take $b^\star \in S$. Let $x_1, \dots, x_n \in T'_{b^\star} S$ and $U \subset T_{b^\star} S$ finite. Then, for all $\varepsilon > 0$, there exists $y \in T_{b^\star} S$ such that for all $u \in U$,*

$$\langle y, u \rangle_{b^\star} \le \sum_{i=1}^n \langle x_i, u \rangle_{b^\star} + \varepsilon,$$

and

$$\|y\|^2 \le \sum_{i,j=1}^n \langle x_i, x_j \rangle_{b^\star} + \varepsilon.$$

Lemma 5 (Approximation). *Let $U \subset T_{b^\star} S$ finite. Take $B \subset S$ measurable and a probability measure $\mathbf{P} \in \mathscr{P}_1(S)$ such that $\mathbf{P} \otimes \mathbf{P}\langle \cdot, \cdot \rangle_{b^\star} = 0$ and $\mathbf{P}(B) > 0$. Then, there exists a sequence $(y^n)_n$ such that for all $u \in U$*

$$\frac{1}{\mathbf{P}(B)} \int_{B^c} \langle u, x \rangle_{b^\star} \mathrm{d}\mathbf{P}(x) \ge \limsup_n \langle u, y^n \rangle_{b^\star} \tag{2}$$

and

$$\frac{1}{\mathbf{P}(B)^2} \int_B \int_B \langle x, y \rangle_{b^\star} \mathrm{d}\mathbf{P} \otimes \mathbf{P}(x,y) = \lim_n d^2(b^\star, y^n). \tag{3}$$

Proof. Using the same arguments as in Lemma 3, we see that the empirical measures $(\mathbf{P}_n)_n$ satisfy

$$\mathbf{P}_n \otimes \mathbf{P}_n f \to \mathbf{P} \otimes \mathbf{P} f,$$

almost surely, for any $f : S \times S \to \mathbf{R} \in L^1(S \times S,\ \mathbf{P} \otimes \mathbf{P})$. In particular, taking $f(x,y) = \langle x, y \rangle_{b^\star} \mathbf{1}_{B \times B}(x,y)$, the following convergence holds in $L^2(\mathbf{P}^{\otimes \infty})$,

$$\int_B \int_B \langle \cdot, \cdot \rangle_{b^\star} \mathrm{d}\mathbf{P}_n \otimes \mathbf{P}_n \to \int_B \int_B \langle \cdot, \cdot \rangle_{b^\star} \mathrm{d}\mathbf{P} \otimes \mathbf{P}, \tag{4}$$

and similarly for B^c. Also, the law of large number ensures that almost surely, for all $u \in U$,

$$\int_B \langle \cdot, u \rangle_{b^\star} \mathrm{d}\mathbf{P}_n \to \int_B \langle \cdot, u \rangle_{b^\star} \mathrm{d}\mathbf{P}, \tag{5}$$

and again, the same for B^c. Thus, there exists a subsequence (of a deterministic realization of) $\mathbf{P}_n$ (that we rename $\mathbf{P}_n$) such that (4) and (5) both hold for all $u \in U$.

Then, applying Lemma 4 to finite sum

$$\frac{1}{\mathbf{P}(B)}\int_{B^c}\langle\cdot,u\rangle_{b^\star}\mathrm{d}\mathbf{P}_n,$$

shows that there exists a sequence $(y^n)_n\in T'_{b^\star}S$ such that (2) holds and for a sequence $(\varepsilon_n)_n$ s.t. $\varepsilon_n\to 0$,

$$\|y^n\|^2_{b^\star}\le\frac{1}{\mathbf{P}(B)^2}\int_{B^c}\int_{B^c}\langle\cdot,\cdot\rangle_{b^\star}\mathrm{d}\mathbf{P}_n\otimes\mathbf{P}_n+\varepsilon_n.$$

Then, applying the same Lemma 4 again shows that there exists a sequence $(z^n)_n\subset T'_{b^\star}S$, such that

$$\begin{aligned}0&\leftarrow\frac{1}{\mathbf{P}(B)^2}\int\int\langle x,y\rangle_{b^\star}\mathrm{d}\mathbf{P}_n\otimes\mathbf{P}_n(x,y)\\&=\frac{1}{\mathbf{P}(B)^2}\left(\int_B\int_B+\int_{B^c}\int_{B^c}+2\int_B\int_{B^c}\right)\langle x,y\rangle_{b^\star}\mathrm{d}\mathbf{P}_n\otimes\mathbf{P}_n(x,y)\\&\ge\|z^n\|^2_{b^\star}+\|y^n\|^2_{b^\star}+2\langle y^n,z^n\rangle_{b^\star}-\varepsilon_n.\end{aligned}\tag{6}$$

Letting $n\to\infty$, one obtains

$$\begin{aligned}0&\ge\lim_n\|z^n\|^2_{b^\star}+2\langle y^n,z^n\rangle_{b^\star}+\|y^n\|^2_{b^\star}\\&\ge\lim_n\|z^n\|^2_{b^\star}-2\|y^n\|_{b^\star}\|z^n\|_{b^\star}+\|y^n\|^2_{b^\star}\\&=\lim_n(\|z^n\|_{b^\star}-\|y^n\|_{b^\star})^2\ge 0.\end{aligned}$$

and which shows $\lim_n\|y^n\|=\lim_n\|z^n\|$ and also that (6) becomes an equality at the limit and therefore

$$\lim_n\|z^n\|^2_{b^\star}=\frac{1}{\mathbf{P}(B)^2}\int_B\int_B\langle x,y\rangle_{b^\star}\mathrm{d}\mathbf{P}\otimes\mathbf{P}(x,y)$$ □

This Lemma appears in a remark of [6].

Lemma 6 (Measurability of the log **map).** *Let (S,d) be a separable Alexandrov space. Let $p\in S$. Then $\log_p:S\to T_pS$ can be chosen to be $\mathfrak{S}_B$-measurable.*

Proof. Denote G_p the space of all constant speed geodesics emanating from p equipped with the sup distance $\|\cdot\|_\infty$. Then $(G_p,\|\cdot\|_\infty)$ is separable and complete too. Using Kuratowski and Ryll-Nardzewski measurable selection theorem, one can choose a Borel map $g:S\to G_p$ such that g maps x to a geodesic from p to x. Then, using the (proof of) Lemma 4.2 of [7], the map $l:G_p\to T_pS$ is measurable T_pS is equipped with the σ-algebra $\mathfrak{S}$ generated by open balls. Therefore, $\log_p:=l\circ g$ is $\mathfrak{S}_B$-measurable. □

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References

[1] A. Ahidar-Coutrix, T. Le Gouic, Q. Paris, "On the rate of convergence of empirical barycentres in metric spaces: curvature, convexity and extendible geodesics", https://arxiv.org/abs/1806.02740v1, 2018.

[2] S. Alexander, V. Kapovitch, A. Petrunin, "Alexandrov geometry", http://arxiv.org/abs/1903.08539, 2019.

[3] D. Burago, Y. Burago, S. Ivanov, *A course in metric geometry*, Graduate Studies in Mathematics, vol. 33, American Mathematical Society, 2001.

[4] S. Halbeisen, "On tangent cones of Alexandrov spaces with curvature bounded below", *Manuscr. Math.* **103** (2000), no. 2, p. 169-182.

[5] U. Lang, V. Schroeder, "Kirszbraun's theorem and metric spaces of bounded curvature", *Geom. Funct. Anal.* **7** (1997), no. 3, p. 535-560.
[6] T. Le Gouic, Q. Paris, P. Rigollet, A. J. Stromme, "Fast convergence of empirical barycenters in Alexandrov spaces and the Wasserstein space", https://arxiv.org/abs/1908.00828, 2019.
[7] S.-I. Ohta, "Barycenters in Alexandrov spaces of curvature bounded below", *Adv. Geom.* **12** (2012), no. 4, p. 571-587.
[8] K.-T. Sturm, "Metric spaces of lower bounded curvature", *Expo. Math.* **17** (1999), no. 1, p. 35-47.
[9] T. Yokota, "A rigidity theorem in Alexandrov spaces with lower curvature bound", *Math. Ann.* **353** (2012), no. 2, p. 305-331.