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Conformally symplectic structures and the Lefschetz condition

Les structures conformément symplectiques et la condition de Lefschetz

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Abstract. This short note provides a symplectic analogue of Vaisman's theorem in complex geometry. Namely, for any compact symplectic manifold satisfying the hard Lefschetz condition in degree 1, every locally conformally symplectic structure is in fact globally conformally symplectic, whenever there is a mutually compatible almost complex structure.

Résumé. Cette courte note fournit un analogue symplectique au théorème de Vaisman en géométrie complexe. Plus précisément, pour toute variété compacte symplectique satisfaisant la condition de Lefschetz en degré 1, chaque structure localement conformément symplectique est en fait globalement conformément symplectique, quand celle-ci et la forme symplectique sont compatibles avec la même structure presque complexe.

Keywords. Locally conformally symplectic structures, hard Lefschetz condition, almost complex structures.

Mots-clés. Structures localement conformément symplectiques, condition de Lefschetz, structures presque complexes.

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This short note provides a symplectic analogue of Vaisman's foundational result for complex manifolds, which states that if a complex manifold is Kähler, then every locally conformally Kähler structure is indeed globally conformally Kähler [20].

Recall that a locally conformally symplectic structure is a pair (η, θ) consisting of a real closed 1-form θ , and a non-degenerate 2-form η , such that $d\eta = \theta \wedge \eta$, where d is the exterior derivative. If θ is d -exact, then (η, θ) is globally conformally symplectic. Moreover, if $\theta = 0$, then η is a symplectic form. For more about locally conformally symplectic structures, we refer the reader

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for instance to [3, 10, 16]. An almost complex structure J is compatible with η if $\eta(J\cdot, J\cdot) = \eta$ and $\eta(\cdot, J\cdot)$ defines a Riemannian metric. The induced Riemannian metric is called an almost-Kähler metric if η is a symplectic form.

Eliashberg and Murphy [11, Theorem 1.11] used the h -principle to prove that compact almost-complex manifolds with a non-zero degree one cohomology class admit locally conformally symplectic structures which are not globally conformally symplectic. In addition to this, symplectic manifolds can have locally conformally symplectic structures which are not globally conformally symplectic, even in the presence of mutually compatible almost complex structures, cf. Example 3 below (we also refer the reader to [2, 4, 5, 18]). A sufficient condition for every locally conformally symplectic structure to be globally conformally symplectic, in the presence of a compatible almost complex structure, is given by the hard Lefschetz condition in degree 1.

Theorem 1. *If a compact symplectic manifold (M, ω) satisfies hard Lefschetz in degree 1, then any locally conformally symplectic structure (η, θ) on (M, ω) is necessarily globally conformally symplectic whenever η and ω share a compatible J .*

Recall that on a symplectic manifold (M, ω) of dimension $2n$ the hard Lefschetz condition holds in degree 1 if

$$L^{n-1} : H_d^1(M) \longrightarrow H_d^{2n-1}(M)$$

is an isomorphism, where $L\alpha = \omega \wedge \alpha$. We begin with a lemma, after establishing some notation.

For an almost complex manifold, we let $d_c := J^{-1}dJ$, where J is extended as an algebra automorphism, and for an odd operator δ define its Laplacian $\Delta_\delta = \delta^*\delta + \delta\delta^*$ with harmonics $\mathcal{H}_\delta^k = \text{Ker } \Delta_\delta \cap \Omega^k$, where Ω^k is the space of real differential k -forms.

Lemma 2. *Let (M, ω) be a compact symplectic manifold of dimension $2n$. The following are equivalent:*

- (1) *The hard Lefschetz condition holds in degree 1.*
- (2) *For any compatible almost-Kähler structure, $\mathcal{H}_d^1 = \mathcal{H}_{d^c}^1$, i.e. every d -harmonic 1-form is d^c -harmonic (and vice versa).*
- (3) *For any compatible almost-Kähler structure, every d -harmonic 1-form is the sum of d -harmonic forms of pure bi-degree,*

$$\mathcal{H}_d^1 = \mathcal{H}_d^{1,0} \oplus \mathcal{H}_d^{0,1}.$$

- (4) *For any compatible almost-Kähler structure,*

$$\mathcal{H}_d^1 = \mathcal{H}_{\bar{\delta}}^1 \cap \mathcal{H}_\mu^1,$$

where $d = \bar{\mu} + \bar{\delta} + \delta + \mu$, with bidegrees

$$|\bar{\mu}| = (-1, 2), |\bar{\delta}| = (0, 1), |\delta| = (1, 0), \text{ and } |\mu| = (2, -1).$$

A proof that the first condition implies the second is given in [14, Proposition 3.3], which we streamline below, using the same essential idea. We note that in [19, Theorem 5.4], Tomassini and Wang show that the second condition implies first (in fact, in all degrees). Namely, for an almost-Kähler manifold, the space \mathcal{H}_d satisfies the hard Lefschetz condition whenever $\mathcal{H}_d = \mathcal{H}_{d^c}$ (in all degrees). The remaining statements are included to give additional characterizations in terms of decompositions of d -harmonic forms.

Proof. To prove the first implication, assume there's a d -harmonic 1-form α so that $J\alpha$ is not d -harmonic. Since J is orthogonal, we may as well assume α is d -harmonic, but $J\alpha$ is orthogonal to the d -harmonics. Then

$$L^{n-1}\alpha = \frac{1}{(n-1)!} \star J\alpha.$$

Since star is an isometry, $L^{n-1}\alpha$ is orthogonal to the d -harmonics, and therefore $[L^{n-1}\alpha]$ is zero in cohomology. This shows L^{n-1} has a non-zero kernel.

Using

$$d^c = -i\bar{\mu} + i\bar{\partial} - i\partial + i\mu$$

we have

$$\mathcal{H}_d^1 \cap \mathcal{H}_{d^c}^1 = \mathcal{H}_d^{1,0} \oplus \mathcal{H}_d^{0,1} \subset \mathcal{H}_d^1.$$

This proves the second condition implies the third.

Next,

$$\mathcal{H}_d^{1,0} \oplus \mathcal{H}_d^{0,1} \subset \mathcal{H}_{\bar{\partial}}^1 \cap \mathcal{H}_{\mu}^1 \subset \mathcal{H}_d^1$$

where the last containment follows from the almost-Kähler identity [8],

$$\Delta_{\bar{\partial}} + \Delta_{\mu} = \Delta_{\theta} + \Delta_{\bar{\mu}}.$$

This proves the third condition implies the fourth.

Finally, by [8, Theorem 5.1], the last condition implies the first, since the space $\mathcal{H}_{\bar{\partial}}^1 \cap \mathcal{H}_{\mu}^1$ always has the Lefschetz $\mathfrak{sl}(2)$ -representation, and so satisfies hard Lefschetz on $H_d^1(M) \cong \mathcal{H}_d^1(M)$ if $\mathcal{H}_{\bar{\partial}}^1 \cap \mathcal{H}_{\mu}^1 = \mathcal{H}_d^1(M)$. \square

The proof of Theorem 1 is now essentially the same as in Vaisman [20], see also Ornea and Verbitsky [16].

Proof. Let (η, θ) be a locally conformally symplectic structure. Suppose there is a symplectic structure (M, ω) satisfying hard Lefschetz in degree 1, and let

$$\theta = h + d\gamma$$

be the Hodge decomposition of θ , with $h \in \mathcal{H}_d^1$. By the conformal change $\eta \mapsto \eta' = e^{-\gamma}\eta$, we have

$$d\eta' = \theta' \wedge \eta'$$

and $\theta' = h$ satisfies $d\theta' = 0$. So, we may as well assume that $d\eta = \theta \wedge \eta$ with θ d -harmonic.

By the Lemma's first implication, $d^c\theta = 0$ for any compatible almost Kähler structure, so that

$$dd^c\eta^{n-1} = -(n-1)^2\theta \wedge J\theta \wedge \eta^{n-1},$$

since $d^c\eta = J^{-1}d\eta = J^{-1}(\theta \wedge \eta) = -J\theta \wedge \eta$. Therefore,

$$0 = \int_M dd^c\eta^{n-1} = -(n-1)^2 \int_M \theta \wedge J\theta \wedge \eta^{n-1}.$$

Since J is compatible with η , we conclude that $\theta = 0$, so that η is symplectic. \square

Example 3. Two examples show the necessity of the assumptions in the theorem.

In [18, Example 4.1] the authors show that the Kodaira–Thurston manifold supports both a symplectic structure ω and a locally conformally symplectic structure (η, θ) , which is not globally conformally symplectic, while η and ω share a compatible J . In this case the hard Lefschetz condition fails in degree 1.

On the other hand, a locally conformally symplectic structure (η, θ) on a symplectic manifold (M, ω) can fail to be globally conformally symplectic if ω and η do not share a compatible J , even if the hard Lefschetz condition holds in degree 1. For example, the solvmanifold with structure equations $dx_1 = x_1x_3$, $dx_2 = -x_2x_3$, and $dx_3 = dx_4 = 0$, with symplectic form $\omega = x_1x_2 + x_3x_4$ has $(\theta = -x_3, \eta = x_1x_4 + x_2x_3)$ which is locally conformally symplectic, but not globally conformally symplectic.

From the theorem we deduce several corollaries, and emphasize that those statements concerning homogeneous spaces apply to all symplectic and locally conformally symplectic structures (even inhomogeneous structures).

Corollary 4. *Let G be a simply connected 4-dimensional Lie group admitting a uniform lattice Γ . Suppose $M = \Gamma \backslash G$ admits a symplectic form ω that satisfies hard Lefschetz in degree 1. Let (η, θ) be a locally conformally symplectic structure on M . Suppose that there is an almost complex structure J compatible with η that is compatible with some symplectic form. Then, (η, θ) is a globally conformal symplectic structure.*

Proof. By hypothesis, ω satisfies hard Lefschetz in degree 1, and therefore all degrees. By [17, Theorem 4.2], any other symplectic form satisfies hard Lefschetz as well, so the corollary follows then from Theorem 1. \square

Corollary 5. *Let M be a $2n$ -dimensional torus or a hyperelliptic surface. Let (η, θ) be a locally conformally symplectic structure on M . Suppose that there is an almost complex structure J compatible with η that is compatible with some symplectic form. Then, (η, θ) is a globally conformal symplectic structure.*

Proof. Any symplectic form on a $2n$ -dimensional torus or a hyperelliptic surface satisfies hard Lefschetz (for the torus see for instance [17, Proposition 4.1], we also refer the reader to [6, 7, 9, 13]). The corollary then follows from Theorem 1. \square

Tan and Tomassini studied in [17] the hard Lefschetz condition on some 6-dimensional compact completely solvable manifolds namely Nakamura manifolds N^6 [15], and $M^6(c)$, $N^6(c)$, $P^6(c)$ (we refer the reader to [1, 12, 17] for their definition). We remark that the manifolds $M^6(c)$, $N^6(c)$, and $P^6(c)$ do not admit any Kähler metric [12].

Corollary 6. *Let M be a Nakamura manifold N^6 or the compact completely solvable manifold $M^6(c)$ or $N^6(c)$ or $P^6(c)$. Let (η, θ) be a locally conformally symplectic structure on M . Suppose there is an almost complex structure J compatible with η that is compatible with some symplectic form. Then, (η, θ) is a globally conformal symplectic structure.*

Proof. It follows from [17, Theorem 5.1, Theorem 6.1, Remark 6.2] that any symplectic form on M satisfies hard Lefschetz. The corollary then follows from Theorem 1. \square

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References

- [1] L. C. de Andrés, M. Fernández, M. de León and J. J. Mencía, “Some six-dimensional compact symplectic and complex solvmanifolds”, *Rend. Mat. Appl.* **12** (1992), no. 1, pp. 59–67.
- [2] D. Angella, G. Bazzoni and M. Parton, “Structure of locally conformally symplectic Lie algebras and solvmanifolds”, *Ann. Sc. Norm. Super. Pisa, Cl. Sci.* **20** (2020), no. 1, pp. 373–411.

- [3] G. Bazzoni, “Locally conformally symplectic and Kähler geometry”, *EMS Surv. Math. Sci.* **5** (2018), no. 1-2, pp. 129–154.
- [4] G. Bazzoni and J. C. Marrero, “Locally conformal symplectic nilmanifolds with no locally conformal Kähler metrics”, *Complex Manifolds* **4** (2017), no. 1, pp. 172–178.
- [5] G. Bazzoni and J. C. Marrero, “On locally conformal symplectic manifolds of the first kind”, *Bull. Sci. Math.* **143** (2018), pp. 1–57.
- [6] C. Benson and C. S. Gordon, “Kähler and symplectic structures on nilmanifolds”, *Topology* **27** (1988), no. 4, pp. 513–518.
- [7] C. Benson and C. S. Gordon, “Kähler structures on compact solvmanifolds”, *Proc. Am. Math. Soc.* **108** (1990), no. 4, pp. 971–980.
- [8] J. Cirici and S. O. Wilson, “Topological and geometric aspects of almost Kähler manifolds via harmonic theory”, *Sel. Math., New Ser.* **26** (2020), no. 3, article no. 35 (27 pages).
- [9] T. Drăghici, “The Kähler cone versus the symplectic cone”, *Bull. Math. Soc. Sci. Math. Roum., Nouv. Sér.* **42(90)** (1999), no. 1, pp. 41–49.
- [10] S. Dragomir and L. Ornea, *Locally conformal Kähler geometry*, Birkhäuser, 1998, pp. xiv+327.
- [11] Y. Eliashberg and E. Murphy, “Making cobordisms symplectic”, *J. Am. Math. Soc.* **36** (2023), no. 1, pp. 1–29.
- [12] M. Fernández, M. de León and M. Saralegui, “A six-dimensional compact symplectic solvmanifold without Kähler structures”, *Osaka J. Math.* **33** (1996), no. 1, pp. 19–35.
- [13] T.-J. Li, “The space of symplectic structures on closed 4-manifolds”, in *Third International Congress of Chinese Mathematicians. Part 1, 2*, American Mathematical Society, 2008, pp. 259–277.
- [14] D. Lin, “Betti numbers and compatible almost complex structures on the hard Lefschetz condition”, 2023. <https://arxiv.org/abs/2312.16948>.
- [15] I. Nakamura, “On complex parallelisable manifolds and their small deformations”, *Proc. Japan Acad.* **48** (1972), pp. 447–449.
- [16] L. Ornea and M. Verbitsky, “Principles of Locally Conformally Kähler Geometry”, 2013. <https://arxiv.org/abs/2208.07188>.
- [17] Q. Tan and A. Tomassini, “Remarks on some compact symplectic solvmanifolds”, *Acta Math. Sin., Engl. Ser.* **39** (2023), no. 10, pp. 1874–1886.
- [18] N. Tardini and A. Tomassini, “ $\bar{\partial}$ -Harmonic forms on 4-dimensional almost-Hermitian manifolds”, 2021.
- [19] A. Tomassini and X. Wang, “Some results on the hard Lefschetz condition”, *Int. J. Math.* **29** (2018), no. 13, article no. 1850095 (30 pages).
- [20] I. Vaisman, “On locally and globally conformal Kähler manifolds”, *Trans. Am. Math. Soc.* **262** (1980), no. 2, pp. 533–542.