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Uniqueness for the Camassa–Holm equation with non-homogeneous boundary conditions

Unicité des solutions de l'équation de Camassa–Holm dans un domaine ouvert

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Abstract. We establish the uniqueness of solutions of the Camassa–Holm equation on a finite interval with non-homogeneous boundary conditions in the case of bounded momentum. A similar result for the higher-order Camassa–Holm system is also given. Our proofs rely on energy-type methods, with some multipliers given as solutions of some auxiliary elliptic systems.

Résumé. Nous montrons l'unicité des solutions de l'équation de Camassa–Holm sur un intervalle fini avec des conditions non-homogènes pour des solutions de moment d'inertie borné. Nous établissons aussi un résultat similaire pour les équations de Camassa–Holm d'ordre supérieur. Nos preuves s'appuient sur des méthodes d'énergies, avec des symétriseurs qui sont donnés comme solutions d'équations elliptiques auxiliaires bien choisies.

Keywords. Camassa–Holm, non-homogeneous boundary conditions, transport-elliptic coupling.

Mots-clés. Camassa–Holm, condition au bord non-homogène, couplage elliptique-hyperbolique.

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1. Introduction

1.1. *Presentation of the models*

The *Camassa–Holm equation* was first introduced by Fokas and Fuchsmeister in [17] for its similarity with the *KdV equation.* It was later re-derived by Camassa and Holm in [5] as a model for water waves in the shallow-water asymptotic under the influence of gravity and no surface tension. It reads as follows:

$$\partial_t v - \partial^3_{txx} v + 2\kappa\,\partial_x v + 3v\,\partial_x v = 2\,\partial_x v\,\partial^2_{xx} v + v\,\partial^3_{xxx} v. \tag{1}$$

We refer to [1,14,19] for a discussion on the physical relevance of this equation in the context of water waves. On the other hand, the Camassa–Holm equation as well as its higher-order generalizations (CH_n) are useful to describe geodesic flow for the Sobolev H^n metric, see [12,13]. The *higher-order Camassa–Holm system* is introduced for $n \geq 1$ integer as

$$\partial_t v = B_n(v, v), \tag{2}$$

where v is the unknown and B_n is defined through

$$B_n(u, v) := -\mathrm{A}_n^{-1}\big(2\partial_x v \mathrm{A}_n(u) + v \mathrm{A}_n(\partial_x u)\big), \tag{3}$$

where the operator A_n is

$$\mathrm{A}_n := \sum_{k=0}^{n} (-\partial_x^2)^k, \tag{4}$$

with suitable boundary conditions. The case $n = 1$ corresponds to the Camassa–Holm equation.

Moreover, the Camassa–Holm equation was studied a lot because of some interesting features it displays: it is *bi-hamiltonian completely integrable*, in the sense that it admits a Lax pair, which allows to construct infinitely many conservations laws, see [11,17]; in the case $\kappa = 0$, it admits *solitons solutions*, which do not evolve in C^1 — as they are peaked at the crest — and are referred to as *peakons*, see [8,21]; it also admits *wave-breaking solutions*, that is solutions whose x derivative gets unbounded in finite time, see [9,10].

The Cauchy problem for (1) was extensively studied both on the torus and on the full line, see for example [6,8,20]. The initial and boundary value problem on a half-line as well as the one on a segment was also studied. Escher and Yin extended the well-posedness result in H^s for $s > \frac{3}{2}$ and homogeneous boundary conditions, see [15,16]. Zhang, Liu, and Qiao tackle the case of inhomogeneous boundary conditions on the whole line, see [27]. Then, Perrolaz proved weak-strong uniqueness in the case of inhomogeneous boundary conditions and regularity $W^{1,\infty}$, see [24]. We refer to [7] for a study of the Cauchy problem for the higher-order Camassa–Holm system on a circle.

The aim of this article is to improve the weak-strong uniqueness for inhomogeneous boundary data stated in [24] into a stability estimate with regards to initial and boundary data, what entails in particular a global in-time uniqueness result, provided that y stays bounded. Our proof differs quite a lot from the previous ones as we do not use any characteristics to obtain our estimates, but rather an inequality on the relative energy between two solutions. This method to derive estimates was initially used for the 2D incompressible Euler equation with non-homogeneous boundary conditions and bounded vorticity, first in [25], then in [23] by the author.

We also deal with the higher-order Camassa–Holm equation, which has the same interesting feature as the Camassa–Holm equation to be recast as a transport-elliptic system. Yet the order of the elliptic part of this system is precisely $2n$, where $n \geq 1$ is the integer introduced in (2)–(4), and the energy methods have to be carried out differently.

We give the precise definition of the notion of solution for both CH in Paragraph 1.2 and higher-order CH in Paragraph 1.3, and state the associated theorems. You can find in Paragraph 1.4 the description of the rest of the article.

1.2. *Definitions and statement of the main result for the Camassa–Holm equation*

As our proof aims to get local in-time estimates only, let us fix once and for all a positive time $T > 0$. In all that follows, T can be arbitrarily large, but we do not want to bother with L^∞ functions not being integrable in time. We denote by Ω_T the space-time domain:

$$\Omega_T := [0, T] \times (0, 1). \tag{5}$$

Let us first remark that Equation (1) can be rewritten into the system

$$\partial_t y + v\cdot\partial_x y = -2y\partial_x v \quad \text{on } \Omega_T, \tag{6a}$$

$$\left(1-\partial_x^2\right)v = y-\kappa \quad \text{on } \Omega_T. \tag{6b}$$

The function y is called the *momentum* associated with v. The first equation (6a) is a *transport equation* with additional *stretching term*, while the second (6b) is elliptic. Once written under this form, the system is analogous to the *incompressible Euler equation* into vorticity form

$$\partial_t u + (u\cdot\nabla)\omega = (\omega\cdot\nabla)u, \tag{7a}$$

$$\operatorname{div} u = 0, \tag{7b}$$

$$\operatorname{curl} u = \omega. \tag{7c}$$

For the 2D Euler equation, it is common to prescribe the flux $u\cdot n$ at the boundary (n is the normal vector to the boundary) as well as the entering vorticity ω, see for example [26]. Using this analogy, we prescribe the flux v on the boundary as well as the *entering momentum*.

Let $v_l, v_r \in C^0(0,T)$. We make the assumption that v_r and v_l are non-zero except on a finite set. Let us define Γ_l and Γ_r similarly to [24] by

$$\Gamma_l := \left\{t\in[0,T];\ v_l>0\right\} \quad \text{and} \quad \Gamma_r := \left\{t\in[0,T];\ v_r<0\right\}. \tag{8}$$

These sets correspond to the sets of times t where the flux is entering the domain at $x=0$ or at $x=1$. Due to our assumption, the sets Γ_l and Γ_r can both be written as a finite union of open intervals.

Let $(y_l^c, y_r^c)\in L^\infty(\Gamma_l)\times L^\infty(\Gamma_r)$ and $y_0\in L^\infty(0,1)$. Following [24], we work with the initial condition

$$y\big|_{t=0} = y_0 \tag{9}$$

as well as the boundary conditions

$$v\big|_{x=0} = v_l \text{ on } (0,T), \quad \text{and} \quad v\big|_{x=1} = v_r \text{ on } (0,T), \tag{10a}$$

$$y\big|_{x=0} = y_l^c \text{ on } \Gamma_l, \quad \text{and} \quad y\big|_{x=1} = y_r^c \text{ on } \Gamma_r. \tag{10b}$$

The letter c refers to boundary condition.

The elliptic system (6b)–(10a) can be solved to express v as a function of y, κ and (v_l, v_r), for $(t,x)\in\Omega_T$,

$$v(t,x) := \cosh(x)v_l(t) + \int_0^x \cosh(x-s)Y(t,s)\,\mathrm{d}s + \frac{\sinh(x)}{\sinh(1)}\left(v_r(t) - \cosh(1)v_l(t) - \int_0^1 \cosh(1-s)Y(t,s)\,\mathrm{d}s\right), \tag{11}$$

where

$$Y(t,x) := \int_0^x y(t,s)\,\mathrm{d}s - \kappa x. \tag{12}$$

Remark that this expression makes sense and defines a function $v\in C^0\big([0,T], W^{1,\infty}(0,1)\big)$ as soon as $y\in C^0\big([0,T], L^1(0,1)\big)$ and $v_r, v_l\in C^0(0,T)$.

We give our definition of a solution to the system (6)–(9)–(10).

Definition 1. *We say that a triple* $(y, y_r, y_l)\in L^\infty(\Omega_T)\cap C^0\big([0,T], L^2(0,1)\big)\times L^\infty([0,T])^2$ *is a weak solution of the Camassa–Holm equation with initial and boundary conditions*

$$(y_0, v_l, v_r, y_r^c, y_l^c)\in L^\infty(0,1)\times C^0([0,T])^2\times L^\infty(\Gamma_l)\times L^\infty(\Gamma_r),$$

when we have the compatibility conditions corresponding to (9) *and* (10b)

$$y\big|_{t=0} = y_0, \quad y_r\big|_{\Gamma_r} = y_r^c \quad \textit{and} \quad y_l\big|_{\Gamma_l} = y_l^c, \tag{13}$$

and when for all $0 \le t_0 \le t_1 < T$ *and for all test functions* $\phi \in H^1\big([t_0,t_1]\times[0,1]\big)$*, we have:*

$$\begin{aligned}\int_{t_0}^{t_1}\int_0^1 &\big(y\partial_t\phi + yv\partial_x\phi - y\partial_x v\phi\big)\mathrm{d}x\mathrm{d}t\\ &= \int_{t_0}^{t_1}\big(y_r(t)v_r(t)\phi(t,1) - y_l(t)v_l(t)\phi(t,0)\big)\mathrm{d}t + \int_0^1\big(\phi(t_1,x)y(t_1,x) - \phi(t_0,x)y(t_0,x)\big)\mathrm{d}x, \end{aligned} \tag{14}$$

where the function v *is given by the formula* (11).

Remark 2. For $\phi \in H^1$, we have $\phi \in C^0$ since we are in dimension 1 (and thus the critical Sobolev embedding is at $H^{1/2}$). In particular, the boundary and initial terms make sense. It is natural to ask for the test function to be in H^1 for H^1 energy estimates, since it is the natural regularity that v itself (the solution of the equation) will have by elliptic regularity, and we want to use it as a test function for energy estimates.

Remark 3. If (y, y_r, y_l) is a weak solution with y smooth, then $y\big|_{x=0} = y_l$ and $y\big|_{x=1} = y_r$. To see this, do the integration by part used to obtain the weak formulation. For y smooth, all the terms will cancel but terms of the form $\int_{t_0}^{t_1}\big(y_r(t)v_r(t) - y(t,1)v(t,1)\big)\phi(\cdot,1)\mathrm{d}t$, then as it is true for all ϕ, we get the wanted equality.

Remark 4. Any solution $y \in L^\infty(\Omega_T)$ in the sense of distribution of equation (6a) for a transporting vector field $v \in L^1\big([0,T], H^2(0,1)\big)$ is in $C^0\big([0,T], L^2(0,1)\big)$. This allows us to evaluate pointwise.

Remark 5. Due to its low regularity, we cannot define the trace of y in 0 and 1 via the standard trace theorems. However, any distribution solution of a transport equation admits a trace in a weaker sense as long as the transporting field does not vanish on the boundary. To look at a complete discussion on the subject, we refer to [2].

We can state our main theorem.

Theorem 6. *Let* (y^1, y^1_r, y^1_l) *and* (y^2, y^2_r, y^2_l) *be two weak solutions of the Camassa–Holm equation with the same boundary conditions* (v_l, v_r, y^c_r, y^c_l) *and initial conditions* y^1_0 *and* y^2_0*. Let us assume that* $v_l, v_r \in H^1(0,T)$*. We call* v^1 *and* v^2 *the solutions of the elliptic problem associated, and* $\widetilde{v} := v^1 - v^2$*. Then there exists* $C > 0$ *such that for any* $0 \le T_0 < T_1 \le T$ *if neither* v_l *nor* v_r *changes sign on the interval* $[T_0, T_1]$*, then one has the estimate*

$$\|\widetilde{v}(T_1,\cdot)\|^2_{H^1} \le \Big(\|\widetilde{v}(T_0,\cdot)\|^2_{H^1} + \big|\partial_x\widetilde{v}(T_0,0)\big|^2 + \big|\partial_x\widetilde{v}(T_0,1)\big|^2\Big)\exp\big(C(T_1 - T_0)\big). \tag{15}$$

In particular, if $y^1_0 = y^2_0$*, then*

$$(y^1, y^1_r, y^1_l) = (y^2, y^2_r, y^2_l)$$

on the interval $[0, T]$.

Remark 7. The existence of a weak solution to the Camassa–Holm equation in the sense of Definition 1, as well as a weak-strong uniqueness property, were tackled in [24].

Remark 8. Theorem 6 assures uniqueness for same initial and boundary data, for as long as v_l and v_r don't change sign. To get uniqueness in general, we use the fact that there is only a finite number of points in time where v_l or v_r changes sign.

Remark 9. The estimate (15) is no longer valid when one of the fluxes v_l or v_r changes sign over time.

1.3. *Definitions and main results for the higher-order Camassa–Holm equation*

Let $n \geq 1$ be an integer. We define the operator A_n by

$$\mathrm{A}_n := \sum_{k=0}^{n} (-\partial_x^2)^k. \tag{16}$$

For example, the operator A_1 is equal to

$$\mathrm{A}_1 = \mathrm{Id} - \partial_x^2, \tag{17}$$

which is the elliptic operator used to describe the standard Camassa–Holm equation, see (6).

We say that a couple (v, y) is a solution of the *higher-order Camassa–Holm equation* when

$$\partial_t y + v\partial_x y = -2y\partial_x v \quad \text{on } (0,1), \tag{18a}$$
$$\mathrm{A}_n v = y \quad \text{on } (0,1). \tag{18b}$$

Remark 10. As mentioned in the presentation of the models, the higher-order Camassa–Holm equations were introduced by Constantin and Kolev in [13]. In their initial formulation, they are written on the torus as

$$\partial_t v = B_n(v, v), \tag{19}$$

where v is the unknown and B_n is defined through

$$B_n(u, v) := -\mathrm{A}_n^{-1}\big(2\partial_x v \mathrm{A}_n(u) + v\mathrm{A}_n(\partial_x u)\big), \tag{20a}$$
$$\mathrm{A}_n := \sum_{k=0}^{n} (-\partial_x^2)^k. \tag{20b}$$

By introducing the *momentum* $y := \mathrm{A}_n(v)$, we obtain the formulation (18), which suits us more in the context of a boundary value problem.

We prescribe the velocity v on the boundary $\{0,1\}$. We also prescribe the momentum y on the part of the boundary where $v_l > 0$ or $v_r < 0$. Moreover, the elliptic problem (18b) is of order n. Therefore, we prescribe more derivatives of v at the boundary. We mean by that, that for two vectors $\mathbf{v_l}$ and $\mathbf{v_r}$ in $\mathbb{R}^n$ with $\mathbf{v_l}_1 = v_l$ and $\mathbf{v_r}_1 = v_r$, we get the system:

$$\mathrm{A}_n v = y, \tag{21a}$$
$$\big(\mathscr{S}_i(v)(0)\big)_{i\in[[0,n-1]]} = \mathbf{v_l}, \tag{21b}$$
$$\big(\mathscr{S}_i(v)(1)\big)_{i\in[[0,n-1]]} = \mathbf{v_r}, \tag{21c}$$

where the operators $\mathscr{S}_i$ are defined through

$$\forall\, x \in \{0,1\}, \quad \mathscr{S}_i(g)(x) = \partial_x^i g(x). \tag{22}$$

With that in mind let us head to the definition of weak solutions.

Definition 11 (Variational solution to the Elliptic problem). *Let* $\mathbf{v_l}, \mathbf{v_r} \in \mathbb{R}^n$ *and* $y \in H^{-n}([0,1]) := H_0^n(0,1)'$. *Let* $\chi \in C^\infty([0,1])$ *be a smooth function equal to* 1 *in a neighborhood of* 0 *and equal to* 0 *in a neighborhood of* 1. *We define* $b = b(\mathbf{v_l}, \mathbf{v_r}, \chi)$ *through*

$$b(\mathbf{v_l}, \mathbf{v_r}, \chi)(x) := \sum_{k=0}^{n-1} \left(\frac{x^k \chi(x)}{k!} \mathbf{v_l}_k + \frac{(1-x)^k \chi(1-x)}{k!} \mathbf{v_r}_k \right). \tag{23}$$

We say that $v \in H^n(0,1)$ *is a solution of the system* (21), *when* $v - b(\mathbf{v_l}, \mathbf{v_r}, \chi)$ *belongs to* $H_0^n(0,1)$ *(closure of* $C_c^\infty(0,1)$ *for the* H^n *norm) and when for all* $g \in H_0^n(0,1)$, *one has*

$$\big\langle y - \mathrm{A}_n b, g \big\rangle_{H^{-n}, H^n} = \int_0^1 \mathbf{A}_n^{\frac{1}{2}}(v - b) \cdot \mathbf{A}_n^{\frac{1}{2}} g, \tag{24}$$

where we define the operator $\mathbf{A}_n^{\frac{1}{2}}$ *through*

$$\mathbf{A}_n^{\frac{1}{2}} := \left(\mathrm{Id}, \partial_x, \dots, \partial_x^n\right), \tag{25}$$

and $\cdot$ *is the standard scalar product in* $\mathbb{R}^n$.

Remark 12. Let g be a test function equal to zero in a neighborhood of 0 and 1, we have

$$\left\langle \mathrm{A}_n b, g \right\rangle_{H^{-n}, H^n} = \int_0^1 \mathbf{A}_n^{\frac{1}{2}} b \cdot \mathbf{A}_n^{\frac{1}{2}} g. \tag{26}$$

Therefore, Definition 11 does not depend on the choice of χ. Moreover, thanks to Lemma 42, any smooth solution of the system (21) is also a variational solution for this system.

Lemma 13. *Let* $y \in H^{-n}([0,1])$ *be a function, and* $\mathbf{v_l}, \mathbf{v_r} \in \mathbb{R}^n$. *There exists a unique solution* v *to the problem* (21) *in the sense of Definition 11. This solution verifies the estimate:*

$$\|v\|_{H^n} \lesssim \|y\|_{H^{-n}} + |\mathbf{v_l}| + |\mathbf{v_r}|. \tag{27}$$

Moreover for $y \in L^\infty(0,1)$, *one has*

$$\|v\|_{W^{2n,\infty}} \lesssim \|y\|_{L^\infty} + |\mathbf{v_l}| + |\mathbf{v_r}|. \tag{28}$$

The constant hidden in the $\lesssim$ *depends on n only.*

Proof. The existence and uniqueness comes from the Lax–Milgram Theorem. The H^{-n}-H^n estimate is straightforward. The L^∞-$W^{2n,\infty}$ estimate comes from Lemma 16 below. □

Remark 14. Note that this last estimate would no longer hold in higher dimensions. However, Schauder's estimates would give us L^p-$W^{2n,p}$ estimates for all p.

Remark 15. The solutions of (21a) form a finite dimensional space of dimension $2n$. This space can be computed explicitly. In total, we have $2n$ boundary conditions in (21b) and (21c), and one can prove that they are linearly independent on the space of solutions to (21a). This means that one could get an explicit formula for our weak solutions similarly to (11).

Lemma 16. *Let* $k, l \in \mathbb{N}$ *be two integers,* $a_0, \dots, a_{k-1} \in \mathbb{R}$ *real numbers, and* $g \in L^\infty(0,1)$ *a function. Let* $f \in H^{-l}$ *be a solution in the sense of distribution over* $(0,1)$ *of the ODE*

$$f^{(k)} + \sum_{i=0}^{k-1} a_i f^{(i)} = g. \tag{29}$$

Then f *belongs to* $W^{k,\infty}(0,1)$.

Proof. We prove Lemma 16 by induction over k.

Initialization ($k = 1$). We prove this initialization by induction over l.

- Initialization ($k = 1$, $l = 0$): If $f' + af = g \in L^\infty$ and $f \in L^2$, then $f' = g - af \in L^2$. Due to that, we have $f \in H^1$, and by Sobolev embedding f belongs to L^∞. Therefore $f' = g - af$ also belongs to L^∞ and f is in $W^{1,\infty}$.
- Induction ($k = 1$, $l \geq 1$): If $f' + af = g \in L^\infty$ and $f \in H^{-l}$, then $f' = g - af \in H^{-l}$. Therefore $f \in H^{-l+1}$.

Induction ($k \geq 2$). Let $k \in \mathbb{N}$ and assume that the lemma is true for $k - 1$. Let $a_0, \dots, a_{k-1} \in \mathbb{R}$ be real numbers and $f \in H^{-l}$ be a solution of

$$f^{(k)} + \sum_{i=0}^{k-1} a_i f^{(i)} = g \tag{30}$$

in the sense of distribution. Let us denote by P the polynomial

$$P := X^k + \sum_{i=0}^{k-1} a_i X^i.$$

We factorize it over $\mathbb{C}$:

$$P = \prod_{j=1}^{k} (X - \lambda_j).$$

We denote by Q the polynomial obtained by canceling the first factor, and by $(b_i)_i$ its coefficients:

$$Q := \prod_{j=2}^{k} (X - \lambda_j) = X^{k-1} + \sum_{i=0}^{k-2} b_i X^i.$$

We denote by h the function

$$h := f' - \lambda_1 f.$$

It is in H^{-l-1} and a solution of

$$h^{k-1} + \sum_{i=0}^{k-2} b_i h^{(i)} = g. \tag{31}$$

Therefore, by the step $k-1$ of the induction, $h \in W^{k-1,\infty}$. Then we get $f \in W^{k,\infty}$, as wanted. □

Definition 17 (Weak solution to the higher-order CH system). *Let $v_l, v_r \in C^0(0,T)$. We make the assumption that v_r and v_l are non-zero except on a finite set. We define the sets Γ_l and Γ_r by* (8).

We say that the boundary conditions

$$(\mathbf{v_l}, \mathbf{v_r}) \in C^0([0,T])^n \times C^0([0,T])^n$$

are admissible*, with respect to (v_l, v_r), when their first respective components $(\mathbf{v_l})_1$ and $(\mathbf{v_r})_1$ are equal respectively to v_r and v_l:*

$$(\mathbf{v_l})_1 = v_l \quad \textit{and} \quad (\mathbf{v_r})_1 = v_r. \tag{32}$$

We say that a triple

$$(y, y_r, y_l) \in L^\infty(\Omega_T) \cap C^0\big([0,T], L^2(0,1)\big) \times L^\infty([0,T])^2$$

is a weak solution of the higher-order Camassa–Holm equation with initial and boundary conditions

$$(y_0, \mathbf{v_l}, \mathbf{v_r}, y_r^c, y_l^c) \in L^\infty(0,1) \times C^0([0,T])^n \times C^0([0,T])^n \times L^\infty(\Gamma_l) \times L^\infty(\Gamma_r)$$

when we have the compatibility condition

$$y_r\big|_{\Gamma_r} = y_r^c \quad \textit{and} \quad y_l\big|_{\Gamma_l} = y_l^c, \tag{33}$$

and when for all $0 \le t_0 \le t_1 < T$ and for all test functions $\phi \in H^1\big([t_0,t_1] \times [0,1]\big)$, we have:

$$\int_{t_0}^{t_1} \int_0^1 \big(y\partial_t\phi + yv\partial_x\phi - y\partial_x v\phi\big) = \int_{t_0}^{t_1} \big(y_r v_r \phi(\cdot,1) - y_l v_l \phi(\cdot,0)\big) + \int_0^1 \phi(t_1,\cdot)y(t_1,\cdot) - \int_0^1 \phi(t_0,\cdot)y(t_0,\cdot), \tag{34}$$

where the function v is given as the unique solution of the elliptic problem (21).

With that in mind, we can formulate a local in-time existence theorem as follows.

Theorem 18. *Let $v_l, v_r \in C^0(0,T)$. We make the assumption that v_r and v_l are non-zero except on a finite set. We define the sets Γ_l and Γ_r through* (8). *Let $(y_0, \mathbf{v_l}, \mathbf{v_r}, y_r^c, y_l^c) \in L^\infty(0,1) \times C^0([0,T])^n \times C^0([0,T])^n \times L^\infty(\Gamma_l) \times L^\infty(\Gamma_r)$ be admissible initial and boundary conditions associated with v_r and v_l (meaning that $(\mathbf{v_l}, \mathbf{v_r})$ is admissible with respect to (v_r, v_l)).*

Then there exists $\widetilde{T} > 0$ and a weak solution $(y, y_r, y_l) \in L^\infty\big([0,\widetilde{T}] \times [0,1]\big) \cap C^0\big([0,\widetilde{T}], L^2(0,1)\big) \times L^\infty([0,\widetilde{T}])^2$ to the higher-order Camassa–Holm equation with $(y_0, \mathbf{v_l}, \mathbf{v_r}, y_r^c, y_l^c)$ as initial and boundary data.

The proof of Theorem 18 follows the lines of the one of Theorem 1 in [24]. We give the sketch of the proof in Appendix B and refer to the article [24] for the details. As for the 3D Euler equation, the proof of existence only constructs solutions on small time intervals (see [22]).

Assuming however the existence of a solution on a given interval $[0,T]$, this solution is unique.

Theorem 19. *Let (y^1, y_r^1, y_l^1) and (y^2, y_r^2, y_l^2) be two solutions in the sense of Definition 17 with the same initial and boundary conditions $(y_0, \mathbf{v_l}, \mathbf{v_r}, y_r^c, y_l^c)$. Then*

$$(y^1, y_r^1, y_l^1) = (y^2, y_r^2, y_l^2)$$

on the interval $[0,T]$.

1.4. *Sketch of the proofs*

To prove Theorem 6, we take two solutions (y^1, y_r^1, y_l^1) and (y^2, y_r^2, y_l^2) with possibly different initial and boundary values, and we analyze the dynamics time evolution on the H_x^1 norm of $v^1 - v^2$, where v^1 and v^2 refer to the solutions of the elliptic problem (21) with respectively y^1 and y^2 instead of y.

The sketch of the proof is the following. In Paragraph 2.1, we provide an energy estimate for the relative energy between two solutions. In Paragraph 2.2, we seek to control the *entering energy fluxes* which were the bad boundary terms (meaning that they cannot be discarded due to their sign and are not trivially bounded by the relative energy) in the relative energy inequality of Paragraph 2.1. To that end, we introduce an auxiliary test function constructed as a well-chosen elliptic multiplier of the equation. In Paragraph 2.3, we conclude the proof with the help of a Gronwall argument.

The proof of Theorem 19 is similar in its structure: in Paragraph 3.1, we derive a relative energy inequality, then in Paragraph 3.2, we control the entering fluxes of the relative energy inequality.

Remark 20. The proof of Theorem 19 is similar to the proof of Theorem 6. However, it has its own difficulties. Heuristically, in the proof of Theorem 6, there is an Energy Inequality, looking like

$$\frac{\mathrm{d}}{\mathrm{d}t}(\text{Energy}) + \text{Energy Fluxes} \le \text{cst} \times \text{Energy},$$

and an Auxiliary Inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}(\text{Energy Fluxes}) \le \text{cst} \times (\text{Energy} + \text{Energy Fluxes}).$$

Combining the two of them we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(\text{Energy} + \text{Energy Fluxes}) \le \text{cst} \times (\text{Energy} + \text{Energy Fluxes}),$$

which allows us to conclude through the help of Gronwall's Lemma. In the proof of Theorem 19, the Energy Inequality still looks like

$$\frac{\mathrm{d}}{\mathrm{d}t}(\text{Energy}) + \text{Exiting Fluxes} - \text{Entering Fluxes} \le \text{cst} \times \text{Energy},$$

but the Auxiliary Inequality is more of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}(\text{Auxiliary}) + \text{Entering Fluxes} \le \text{cst} \times (\text{Energy} + \text{Auxiliary}).$$

Combining the two of them we get

$$\tfrac{\mathrm{d}}{\mathrm{d}t}(\text{Energy}+\text{Auxiliary})+\left|\text{Energy Fluxes}\right|\leq \text{cst}\times(\text{Energy}+\text{Auxiliary}),$$

which still allows us to conclude through the help of Gronwall's Lemma.

The proof is quite different because of the construction of the auxiliary test function. We solve the same dual elliptic problem to construct it, but in the case of Camassa–Holm, the boundary data for this elliptic problem are bounded by the energy fluxes. This is not true in the higher-order case. That is the main reason why the proof for Camassa–Holm is easier and stronger, i.e. it gives bounds on the derivative of the entering flux.

Remark 21. In the case of periodic boundary condition, the Camassa–Holm equation was proved to be locally well-posed in $H^s(\mathbb{T})$ for $1\leq s<2$, for any initial data $v_0\in W^{1,\infty}$ (see [20]). In the case of the full line, the existence and uniqueness of conservative solutions in $H^1(\mathbb{R})$ was tackled by Constantin, Bressan, Chen and Zhang in [3] and [4]. To define weak solutions for which the momentum y is not L^∞ or even L^1, one writes the equation as

$$\partial_t v+\partial_x\left(\tfrac{v^2}{2}\right)=\partial_x P, \tag{35a}$$

$$P=\left(1-\partial_x^2\right)^{-1}\left(v^2+\tfrac{(\partial_x v)^2}{2}\right). \tag{35b}$$

This formulation is easier to define in the case of the line or the torus as one does not need additional boundary conditions for P. To go into lower regularity than what we do in this article (meaning solution with bounded momentum), one could explore this formulation of the equation and in particular, ask ourselves which boundary conditions are needed for it to make sense. One caveat for the construction of a conservative solution in our case is that we don't have conservation of energy anymore. One will need to take into account how the energy fluxes are treated for less regular solutions.

Remark 22. One can also remark that a formulation similar to (35) exists for the higher-order Camassa–Holm system (18). One can recast this system as:

$$\partial_t v+\partial_x\left(\tfrac{v^2}{2}\right)=\partial_x P, \tag{36a}$$

$$A_n(P)=\mathscr{F}_n[v], \tag{36b}$$

where $\mathscr{F}_n[v]$ is a differential polynomial in v depending on n that we will not describe here.

It is using this formulation as well as the elliptic regularization of the equation that Coclite, Holden and Karlsen tackled the existence of a solution for the higher-order Camassa–Holm system on the circle (see [7]). They also obtained a weak-strong uniqueness result.

Remark 23. The problem of stability estimates on the whole interval when the flux v_r or v_l changes sign is still open.

2. Proof of Theorem 6

2.1. *Energy estimate for the difference of two solutions*

Let us take two weak solutions (y^1,y^1_r,y^1_l) and (y^2,y^2_r,y^2_l) of the Camassa–Holm equation with initial and boundary conditions $(y^1_0,v_l,v_r,y^{1,c}_r,y^{1,c}_l)$ and $(y^2_0,v_l,v_r,y^{2,c}_r,y^{2,c}_l)$. We define the following functions

$$\widetilde{y}:=y^1-y^2,\qquad \widetilde{v}:=v^1-v^2, \tag{37}$$

$$\widehat{y}:=\frac{y^1+y^2}{2},\qquad \widehat{v}:=\frac{v^1+v^2}{2}, \tag{38}$$

$$\widetilde{y}_l:=y^1_l-y^2_l,\qquad \widetilde{y}_r:=y^1_r-y^2_r, \tag{39}$$

where the functions v^1 and v^2 are given through (11).

We take the difference of Equation (14) for the solutions y^1 and y^2. The function $\widetilde{y}$ verifies the following equality for all $0 \le t_0 \le t_1 < T$ and for all test functions $\phi \in H^1\big([t_0,t_1]\times[0,1]\big)$:

$$\int_{t_0}^{t_1}\int_0^1 \Big(\widetilde{y}\partial_t\phi + \big(\widetilde{y}\widehat{v}+\widehat{y}\widetilde{v}\big)\partial_x\phi - \big(\widetilde{y}\partial_x\widehat{v}+\widehat{y}\partial_x\widetilde{v}\big)\phi\Big) = \int_{t_0}^{t_1}\widetilde{y}_r v_r\phi(\cdot,1) - \int_{t_0}^{t_1}\widetilde{y}_l v_l\phi(\cdot,0) + \int_0^1\phi(t_1,\cdot)\widetilde{y}(t_1,\cdot) - \int_0^1\phi(t_0,\cdot)\widetilde{y}(t_0,\cdot). \quad (40)$$

Furthermore the functions $\widetilde{v}$ and $\widehat{v}$ are solutions of the following elliptic problems:

$$\big(1-\partial_x^2\big)\widetilde{v} = \widetilde{y} \quad \text{on } (0,1), \qquad \big(1-\partial_x^2\big)\widehat{v} = \widehat{y}-\kappa \quad \text{on } (0,1), \quad (41a)$$

$$\widetilde{v}\big|_{x=0} = 0, \qquad \widehat{v}\big|_{x=0} = v_l, \quad (41b)$$

$$\widetilde{v}\big|_{x=1} = 0, \qquad \widehat{v}\big|_{x=1} = v_r. \quad (41c)$$

With that in mind, we prove the following lemma.

Lemma 24. *The functions $\widehat{v}$ and $\widetilde{v}$ lie in $L^\infty\big([0,T],W^{2,\infty}([0,1])\big)\cap C^0\big([0,T],H^2(0,1)\big)$. Moreover, the function $\widetilde{v}$ lies in $W^{1,\infty}\big([0,T],H^1(0,1)\big)$.*

Proof. To obtain the space regularity of $\widehat{v}$ and $\widetilde{v}$, remark that the primitives Y^1 and Y^2 of y^1 and y^2, which are in $L^\infty\big([0,T]\times(0,1)\big)\cap C^0\big([0,T],L^2(0,1)\big)$, defined as in (12) with y^1 and y^2 instead of y, are in $L^\infty\big([0,T],W^{1,\infty}(0,1)\big)\cap L^\infty\big([0,T],H^1(0,1)\big)$. Then use the formula (11).

Let us now prove the time regularity of $\widetilde{v}$. Let us fix two times $t_0 < t_1$, and denote

$$a_{t_0}^{t_1}(x) := \widetilde{v}(t_1,x) - \widetilde{v}(t_0,x).$$

Recalling that $\widetilde{v}$ verifies (41a) with homogeneous boundary conditions, we obtain that

$$\int_0^1 \big|\partial_x a_{t_0}^{t_1}\big|^2 = -\int_0^1 \Big(a_{t_0}^{t_1} - \big(\widetilde{y}(t_1,\cdot)-\widetilde{y}(t_0,\cdot)\big)\Big)a_{t_0}^{t_1},$$

which can be rewritten as

$$\int_0^1 \big|\partial_x a_{t_0}^{t_1}\big|^2 + \int_0^1 \big|a_{t_0}^{t_1}\big|^2 = \int_0^1 \big(\widetilde{y}(t_1,\cdot)-\widetilde{y}(t_0,\cdot)\big)a_{t_0}^{t_1}.$$

Hence, we get the inequality

$$\|a_{t_0}^{t_1}\|_{H^1(0,1)}^2 \le \left|\int_0^1 \big(\widetilde{y}(t_1,\cdot)-\widetilde{y}(t_0,\cdot)\big)a_{t_0}^{t_1}\right|. \quad (42)$$

Using (40) with $a_{t_0}^{t_1}$ instead of ϕ (considered as a function constant in time), we obtain that:

$$\int_0^1 \big(\widetilde{y}(t_1,\cdot)-\widetilde{y}(t_0,\cdot)\big)a_{t_0}^{t_1} = \int_{t_0}^{t_1}\int_0^1 \Big(\big(\widetilde{y}\widehat{v}+\widehat{y}\widetilde{v}\big)\partial_x a_{t_0}^{t_1} - \big(\widetilde{y}\partial_x\widehat{v}+\widehat{y}\partial_x\widetilde{v}\big)a_{t_0}^{t_1}\Big). \quad (43)$$

Combining (42) and (43), we get that:

$$\|a_{t_0}^{t_1}\|_{H^1(0,1)} \le |t_1-t_0|\big(\|\widetilde{y}\|_{L^\infty([0,T]\times[0,1])}\|\widehat{v}\|_{L^\infty([0,T],H^1(0,1))} + \|\widehat{y}\|_{L^\infty([0,T]\times[0,1])}\|\widetilde{v}\|_{L^\infty([0,T],H^1(0,1))}\big).$$

Recalling that $a_{t_0}^{t_1}(x) = \widetilde{v}(t_1,x)-\widetilde{v}(t_0,x)$, we conclude that $\widetilde{v}\in W^{1,\infty}\big([0,T],H^1(0,1)\big)$. □

Now that we have Lemma 24, we prove the following relative energy equality:

Proposition 25. *For all $0 \le t_0 < t_1 \le T$, we have the following equality:*

$$\|\widetilde{v}(t_1,\cdot)\|_{H^1(0,1)}^2 - \|\widetilde{v}(t_0,\cdot)\|_{H^1(0,1)}^2 + \int_{t_0}^{t_1}\big|\partial_x\widetilde{v}(t,1)\big|^2 v_r - \int_{t_0}^{t_1}\big|\partial_x\widetilde{v}(t,0)\big|^2 v_l + \int_{t_0}^{t_1}\int_0^1\Big(3|\widetilde{v}|^2+|\partial_x\widetilde{v}|^2\Big)\partial_x\widehat{v} + \int_{t_0}^{t_1}\int_0^1\partial_x\big(|\widetilde{v}|^2\big)\big(\widehat{v}-\widehat{y}+\kappa\big) = 0. \quad (44)$$

Proof. Thanks to Lemma 24, we can take $\widetilde{v}$ as a test function in (40), which we do. For all $t_0 < t_1$, we have that

$$\int_{t_0}^{t_1}\int_0^1 \big(\widetilde{y}\partial_t\widetilde{v} + \widetilde{y}\widehat{v}\partial_x\widetilde{v} - \widetilde{y}\widetilde{v}\partial_x\widehat{v}\big)$$
$$= \int_{t_0}^{t_1} \widetilde{y}_r v_r \widetilde{v}(\cdot,1) - \int_{t_0}^{t_1} \widetilde{y}_l v_l \widetilde{v}(\cdot,0) + \int_0^1 \widetilde{v}(t_1,\cdot)\widetilde{y}(t_1,\cdot) - \int_0^1 \widetilde{v}(t_0,\cdot)\widetilde{y}(t_0,\cdot). \tag{45}$$

We cancel the boundary terms, because $\widetilde{v}\big|_{x=0} = \widetilde{v}\big|_{x=1} = 0$, to get

$$\int_{t_0}^{t_1}\int_0^1 \big(\widetilde{y}\partial_t\widetilde{v} + \widetilde{y}\widehat{v}\partial_x\widetilde{v} - \widetilde{y}\partial_x\widehat{v}\widetilde{v}\big) = \int_0^1 \widetilde{v}(t_1,\cdot)\widetilde{y}(t_1,\cdot) - \int_0^1 \widetilde{v}(t_0,\cdot)\widetilde{y}(t_0,\cdot). \tag{46}$$

Now, we reformulate each term of (46) by using integration by parts as well as (41).

- First let us look at $\int_0^1 \widetilde{v}(t,\cdot)\widetilde{y}(t,\cdot)$ (which will be used for $t = t_0$ and $t = t_1$):

$$\begin{aligned}\int_0^1 \widetilde{v}(t,\cdot)\widetilde{y}(t,\cdot) &= \int_0^1 \widetilde{v}\big(1-\partial_x^2\big)\widetilde{v}\\ &= \int_0^1 |\widetilde{v}|^2 + \int_0^1 |\partial_x\widetilde{v}|^2\\ &= \big\|\widetilde{v}(t,\cdot)\big\|^2_{H^1(0,1)}.\end{aligned} \tag{47}$$

  By Lemma 24, $\widetilde{v}(t,\cdot)$ is in H^2, which allows the computations above.
- The term $\int_{t_0}^{t_1}\int_0^1 \widetilde{y}\partial_t\widetilde{v}$ is dealt with similarly:

$$\int_{t_0}^{t_1}\int_0^1 \widetilde{y}\partial_t\widetilde{v} = \frac{1}{2}\Big(\big\|\widetilde{v}(t_1,\cdot)\big\|^2_{H^1(0,1)} - \big\|\widetilde{v}(t_0,\cdot)\big\|^2_{H^1(0,1)}\Big). \tag{48}$$

- Now, we deal with the two bilinear terms. Let us recast the first one:

$$\begin{aligned}\int_{t_0}^{t_1}\int_0^1 \widetilde{y}\widehat{v}\partial_x\widetilde{v} &= \int_{t_0}^{t_1}\int_0^1 \big(1-\partial_x^2\big)\widetilde{v}\,\widehat{v}\,\partial_x\widetilde{v}\\ &= \frac{1}{2}\int_{t_0}^{t_1}\int_0^1 \partial_x\big(|\widetilde{v}|^2 - |\partial_x\widetilde{v}|^2\big)\widehat{v}\\ &= -\frac{1}{2}\int_{t_0}^{t_1}\int_0^1 \big(|\widetilde{v}|^2 - |\partial_x\widetilde{v}|^2\big)\partial_x\widehat{v} - \frac{1}{2}\int_{t_0}^{t_1}\Big(\big|\partial_x\widetilde{v}(t,1)\big|^2 v_r - \big|\partial_x\widetilde{v}(t,0)\big|^2 v_l\Big).\end{aligned} \tag{49}$$

  Since $\widetilde{v} \in L_t^\infty W_x^{2,\infty}$, we have $\partial_x\widetilde{v} \in L_t^\infty W_x^{1,\infty}$, and therefore the above integrations are done in a strong sense.
- Let us reformulate the second one:

$$\begin{aligned}\int_{t_0}^{t_1}\int_0^1 \widetilde{y}\widetilde{v}\partial_x\widehat{v} &= \int_{t_0}^{t_1}\int_0^1 \big(1-\partial_x^2\big)\widetilde{v}\,\widetilde{v}\,\partial_x\widehat{v}\\ &= \int_{t_0}^{t_1}\int_0^1 \big(|\widetilde{v}|^2\partial_x\widehat{v} + |\partial_x\widetilde{v}|^2\partial_x\widehat{v} + \widetilde{v}\partial_x\widetilde{v}\partial_x^2\widehat{v}\big),\end{aligned} \tag{50}$$

  and we use (41a) to get rid of the second order derivative:

$$\int_{t_0}^{t_1}\int_0^1 \widetilde{v}\partial_x\widetilde{v}\partial_x^2\widehat{v} = \frac{1}{2}\int_{t_0}^{t_1}\int_0^1 \partial_x\big(|\widetilde{v}|^2\big)\big(\widehat{v} - \widehat{y} + \kappa\big). \tag{51}$$

By combining (46) with (47), (48), (49), (50) and (51), we obtain the wanted result. □

We deduce the following corollary.

Corollary 26. *There exists a constant $C > 0$, depending only on $\|\widehat{y}\|_{L_t^\infty L_x^\infty}$, such that for almost every $0 < t < T$, we have the following inequality:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\big\|\widetilde{v}(t,\cdot)\big\|^2_{H^1(0,1)}\Big) + \big|\partial_x\widetilde{v}(t,1)\big|^2 v_r(t) - \big|\partial_x\widetilde{v}(t,0)\big|^2 v_l(t) \le C\big\|\widetilde{v}(t,\cdot)\big\|^2_{H^1(0,1)}. \tag{52}$$

Proof. One starts from equality (44), with $t_0 = t$ and $t_1 = t + \varepsilon$ for $t \in [0, T]$ and $\varepsilon > 0$.

Since $\widetilde{v}$ lies in $W^{1,\infty}_t H^1_x$, by Rademacher's Theorem the fraction

$$\frac{\|\widetilde{v}(t+\varepsilon,\cdot)\|^2_{H^1(0,1)} - \|\widetilde{v}(t,\cdot)\|^2_{H^1(0,1)}}{\varepsilon}$$

converges for almost every t towards

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\widetilde{v}(t,\cdot)\|^2_{H^1(0,1)}\right).$$

The quantities $\partial_x \widetilde{v}(\cdot,0)^2 v_l$ and $\partial_x \widetilde{v}(\cdot,1)^2 v_r$ are both L^∞. Therefore, by Lebesgue Point Theorem, for almost every $t \in [0, T]$, the integral

$$\frac{1}{\varepsilon}\int_t^{t+\varepsilon} \partial_x \widetilde{v}(\cdot,0)^2 v_l$$

converges towards

$$\partial_x \widetilde{v}(t,0)^2 v_l(t).$$

Similarly $\frac{1}{\varepsilon}\int_t^{t+\varepsilon} \partial_x \widetilde{v}(\cdot,1)^2 v_r$ converges towards $\partial_x \widetilde{v}(t,1)^2 v_r(t)$.

Moreover, using the Cauchy–Schwarz inequality, one gets that for all $t \in [0, T]$ and for all $\varepsilon > 0$

$$\int_t^{t+\varepsilon}\int_0^1 \left(3|\widetilde{v}|^2 + |\partial_x \widetilde{v}|^2\right)\partial_x \widehat{v} \le 4\varepsilon \|\widehat{v}\|_{L^\infty_t W^{1,\infty}_x} \|\widetilde{v}\|^2_{L^\infty([t-\varepsilon,t+\varepsilon],H^1(0,1))},$$

$$\int_t^{t+\varepsilon}\int_0^1 \partial_x\left(|\widetilde{v}|^2\right)\left(\widehat{v} - \widehat{y} + \kappa\right) \le \varepsilon\left(\|\widehat{v}\|_{L^\infty_t L^\infty_x} + \|\widehat{y}\|_{L^\infty_t L^\infty_x} + \kappa\right)\|\widetilde{v}\|^2_{L^\infty([t-\varepsilon,t+\varepsilon],H^1(0,1))}. \qquad \square$$

2.2. *Auxiliary inequality*

We define two functions u_l and u_r by setting for all $x \in [0, 1]$

$$u_l(x) := -\sinh(x) + \cosh(x)\tanh(1) \quad \text{and} \quad u_r(x) := -\frac{\sinh(x)}{\cosh(1)}. \tag{53}$$

They are the solutions to the non-homogeneous Zaremba-type problems:

$$\left(1-\partial_x^2\right)u_l = \left(1-\partial_x^2\right)u_r = 0, \tag{54a}$$

$$-\partial_x u_l(0) = \partial_x u_r(1) = 1, \tag{54b}$$

$$u_l(1) = u_r(0) = 0. \tag{54c}$$

We want to bound $\partial_x \widetilde{v}$ at the boundary with the help of a Gronwall argument. Let us begin by showing that $\partial_x \widetilde{v}(\cdot,0)$ and $\partial_x \widetilde{v}(\cdot,1)$ are Lipschitz functions.

Lemma 27. *The functions $\partial_x \widetilde{v}(\cdot,0)$ and $\partial_x \widetilde{v}(\cdot,1)$ are Lipschitz functions with respect to time.*

Proof. By differentiating (11) in x, we obtain:

$$\partial_x \widetilde{v}(t,x) = \int_0^x \sinh(x-s)\widetilde{Y}(t,s)\mathrm{d}s + \widetilde{Y}(t,x) - \frac{\cosh(x)}{\sinh(1)}\int_0^1 \cosh(1-s)\widetilde{Y}(t,s)\mathrm{d}s. \tag{55}$$

To prove the regularity in time of $\partial_x \widetilde{v}(\cdot,0)$, we prove the time regularity of the function $\widetilde{Y}$. More precisely, we prove that $\widetilde{Y} \in W^{1,\infty}_t L^2_x$, which is enough since $\widetilde{Y}(t,0) = 0$ for all t.

Let $\phi \in L^2(0,1)$ be a function. We denote by Φ the primitive of ϕ verifying $\Phi(1) = 0$:

$$\Phi(x) := -\int_x^1 \phi(s)\mathrm{d}s.$$

Let us remark that we take this choice of primitive to cancel the term involving $\widetilde{y}_r$. Using Φ, (considered as a constant function in time) as a test function in (40), we obtain that

$$\int_{t_0}^{t_1} \widetilde{y}_l v_l \Phi(0) + \int_0^1 \Phi\widetilde{y}(t_1,\cdot) - \int_0^1 \Phi\widetilde{y}(t_0,\cdot) = \int_{t_0}^{t_1}\int_0^1 \Big(\left(\widetilde{y}\widehat{v} + \widehat{y}\widetilde{v}\right)\phi - \left(\widetilde{y}\partial_x\widehat{v} + \widehat{y}\partial_x\widetilde{v}\right)\Phi\Big). \tag{56}$$

Moreover, by integration by parts, we have

$$\int_0^1 \Phi \widetilde{y}(t,\cdot) = -\int_0^1 \phi \widetilde{Y}(t,\cdot). \tag{57}$$

Hence, by combining (56) and (57), we obtain that

$$\begin{aligned}\left\| \widetilde{Y}(t_1,\cdot) - \widetilde{Y}(t_0,\cdot)\right\|_{L^2(0,1)} \le |t_1 - t_0| \Big(|\Phi(0)| \, \|\widetilde{y}_l\|_{L^\infty([0,T])} \\ + \left\| \widetilde{y}\widehat{v} + \widehat{y}\widetilde{v}\right\|_{L^\infty([0,T],L^2(0,1))} + \left\| \widetilde{y}\partial_x\widehat{v} + \widehat{y}\partial_x\widetilde{v}\right\|_{L^\infty([0,T],L^2(0,1))}\Big). \quad \square\end{aligned}$$

We prove the following auxiliary inequalities.

Proposition 28. *There exists a constant $C>0$ such that, we have the inequalities*

$$\forall^{\text{a.e.}}\ t\in\Gamma_l, \quad \frac{\mathrm{d}}{\mathrm{d}t}\Big(|\partial_x\widetilde{v}(t,0)|^2\Big) \le C\Big(\|\widetilde{v}(t,\cdot)\|^2_{H^1(0,1)} + |\partial_x \widetilde{v}(t,0)|^2\Big) + \frac{1}{2}|\partial_x\widetilde{v}(t,1)|^2 |v_r(t)|, \tag{58}$$

$$\forall^{\text{a.e.}}\ t\in\Gamma_r, \quad \frac{\mathrm{d}}{\mathrm{d}t}\Big(|\partial_x\widetilde{v}(t,1)|^2\Big) \le C\Big(\|\widetilde{v}(t,\cdot)\|^2_{H^1(0,1)} + |\partial_x \widetilde{v}(t,1)|^2\Big) + \frac{1}{2}|\partial_x\widetilde{v}(t,0)|^2 |v_l(t)|, \tag{59}$$

where we recall that Γ_l/Γ_r are the sets of times of entering flux at the left/right defined in (8).

Proof. The two inequalities (58) and (59) have the same proof, we prove inequality (58) here.

We define the auxiliary test function $\widetilde{v}^{\text{aux},l}$ through

$$\forall\ t\in[0,T], \forall\ x\in[0,1], \quad \widetilde{v}^{\text{aux},l}(t,x) := \partial_x\widetilde{v}(t,0)\, v_l^+(t)\, u_l(x). \tag{60}$$

Let $0<t_0<t_1<T$ be two positive times (at the end of the proof, we will take $t_0=t$ and $t_1=t+\varepsilon$ and make ε go to 0) such that $[t_0,t_1]\subset\Gamma_l$. Using Lemma 27, we know that we can take $\widetilde{v}^{\text{aux},l}$ as test function in (40), which leads to

$$\begin{aligned}&\int_{t_0}^{t_1}\int_0^1 \Big(\widetilde{y}\partial_t\widetilde{v}^{\text{aux},l} + \big(\widetilde{y}\widehat{v}+\widehat{y}\widetilde{v}\big)\partial_x\widetilde{v}^{\text{aux},l} - \big(\widetilde{y}\partial_x\widehat{v}+\widehat{y}\partial_x\widetilde{v}\big).\widetilde{v}^{\text{aux},l}\Big) \\ &= \int_{t_0}^{t_1} \widetilde{y}_r v_r \widetilde{v}^{\text{aux},l}(\cdot,1) - \int_{t_0}^{t_1}\widetilde{y}_l v_l \widetilde{v}^{\text{aux},l}(\cdot,0) + \int_0^1 \widetilde{v}^{\text{aux},l}(t_1,\cdot)\widetilde{y}(t_1,\cdot) - \int_0^1 \widetilde{v}^{\text{aux},l}(t_0,\cdot)\widetilde{y}(t_0,\cdot).\end{aligned} \tag{61}$$

The boundary term $\int_{t_0}^{t_1}\widetilde{y}_r v_r \widetilde{v}^{\text{aux},l}(\cdot,1)$ is equal to 0 due to the assumption $u_l(1)=0$, and therefore $\widetilde{v}^{\text{aux},l}(\cdot,1)=0$. The boundary term $\int_{t_0}^{t_1}\widetilde{y}_l v_l \widetilde{v}^{\text{aux},l}(\cdot,0)$ is also equal to 0 since $\widetilde{y}_l\big|_{\Gamma_l}=0$.

$$\begin{aligned}\int_{t_0}^{t_1}\int_0^1 \Big(\widetilde{y}\partial_t\widetilde{v}^{\text{aux},l} + \big(\widetilde{y}\widehat{v}+\widehat{y}\widetilde{v}\big)\partial_x\widetilde{v}^{\text{aux},l} - \big(\widetilde{y}\partial_x\widehat{v}+\widehat{y}\partial_x\widetilde{v}\big).\widetilde{v}^{\text{aux},l}\Big) \\ = \int_0^1 \widetilde{v}^{\text{aux},l}(t_1,\cdot)\widetilde{y}(t_1,\cdot) - \int_0^1 \widetilde{v}^{\text{aux},l}(t_0,\cdot)\widetilde{y}(t_0,\cdot).\end{aligned} \tag{62}$$

The boundary term $\int_{t_0}^{t_1}\widetilde{y}_l v_l \widetilde{v}^{\text{aux},l}(\cdot,0)$ is also equal to 0 due to the assumption $[t_0,t_1]\subset\Gamma_l$. Indeed, if $y_l^{1,c}=y_l^{2,c}$ (meaning that the two solutions have the same boundary condition) then $\widetilde{y}_l=0$.

We simplify each term similarly to the proof of Proposition 25.

For $a\colon [0,1]\to\mathbb{R}$ continuous, we use the notation $[a]_0^1$ for

$$[a]_0^1 := a(1)-a(0)$$

- First, let us simplify $\int_{t_0}^{t_1}\int_0^1 \widetilde{y}\partial_t\widetilde{v}^{\text{aux},l}$. To do so, we replace $\widetilde{y}$ by $\big(1-\partial_x^2\big)\widetilde{v}$ using (41a). Then, we integrate by parts:

$$\begin{aligned}\int_{t_0}^{t_1}\int_0^1 \widetilde{y}\partial_t\widetilde{v}^{\text{aux},l} &= \int_{t_0}^{t_1}\int_0^1 \big(1-\partial_x^2\big)\widetilde{v}\partial_t\widetilde{v}^{\text{aux},l} \\ &= \int_{t_0}^{t_1}\int_0^1 \widetilde{v}\partial_t\widetilde{v}^{\text{aux},l} - \int_{t_0}^{t_1}\int_0^1 \widetilde{v}\partial_t\partial_x^2\widetilde{v}^{\text{aux},l} + \int_{t_0}^{t_1}\Big[\widetilde{v}\partial_t\partial_x\widetilde{v}^{\text{aux},l}\Big]_0^1 - \int_{t_0}^{t_1}\Big[\partial_x\widetilde{v}\partial_t\widetilde{v}^{\text{aux},l}\Big]_0^1.\end{aligned}$$

By definition of $\widetilde{v}^{\text{aux},l}$, we have $\left(1-\partial_x^2\right)\widetilde{v}^{\text{aux},l}=0$, which allows us to cancel the first two terms. Moreover, $\widetilde{v}\big|_{x=0}=\widetilde{v}\big|_{x=1}=0$, which allows to forget the third term. For the last term, we use the facts that $\widetilde{v}^{\text{aux},l}\big|_{x=1}=0$, $u_l(0)=\tanh(1)$ and $\partial_x\widetilde{v}^{\text{aux},l}\big|_{x=0}=-\partial_x\widetilde{v}\big|_{x=0}v_l$. This gives

$$\begin{aligned}\int_{t_0}^{t_1}\int_0^1\widetilde{y}\partial_t\widetilde{v}^{\text{aux},l}&=-\int_{t_0}^{t_1}\left[\partial_x\widetilde{v}\partial_t\widetilde{v}^{\text{aux},l}\right]_0^1\\&=\tanh(1)\int_{t_0}^{t_1}\partial_x\widetilde{v}(t,0)\frac{\mathrm{d}}{\mathrm{d}t}\left(\partial_x\widetilde{v}(t,0)v_l(t)\right)\\&=\tanh(1)\Big(\left|\partial_x\widetilde{v}(t_1,0)\right|^2v_l(t_1)-\left|\partial_x\widetilde{v}(t_0,0)\right|^2v_l(t_0)\Big)\\&\qquad-\frac{\tanh(1)}{2}\int_{t_0}^{t_1}v_l(t)\frac{\mathrm{d}}{\mathrm{d}t}\Big(\left|\partial_x\widetilde{v}(t,0)\right|^2\Big).\end{aligned}\tag{63}$$

- For all $t\in[0,T]$, and in particular for $t=t_0$ and $t=t_1$, we simplify $\int_0^1\widetilde{y}(t,\cdot).\widetilde{v}^{\text{aux},l}(t,\cdot)$ similarly:

$$\begin{aligned}\int_0^1\widetilde{y}(t,\cdot)\widetilde{v}^{\text{aux},l}(t,\cdot)&=\int_0^1\left(1-\partial_x^2\right)\widetilde{v}(t,\cdot)\widetilde{v}^{\text{aux},l}(t,\cdot)\\&=\int_0^1\widetilde{v}\,\widetilde{v}^{\text{aux},l}-\int_0^1\widetilde{v}\partial_x^2\widetilde{v}^{\text{aux},l}+\left[\widetilde{v}\partial_x\widetilde{v}^{\text{aux},l}\right]_0^1-\left[\partial_x\widetilde{v}\,\widetilde{v}^{\text{aux},l}\right]_0^1\\&=-\left[\partial_x\widetilde{v}\,\widetilde{v}^{\text{aux},l}\right]_0^1\\&=\tanh(1)\left|\partial_x\widetilde{v}(t,0)\right|^2v_l(t).\end{aligned}\tag{64}$$

- We bound $\int_{t_0}^{t_1}\int_0^1\widehat{y}\widetilde{v}\partial_x\widetilde{v}^{\text{aux},l}$ and $\int_{t_0}^{t_1}\int_0^1\widehat{y}\partial_x\widetilde{v}.\widetilde{v}^{\text{aux},l}$ using the Cauchy–Schwarz inequality:

$$\left|\int_{t_0}^{t_1}\int_0^1\widehat{y}\widetilde{v}\partial_x\widetilde{v}^{\text{aux},l}\right|\le\|\widehat{y}\|_{L^\infty(\Omega_T)}\int_{t_0}^{t_1}\|\widetilde{v}\|_{L^2(0,1)}\left\|\widetilde{v}^{\text{aux},l}\right\|_{H^1(0,1)},\tag{65}$$

$$\left|\int_{t_0}^{t_1}\int_0^1\widehat{y}\partial_x\widetilde{v}.\widetilde{v}^{\text{aux},l}\right|\le\|\widehat{y}\|_{L^\infty(\Omega_T)}\int_{t_0}^{t_1}\|\widetilde{v}\|_{H^1(0,1)}\left\|\widetilde{v}^{\text{aux},l}\right\|_{L^2(0,1)}.\tag{66}$$

We simplify this expression using the fact that for $a,b,c\ge0$, one has $a^2c+b^2c\ge 2abc$:

$$\left\|\widetilde{v}(t,\cdot)\right\|_{L^2(0,1)}\left\|\widetilde{v}^{\text{aux},l}(t,\cdot)\right\|_{H^1(0,1)}\le\left\|\widetilde{v}(t,\cdot)\right\|_{L^2(0,1)}^2v_l(t)+\|u_l\|_{H^1(0,1)}^2\left|\partial_x\widetilde{v}(t,0)\right|^2v_l(t),$$
$$\left\|\widetilde{v}(t,\cdot)\right\|_{H^1(0,1)}\left\|\widetilde{v}^{\text{aux},l}(t,\cdot)\right\|_{L^2(0,1)}\le\left\|\widetilde{v}(t,\cdot)\right\|_{H^1(0,1)}^2v_l(t)+\|u_l\|_{L^2(0,1)}^2\left|\partial_x\widetilde{v}(t,0)\right|^2v_l(t).$$

Therefore:

$$\left|\int_{t_0}^{t_1}\int_0^1\widehat{y}\widetilde{v}\partial_x\widetilde{v}^{\text{aux},l}\right|+\left|\int_{t_0}^{t_1}\int_0^1\widehat{y}\partial_x\widetilde{v}.\widetilde{v}^{\text{aux},l}\right|\le C\int_{t_0}^{t_1}\Big(\left\|\widetilde{v}(t,\cdot)\right\|_{H^1(0,1)}^2+\left|\partial_x\widetilde{v}(t,0)\right|^2\Big)v_l(t)\,\mathrm{d}t.\tag{67}$$

- We simplify the term $\int_{t_0}^{t_1}\int_0^1\widetilde{y}\widehat{v}\partial_x\widetilde{v}^{\text{aux},l}$:

$$\begin{aligned}\int_{t_0}^{t_1}\int_0^1\widetilde{y}\widehat{v}\partial_x\widetilde{v}^{\text{aux},l}&=\int_{t_0}^{t_1}\int_0^1\left(1-\partial_x^2\right)\widetilde{v}\widehat{v}\partial_x\widetilde{v}^{\text{aux},l}\\&=\int_{t_0}^{t_1}\int_0^1\widetilde{v}\widehat{v}\partial_x\widetilde{v}^{\text{aux},l}-\int_{t_0}^{t_1}\int_0^1\partial_x^2\widetilde{v}\widehat{v}\partial_x\widetilde{v}^{\text{aux},l}.\end{aligned}\tag{68}$$

We bound the first term $\int_{t_0}^{t_1}\int_0^1\widetilde{v}\widehat{v}\partial_x\widetilde{v}^{\text{aux},l}$ of the right-hand side of (68) by

$$\begin{aligned}\left|\int_{t_0}^{t_1}\int_0^1\widetilde{v}\widehat{v}\partial_x\widetilde{v}^{\text{aux},l}\right|&\le\|\widehat{v}\|_{L^\infty(\Omega_T)}\int_{t_0}^{t_1}\|\widetilde{v}\|_{L^2(0,1)}\left\|\widetilde{v}^{\text{aux},l}\right\|_{H^1(0,1)}\\&\le C\int_{t_0}^{t_1}\Big(\|\widetilde{v}\|_{H^1(0,1)}^2+\left|\partial_x\widetilde{v}(\cdot,0)\right|^2\Big)v_l.\end{aligned}\tag{69}$$

For the second term $\int_{t_0}^{t_1}\int_0^1 \partial_x^2 \widetilde{v}\widehat{v}\partial_x \widetilde{v}^{\text{aux},l}$ of the right-hand side of (68), we have

$$\int_{t_0}^{t_1}\int_0^1 \partial_x^2 \widetilde{v}\widehat{v}\partial_x \widetilde{v}^{\text{aux},l} = \int_{t_0}^{t_1}\left[\partial_x \widetilde{v}\partial_x \widetilde{v}^{\text{aux},l}\widehat{v}\right]_0^1 - \int_{t_0}^{t_1}\int_0^1 \partial_x \widetilde{v}\partial_x\left(\widehat{v}\partial_x \widetilde{v}^{\text{aux},l}\right). \tag{70}$$

We bound the trilinear term using the fact that $\left\|\widetilde{v}^{\text{aux},l}\right\|_{H^2} = \left|\partial_x \widetilde{v}(\cdot,0)\right| v_l \|u_l\|_{H^2}$.

$$\left|\int_{t_0}^{t_1}\int_0^1 \partial_x \widetilde{v}\partial_x\left(\widehat{v}\partial_x \widetilde{v}^{\text{aux},l}\right)\right| \leq C\int_{t_0}^{t_1}\left(\|\widetilde{v}\|_{H^1(0,1)}^2 + \left|\partial_x \widetilde{v}(\cdot,0)\right|^2\right) v_l. \tag{71}$$

The boundary term $\int_{t_0}^{t_1}\left(\partial_x \widetilde{v}\partial_x \widetilde{v}^{\text{aux},l}\widehat{v}\right)\big|_{x=0} = -\int_{t_0}^{t_1}\left|\partial_x \widetilde{v}(\cdot,0)\right|^2 (v_l)^2$ can be left as is (as it is negative), whereas the term $\int_{t_0}^{t_1}\left(\partial_x \widetilde{v}\partial_x \widetilde{v}^{\text{aux},l}\widehat{v}\right)\big|_{x=1}$ can be bounded through

$$\begin{aligned}\left|\int_{t_0}^{t_1}\left(\partial_x \widetilde{v}\partial_x \widetilde{v}^{\text{aux},l}\widehat{v}\right)\Big|_{x=1}\right| &\leq \int_{t_0}^{t_1}\left(\frac{\tanh(1)}{4}\left|\partial_x \widetilde{v}(\cdot,1)\right|^2 |v_r| v_l + \frac{4}{\tanh(1)}\left|\partial_x \widetilde{v}(\cdot,0)\right|^2\left|\partial_x u_l(1)\right|^2 |v_r| v_l\right)\\ &\leq \int_{t_0}^{t_1}\left(\frac{\tanh(1)}{4}\left|\partial_x \widetilde{v}(\cdot,1)\right|^2 |v_r| v_l + C\left|\partial_x \widetilde{v}(\cdot,0)\right|^2 v_l\right).\end{aligned} \tag{72}$$

Combining (68)–(72), we get

$$\left|\int_{t_0}^{t_1}\int_0^1 \widetilde{y}\widehat{v}\partial_x \widetilde{v}^{\text{aux},l}\right| \leq C\int_{t_0}^{t_1}\left(\|\widetilde{v}\|_{H^1(0,1)}^2 + \left|\partial_x \widetilde{v}(\cdot,0)\right|^2\right) v_l + \frac{\tanh(1)}{4}\int_{t_0}^{t_1}\left|\partial_x \widetilde{v}(\cdot,1)\right|^2 |v_r| v_l. \tag{73}$$

Let us recall here that $\tanh(1) \leq 1$, therefore, if $v_r(t) > 0$, we can absorb the last term of this inequality with the "exiting flux" term of the energy inequality. If $v_r(t) > 0$, the last term is controlled by the second auxiliary inequality.

- We simplify the term $\int_{t_0}^{t_1}\int_0^1 \widetilde{y}\partial_x \widehat{v}.\widetilde{v}^{\text{aux},l}$ of (62):

$$\begin{aligned}\int_{t_0}^{t_1}\int_0^1 \widetilde{y}\widetilde{v}^{\text{aux},l}\partial_x \widehat{v} &= \int_{t_0}^{t_1}\int_0^1 \left(1-\partial_x^2\right)\widetilde{v}\widetilde{v}^{\text{aux},l}\partial_x \widehat{v}\\ &= \int_{t_0}^{t_1}\int_0^1 \widetilde{v}\widetilde{v}^{\text{aux},l}\partial_x \widehat{v} - \int_{t_0}^{t_1}\int_0^1 \partial_x^2 \widetilde{v}\widetilde{v}^{\text{aux},l}\partial_x \widehat{v}.\end{aligned} \tag{74}$$

We bound the first term $\int_{t_0}^{t_1}\int_0^1 \widetilde{v}\widetilde{v}^{\text{aux},l}\partial_x \widehat{v}$ of the right-hand side of (74) by

$$\begin{aligned}\left|\int_{t_0}^{t_1}\int_0^1 \widetilde{v}\widetilde{v}^{\text{aux},l}\partial_x \widehat{v}\right| &\leq \|\widehat{v}\|_{L^\infty([0,T],W^{1,\infty}(0,1))}\int_{t_0}^{t_1}\left(\|\widetilde{v}\|_{L^2(0,1)}\left\|\widetilde{v}^{\text{aux},l}\right\|_{L^2(0,1)}\right)\\ &\leq C\int_{t_0}^{t_1}\left(\|\widetilde{v}\|_{H^1(0,1)}^2 + \left|\partial_x \widetilde{v}(t\cdot,0)\right|^2\right) v_l.\end{aligned} \tag{75}$$

For the second term $\int_{t_0}^{t_1}\int_0^1 \partial_x^2 \widetilde{v}\widetilde{v}^{\text{aux},l}\partial_x \widehat{v}$ of the right-hand side of (74), we have:

$$\int_{t_0}^{t_1}\int_0^1 \partial_x^2 \widetilde{v}\widetilde{v}^{\text{aux},l}\partial_x \widehat{v} = \int_{t_0}^{t_1}\left[\partial_x \widetilde{v}\widetilde{v}^{\text{aux},l}\partial_x \widehat{v}\right]_0^1 - \int_{t_0}^{t_1}\int_0^1 \partial_x \widetilde{v}\partial_x\left(\widetilde{v}^{\text{aux},l}\partial_x \widehat{v}\right). \tag{76}$$

Once again, we bound the trilinear term by using $\left\|\widetilde{v}^{\text{aux},l}\right\|_{H^2} = \left|\partial_x \widetilde{v}(\cdot,0)\right| v_l \|u_l\|_{H^2}$:

$$\left|\int_{t_0}^{t_1}\int_0^1 \partial_x \widetilde{v}\partial_x\left(\widetilde{v}^{\text{aux},l}\partial_x \widehat{v}\right)\right| \leq C\int_{t_0}^{t_1}\left(\|\widetilde{v}\|_{H^1(0,1)}^2 + \left|\partial_x \widetilde{v}(\cdot,0)\right|^2\right) v_l. \tag{77}$$

The boundary term $\int_{t_0}^{t_1}\left(\partial_x \widetilde{v}\widetilde{v}^{\text{aux},l}\partial_x \widehat{v}\right)\big|_{x=1}$ is equal to 0 as $\widetilde{v}^{\text{aux},l}\big|_{x=1} = 0$ by definition. The term $\int_{t_0}^{t_1}\left(\partial_x \widetilde{v}\widetilde{v}^{\text{aux},l}\partial_x \widehat{v}\right)\big|_{x=0}$ can be bounded through

$$\left|\int_{t_0}^{t_1}\left(\partial_x \widetilde{v}\widetilde{v}^{\text{aux},l}\partial_x \widehat{v}\right)\Big|_{x=0}\right| \leq C\int_{t_0}^{t_1}\left|\partial_x \widetilde{v}(\cdot,0)\right|^2 v_l. \tag{78}$$

Combining (74)–(78), we get

$$\left|\int_{t_0}^{t_1}\int_0^1 \widetilde{y}\widehat{v}\partial_x \widetilde{v}^{\text{aux},l}\right| \leq C\int_{t_0}^{t_1}\left(\|\widetilde{v}\|_{H^1(0,1)}^2 + \left|\partial_x \widetilde{v}(\cdot,0)\right|^2\right) v_l. \tag{79}$$

Combining all the estimates (63), (64), (67), (73) and (79) for all the terms of (62), we get that there exists a constant $C > 0$ independent of t_0, t_1 such that:

$$\int_{t_0}^{t_1} v_l \frac{\mathrm{d}}{\mathrm{d}t}\left(\left|\partial_x \widetilde{v}(\cdot,0)\right|^2\right) \leq C \int_{t_0}^{t_1} \left(\|\widetilde{v}\|^2_{H^1(0,1)} + \left|\partial_x \widetilde{v}(\cdot,0)\right|^2\right) v_l + \frac{1}{2}\int_{t_0}^{t_1} \left|\partial_x \widetilde{v}(\cdot,1)\right|^2 |v_r| v_l. \tag{80}$$

Using Lemma 27, we get that $\left|\partial_x \widetilde{v}(\cdot,0)\right|^2 \in W^{1,\infty}(0,T)$. Therefore the function

$$U\colon t \longmapsto \int_0^t v_l(s) \frac{\mathrm{d}}{\mathrm{d}t}\left(\left|\partial_x \widetilde{v}(s,0)\right|^2\right) \mathrm{d}s$$

is also $W^{1,\infty}$ and its derivative in the weak sense is equal to

$$t \longmapsto v_l(t) \frac{\mathrm{d}}{\mathrm{d}t}\left(\left|\partial_x \widetilde{v}(t,0)\right|^2\right),$$

which is in L^∞. By Lebesgue Point Theorem, U is differentiable in the classical sense for almost every $t \in]0,T[$. For such a t, and for $\varepsilon > 0$, we can take $t_0 = t$ and $t_1 = t+\varepsilon$ in (80), which we do. When ε goes to zero, every term converges, and we get

$$v_l \frac{\mathrm{d}}{\mathrm{d}t}\left(\left|\partial_x \widetilde{v}(\cdot,0)\right|^2\right) \leq C\left(\|\widetilde{v}\|^2_{H^1(0,1)} + \left|\partial_x \widetilde{v}(\cdot,0)\right|^2\right) v_l + \frac{1}{2}\left|\partial_x \widetilde{v}(\cdot,1)\right|^2 |v_r| v_l. \tag{81}$$

We can divide by v_l whenever we are in Γ_l, which gives the inequality (58), as wanted.

The proof of inequality (59) is similar at each step, except we use the test function

$$\widetilde{v}^{\text{aux},r} := \partial_x \widetilde{v}(t,1)\, v_r^-(t)\, u_r(x) \tag{82}$$

instead of $\widetilde{v}^{\text{aux},l}$. You can also get (59) using (58) by symmetry with the change of variable $x \mapsto 1-x$. □

Remark 29. If we are ready to increase the constant C in front of $\left(\|\widetilde{v}(t,\cdot)\|^2_{H^1(0,1)} + |\partial_x \widetilde{v}(t,1)|^2\right)$ in (58), then we could change the constants in front of $|\partial_x \widetilde{v}(t,1)|^2 |v_r(t)|$ and $|\widetilde{y}_l|^2 v_l$ in this inequality.

2.3. *Gronwall argument and end of the proof*

We define the functions E, E_l and E_r by

$$E(t) := \left\|\widetilde{v}(t,\cdot)\right\|^2_{H^1(0,1)}, \quad E_l(t) := \left|\partial_x \widetilde{v}(t,0)\right|^2, \quad E_r(t) := \left|\partial_x \widetilde{v}(t,1)\right|^2. \tag{83}$$

By Lemmas 24 and 27, we know that E is well-defined and Lipschitz. Moreover, in the case where the boundary conditions for y^1 and y^2 are the same,

$$y_l^{1,c} = y_l^{2,c} \quad \text{and} \quad y_r^{1,c} = y_r^{2,c}, \tag{84}$$

we can combine (52), (58) and (59), to get

$$E' + E_l' + E_r' \leq C(E + E_l + E_r) \qquad \text{on } \Gamma_l \cap \Gamma_r, \tag{85a}$$
$$E' + E_l' + \tfrac{1}{2} E_r |v_r| \leq C(E + E_l) \qquad \text{on } \Gamma_l \setminus (\Gamma_l \cap \Gamma_r), \tag{85b}$$
$$E' + E_r' + \tfrac{1}{2} E_l |v_l| \leq C(E + E_r) \qquad \text{on } \Gamma_r \setminus (\Gamma_l \cap \Gamma_r), \tag{85c}$$
$$E' + \tfrac{1}{2} E_l |v_l| + \tfrac{1}{2} E_r |v_r| \leq CE \qquad \text{on } [0,T] \setminus (\Gamma_l \cup \Gamma_r). \tag{85d}$$

Therefore, we can use the Gronwall inequality to get uniqueness on each time interval where neither v_l nor v_r changes sign. On such an interval $I = [T_0, T_1]$, one gets:

$$(E + E_l + E_r)(T_1) \leq \exp\big(C(T_1 - T_0)\big)\,(E + E_l + E_r)(T_0) \qquad \text{if } I \subset \Gamma_l \cap \Gamma_r, \tag{86a}$$
$$(E + E_l)(T_1) \leq \exp\big(C(T_1 - T_0)\big)\,(E + E_l)(T_0) \qquad \text{if } I \subset \Gamma_l \setminus (\Gamma_l \cap \Gamma_r), \tag{86b}$$
$$(E + E_r)(T_1) \leq \exp\big(C(T_1 - T_0)\big)\,(E + E_r)(T_0) \qquad \text{if } I \subset \Gamma_r \setminus (\Gamma_l \cap \Gamma_r), \tag{86c}$$
$$E(T_1) \leq \exp\big(C(T_1 - T_0)\big)\,E(T_0) \qquad \text{if } I \subset [0,T] \setminus (\Gamma_l \cup \Gamma_r). \tag{86d}$$

This implies that

$$E(T_1) \le \exp\big(C(T_1 - T_0)\big)\,(E + E_l + E_r)(T_0). \tag{87}$$

This concludes the proof of the first part of Theorem 6.

Now let us assume that $y_0^1 = y_0^2$. Let us denote by $T_0 < T_1 < \cdots < T_n < T$ the times where v_l or v_r changes sign. On $[0, T_0]$, and on each interval $[T_i, T_{i+1}]$, one has the estimate (87). By induction, we obtain that v is equal to zero for each T_i and on $[0, T]$.

Remark 30. Gronwall argument normally provides stability estimates. However, in our case, if initial data are non-zero, they could degenerate.

For example, take $T = 1$ and v_r and v_l given by $v_l(t) = -1 + t$ and $v_r(t) = 1$. For the sake of simplicity, we assume that $C = 1$ here. Then the functions E, E_l and E_r defined by $E(t) := \frac{e^t+1}{2}$, $E_l(t) := \frac{1}{2(1-t)}$ and $E_r(t) := 0$ verify the system (85) on $[0,1]$. But E_l is going to infinity so we cannot continue estimates on E after $t = 1$.

This phenomenon cannot happen in the case of an initial data equal to zero, because in this case, $E_l = E_r = 0$ for all t.

Remark 31. The aforementioned constant C does depend on $\|y^1\|_{L^\infty}$ and $\|y^2\|_{L^\infty}$. Due to this, our estimates cannot be used to prove the existence or uniqueness of a lower class of regularity than the one we use.

Remark 32. If the boundary conditions $(y_l^{1,c}, y_r^{1,c})$ and $(y_l^{2,c}, y_r^{2,c})$ are not the same, one still gets an *a priori* estimate. However, due to Remark 30, one can see that this estimate no longer provides uniqueness in the case where v_l or v_r changes sign.

A question that is still open is to determine whether or not one could still get estimates if the two solutions we are comparing do not have the same boundary fluxes v_l and v_r.

3. Proof of Theorem 19

3.1. *Energy estimate*

Let us take two weak solutions (y^1, y_r^1, y_l^1) and (y^2, y_r^2, y_l^2) of the transport-elliptic system associated with A_n with initial and boundary conditions $(y_0^1, \mathbf{v_l}, \mathbf{v_r}, y_r^{1,c}, y_l^{1,c})$ and $(y_0^2, \mathbf{v_l}, \mathbf{v_r}, y_r^{2,c}, y_l^{2,c})$. We define the following functions

$$\widetilde{y} := y^1 - y^2, \qquad \widetilde{v} := v^1 - v^2, \tag{88}$$

$$\widehat{y} := \frac{y^1 + y^2}{2}, \qquad \widehat{v} := \frac{v^1 + v^2}{2}, \tag{89}$$

$$\widetilde{y}_l := y_l^1 - y_l^2, \qquad \widetilde{y}_r := y_r^1 - y_r^2, \tag{90}$$

where the functions v^1 and v^2 are given through (21).

Remark 33. Let us remark here that we have

$$\big(v^1 - v^2\big)\big|_{x=0} = \big(v^1 - v^2\big)\big|_{x=1} = 0, \ldots, \partial_x^{n-1}\big(v^1 - v^2\big)\big|_{x=0} = \partial_x^{n-1}\big(v^1 - v^2\big)\big|_{x=1} = 0,$$

and $\left(\left(\frac{v^1+v^2}{2}\right)\Big|_{x=0}, \ldots, \partial_x^{n-1}\left(\frac{v^1+v^2}{2}\right)\Big|_{x=0}\right) = \mathbf{v_l}$ as well as $\left(\left(\frac{v^1+v^2}{2}\right)\Big|_{x=1}, \ldots, \partial_x^{n-1}\left(\frac{v^1+v^2}{2}\right)\Big|_{x=1}\right) = \mathbf{v_r}$. Let us also remark that $\mathrm{A}_n\,\widetilde{v} = \widetilde{y}$ and $\mathrm{A}_n\,\widehat{v} = \widehat{y}$.

We take the difference of Equation (34) for the solutions y^1 and y^2. The function $\widetilde{y}$ verifies the following equality for all $0 \le t_0 \le t_1 < T$ and for all test functions $\phi \in H^1\big([t_0, t_1] \times [0,1]\big)$:

$$\int_{t_0}^{t_1}\int_0^1 \Big(\widetilde{y}\,\partial_t\phi + \big(\widetilde{y}\,\widehat{v} + \widehat{y}\,\widetilde{v}\big)\,\partial_x\phi - \big(\widetilde{y}\,\partial_x\widehat{v} + \widehat{y}\,\partial_x\widetilde{v}\big)\,\widetilde{v}\Big)$$
$$= \int_{t_0}^{t_1} \widetilde{y}_r\,v_r\,\phi(\cdot,1) - \int_{t_0}^{t_1} \widetilde{y}_l\,v_l\,\phi(\cdot,0) + \int_0^1 \phi(t_1,\cdot)\,\widetilde{y}(t_1,\cdot) - \int_0^1 \phi(t_0,\cdot)\,\widetilde{y}(t_0,\cdot). \tag{91}$$

The following Lemma is the generalization of Lemma 24, and its proof is similar.

Lemma 34. *The functions $\widehat{v}$ and $\widetilde{v}$ lie in $L^\infty\big([0,T],W^{2n,\infty}(0,1)\big)$. Moreover the function $\widetilde{v}$ lies in $W^{1,\infty}\big([0,T],H^n([0,1])\big)$.*

Proof. For the regularity in space of $\widetilde{v}$ and $\widehat{v}$, write

$$(-1)^n\partial_x^{2n}\widetilde{v}=-\sum_{k=0}^{n-1}(-\partial_x^2)^k\widetilde{v}+\widetilde{y}. \tag{92}$$

We can conclude using Lemma 16.

Let us now prove the regularity in time of $\widetilde{v}$. Let us fix two times $t_0<t_1$, and denote

$$a_{t_0}^{t_1}(x):=\widetilde{v}(t_1,x)-\widetilde{v}(t_0,x).$$

Recalling that $\widetilde{v}$ verifies (21) with homogeneous boundary conditions, we obtain that for every function $g\in H_0^n(0,1)$

$$\int_0^1\big(\widetilde{y}(t_1,\cdot)-\widetilde{y}(t_0,\cdot)\big)g=\int_0^1\mathbf{A}_n^{\frac12}a_{t_0}^{t_1}\cdot\mathbf{A}_n^{\frac12}g. \tag{93}$$

We apply it with $a_{t_0}^{t_1}$ instead of g:

$$\int_0^1\big(\widetilde{y}(t_1,\cdot)-\widetilde{y}(t_0,\cdot)\big)a_{t_0}^{t_1}=\int_0^1\big|\mathbf{A}_n^{\frac12}a_{t_0}^{t_1}\big|^2. \tag{94}$$

Hence, we get the inequality

$$\|a_{t_0}^{t_1}\|_{H^n(0,1)}^2\le\left|\int_0^1\big(\widetilde{y}(t_1,\cdot)-\widetilde{y}(t_0,\cdot)\big)a_{t_0}^{t_1}\right|. \tag{95}$$

Using (34) with $a_{t_0}^{t_1}$ instead of ϕ (considered as a function constant in time), we obtain that:

$$\int_0^1\big(\widetilde{y}(t_1,\cdot)-\widetilde{y}(t_0,\cdot)\big)a_{t_0}^{t_1}=\int_{t_0}^{t_1}\int_0^1\Big(\big(\widetilde{y}\widehat{v}+\widehat{y}\widetilde{v}\big)\partial_x a_{t_0}^{t_1}-\big(\widetilde{y}\partial_x\widehat{v}+\widehat{y}\partial_x\widetilde{v}\big)a_{t_0}^{t_1}\Big). \tag{96}$$

Combining (95) and (96), using that $n\ge1$, we get that

$$\|a_{t_0}^{t_1}\|_{H^n(0,1)}\le|t_1-t_0|\big(\|\widetilde{y}\|_{L^\infty([0,T]\times[0,1])}\|\widehat{v}\|_{L^\infty([0,T],H^1(0,1))}+\|\widehat{y}\|_{L^\infty([0,T]\times[0,1])}\|\widetilde{v}\|_{L^\infty([0,T],H^1(0,1))}\big).$$

Recalling that $a_{t_0}^{t_1}(x)=\widetilde{v}(t_1,x)-\widetilde{v}(t_0,x)$, we conclude that $\widetilde{v}\in W^{1,\infty}\big([0,T],H^n([0,1])\big)$. □

Remark 35. Lemma 34 expresses the fact that $\partial_t\mathrm{A}_n\widetilde{v}=\partial_x\big(\widetilde{v}\widehat{y}+\widehat{v}\widetilde{y}\big)$, which is in $L_t^\infty H_x^{-1}$. By elliptic regularity, we could obtain a higher regularity for $\partial_t\widetilde{v}$, but it is not needed here.

Proposition 36. *There exists a constant $C>0$ such that the following inequality holds for almost every $t\in[0,T]$:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\widetilde{v}\|_{H^n(0,1)}^2\Big)+\big|\partial_x^n\widetilde{v}(\cdot,1)\big|^2v_r-\big|\partial_x^n\widetilde{v}(\cdot,0)\big|^2v_l\le C\|\widehat{v}\|_{W^{2n,\infty}(0,1)}\|\widetilde{v}\|_{H^n(0,1)}^2. \tag{97}$$

Proof. Using Lemma 34, we can take $\widetilde{v}$ as a test function in (91), which we do, giving us

$$\int_{t_0}^{t_1}\int_0^1\Big(\widetilde{y}\partial_t\widetilde{v}+\big(\widetilde{y}\widehat{v}+\widehat{y}\widetilde{v}\big)\partial_x\widetilde{v}-\big(\widetilde{y}\partial_x\widehat{v}+\widehat{y}\partial_x\widetilde{v}\big)\widetilde{v}\Big)=\int_0^1\widetilde{v}(t_1,\cdot)\widetilde{y}(t_1,\cdot)-\int_0^1\widetilde{v}(t_0,\cdot)\widetilde{y}(t_0,\cdot). \tag{98}$$

Then, we simplify each term.

- The term $\int_{t_0}^{t_1}\int_0^1\widetilde{y}\partial_t\widetilde{v}$ can be treated, using Remark 33 and Lemma 42, as follows:

$$\begin{aligned}\int_{t_0}^{t_1}\int_0^1\widetilde{y}\partial_t\widetilde{v}&=\int_{t_0}^{t_1}\int_0^1\mathrm{A}_n\widetilde{v}\partial_t\widetilde{v}\\&=\int_{t_0}^{t_1}\int_0^1\mathbf{A}_n^{\frac12}\widetilde{v}\cdot\partial_t\mathbf{A}_n^{\frac12}\widetilde{v}\\&=\frac12\Big[\|\mathbf{A}_n^{\frac12}\widetilde{v}\|_{L^2}^2\Big]_{t_0}^{t_1}.\end{aligned} \tag{99}$$

- Similarly, we get

$$\int_0^1 \widetilde{y}\widetilde{v} = \left\|\mathbf{A}_n^{\frac{1}{2}}\,\widetilde{v}\right\|_{L^2}^2. \tag{100}$$

- The trilinear term $\int_0^1 \widehat{y}\widetilde{v}\partial_x\widetilde{v}$ cancels with $\int_0^1 \widehat{y}\partial_x\widetilde{v}\,\widetilde{v}$.
- To simplify the trilinear term $\int_0^1 \widetilde{y}\widehat{v}\partial_x\widetilde{v}$, we first use Lemma 42 and Remark 33:

$$\begin{aligned}\int_0^1 \widetilde{y}\widehat{v}\partial_x\widetilde{v} &= \int_0^1 \mathrm{A}_n\,\widetilde{v}\widehat{v}\partial_x\widetilde{v}\\ &= \int_0^1 \mathbf{A}_n^{\frac{1}{2}}\,\widetilde{v}\cdot\mathbf{A}_n^{\frac{1}{2}}(\widehat{v}\partial_x\widetilde{v}) - \left[\widehat{v}|\partial_x^n\widetilde{v}|^2\right]_0^1.\end{aligned} \tag{101}$$

  Then we put all the derivatives on $\widetilde{v}$, which can be done by introducing commutators

$$\int_0^1 \mathbf{A}_n^{\frac{1}{2}}\,\widetilde{v}\cdot\mathbf{A}_n^{\frac{1}{2}}(\widehat{v}\partial_x\widetilde{v}) = \int_0^1 \mathbf{A}_n^{\frac{1}{2}}\,\widetilde{v}\cdot\widehat{v}\partial_x\mathbf{A}_n^{\frac{1}{2}}\,\widetilde{v} + \int_0^1 \mathbf{A}_n^{\frac{1}{2}}\,\widetilde{v}\cdot\left[\widehat{v},\mathbf{A}_n^{\frac{1}{2}}\right]\partial_x\widetilde{v}. \tag{102}$$

  Finally, we integrate by part to once again put all derivatives on $\widetilde{v}$

$$\begin{aligned}\int_0^1 \mathbf{A}_n^{\frac{1}{2}}\,\widetilde{v}\cdot\widehat{v}\partial_x\mathbf{A}_n^{\frac{1}{2}}\,\widetilde{v} &= \frac{1}{2}\int_0^1 \widehat{v}\partial_x\left|\mathbf{A}_n^{\frac{1}{2}}\,\widetilde{v}\right|^2\\ &= -\frac{1}{2}\int_0^1 \left|\mathbf{A}_n^{\frac{1}{2}}\,\widetilde{v}\right|^2\partial_x\widehat{v} + \frac{1}{2}\left[\widehat{v}|\partial_x^n\widetilde{v}|^2\right]_0^1.\end{aligned} \tag{103}$$

  Combining (101), (102) and (103), we get

$$\int_0^1 \widetilde{y}\widehat{v}\partial_x\widetilde{v} = -\frac{1}{2}\int_0^1 \left|\mathbf{A}_n^{\frac{1}{2}}\,\widetilde{v}\right|^2\partial_x\widehat{v} + \int_0^1 \mathbf{A}_n^{\frac{1}{2}}\,\widetilde{v}\cdot\left[\widehat{v},\mathbf{A}_n^{\frac{1}{2}}\right]\partial_x\widetilde{v} - \frac{1}{2}\left[\widehat{v}|\partial_x^n\widetilde{v}|^2\right]_0^1. \tag{104}$$

- To simplify the trilinear term $\int_0^1 \widetilde{y}\partial_x\widehat{v}\widetilde{v}$, we use Lemma 42 and Remark 33

$$\begin{aligned}\int_0^1 \widetilde{y}\partial_x\widehat{v}\widetilde{v} &= \int_0^1 \mathrm{A}_n\,\widetilde{v}\partial_x\widehat{v}\widetilde{v}\\ &= \int_0^1 \mathbf{A}_n^{\frac{1}{2}}\,\widetilde{v}\cdot\mathbf{A}_n^{\frac{1}{2}}(\partial_x\widehat{v}\widetilde{v}).\end{aligned} \tag{105}$$

We substitute (99), (100), (104) and (105) into (98) to get

$$\begin{aligned}&\left[\left\|\mathbf{A}_n^{\frac{1}{2}}\,\widetilde{v}\right\|_{L^2}^2\right]_{t_0}^{t_1} + \int_{t_0}^{t_1}\left[\widehat{v}|\partial_x^n\widetilde{v}|^2\right]_0^1\\ &\quad + \int_{t_0}^{t_1}\int_0^1\left(-2\mathbf{A}_n^{\frac{1}{2}}\,\widetilde{v}\cdot\mathbf{A}_n^{\frac{1}{2}}(\partial_x\widehat{v}\widetilde{v}) + \left|\mathbf{A}_n^{\frac{1}{2}}\,\widetilde{v}\right|^2\partial_x\widehat{v} - 2\mathbf{A}_n^{\frac{1}{2}}\,\widetilde{v}\cdot\left[\widetilde{v},\mathbf{A}_n^{\frac{1}{2}}\right]\partial_x\widetilde{v}\right) = 0.\end{aligned} \tag{106}$$

Using Lemma 43, we get that for all $0 < t_0 < t_1 < T$, one has

$$\left[\|\widetilde{v}\|_{H^n}^2\right]_{t_0}^{t_1} + \int_{t_0}^{t_1}\left|\partial_x^n\widetilde{v}(\cdot,1)\right|^2 v_r - \int_{t_0}^{t_1}\left|\partial_x^n\widetilde{v}(\cdot,0)\right|^2 v_l \le C\int_{t_0}^{t_1}\int_0^1 \|\widehat{v}\|_{W^{2n,\infty}(0,1)}\|\widetilde{v}\|_{H^n(0,1)}^2.$$ □

3.2. *Auxiliary estimate*

In this paragraph, we choose $I = [T_0, T_1] \subset [0, T]$ an interval such that v_l and v_r do not change sign on I. Without loss of generality, we assume that:

$$\forall\ t \in I, v_l(t) > 0 \text{ and } v_r(t) > 0. \tag{107}$$

The case where $v_l, v_r < 0$ can be done by symmetry. The case $v_l < 0$ and $v_r > 0$ can be done with the energy estimates alone. The case $v_l > 0$ and $v_r < 0$ can be done combining the auxiliary inequalities from the cases $v_l, v_r < 0$ and $v_l, v_r > 0$.

We construct the auxiliary test function $\widetilde{v}_n^{\text{aux},l}$, which is a generalization of $\widetilde{v}^{\text{aux},l}$. It is the solution to the following elliptic problem:

$$\mathrm{A}_n\, \widetilde{v}_n^{\text{aux},l} = 0, \tag{108a}$$

$$\forall\, i \in [[0, n-1]], \quad \mathscr{S}_i(\widetilde{v}_n^{\text{aux},l})(\cdot, 1) = 0, \tag{108b}$$

$$\forall\, i \in [[0, n-1]], \quad \mathscr{B}_i(\widetilde{v}_n^{\text{aux},l})(\cdot, 0) = -\mathscr{B}_i(\widetilde{v})(\cdot, 0), \tag{108c}$$

where the operators $\mathscr{B}_i$ where defined in Appendix A through (150a). This elliptic problem is a generalization of (54).

Let us introduce the space $H^n_{0,r}(0,1)$ as the closure of $C_c^\infty([0,1))$ for the H^n-norm:

$$H^n_{0,r}(0,1) = \left\{g \in H^n(0,1);\ \forall\, i \in [[0, n-1]],\ \partial_x^i g(1) = 0\right\}. \tag{109}$$

It is the natural space to define $\widetilde{v}_n^{\text{aux},l}$ as it is a solution to a Zaremba problem (Dirichlet on one side and Neumann on the other).

Lemma 37. *The function $\widetilde{v}_n^{\text{aux},l}$ exists and is unique in $L^\infty\big(I, H^n_{0,r}(0,1)\big)$ as the solution of the following variational problem:*

$$\forall\, g \in H^n_{0,r}(0,1),\ \forall^{\text{a.e.}}\ t \in I, \quad \int_0^1 \mathbf{A}_n^{\frac{1}{2}}\, \widetilde{v}_n^{\text{aux},l}(t,\cdot) \cdot \mathbf{A}_n^{\frac{1}{2}}\, g = -\sum_{i=0}^{n-1} \mathscr{B}_i(\widetilde{v})(t,0)\, \mathscr{S}_i(g)(0). \tag{110}$$

Moreover, the function $\widetilde{v}_n^{\text{aux},l}$ lies in $L^\infty\big(I, W^{2n,\infty}(0,1)\big)$.

Proof. Since $\widetilde{v}$ belongs to $L^\infty\big(I, H^{2n}(0,1)\big)$, for each i, $t \mapsto \mathscr{B}_i(\widetilde{v})(t,0)$ belongs to $L^\infty(I)$. Hence by Lax–Milgram, $\widetilde{v}_n^{\text{aux},l}$ belongs to $L^\infty\big(I, H^n_{0,r}(0,1)\big)$ and is the unique solution of problem (110) in this space. Using Lemma 16, one gets that $\widetilde{v}_n^{\text{aux},l}$ belongs to $L^\infty\big(I, W^{2n,\infty}(0,1)\big)$. □

Let $g \in H^n(0,1)$ be a function. Using Lemma 42, with $\widetilde{v}_n^{\text{aux},l}$ instead of f and g instead of g one has

$$\int_0^1 \mathrm{A}_n\, \widetilde{v}_n^{\text{aux},l}\, g = \int_0^1 \mathbf{A}_n^{\frac{1}{2}}\, \widetilde{v}_n^{\text{aux},l} \cdot \mathbf{A}_n^{\frac{1}{2}}\, g + \sum_{i=0}^{n-1} \left[\mathscr{B}_i(\widetilde{v}_n^{\text{aux},l}) \mathscr{S}_i(g)\right]_0^1. \tag{111}$$

Now due to (108a), one has $\int_0^1 \mathrm{A}_n\, \widetilde{v}_n^{\text{aux},l}\, g = 0$ and due to (108c), one has

$$\sum_{i=0}^{n-1} \mathscr{B}_i(\widetilde{v}_n^{\text{aux},l})(\cdot,0) \mathscr{S}_i(g)(0) = -\sum_{i=0}^{n-1} \mathscr{B}_i(\widetilde{v})(\cdot,0) \mathscr{S}_i(g)(0). \tag{112}$$

Hence,

$$\int_0^1 \mathbf{A}_n^{\frac{1}{2}}\, \widetilde{v}_n^{\text{aux},l} \cdot \mathbf{A}_n^{\frac{1}{2}}\, g = -\sum_{i=0}^{n-1} \mathscr{B}_i(\widetilde{v})(\cdot,0) \mathscr{S}_i(g)(0). \tag{113}$$

In particular, for $g \in H^n_0(0,1)$, one has

$$\int_0^1 \mathbf{A}_n^{\frac{1}{2}}\, \widetilde{v}_n^{\text{aux},l}(t,\cdot) \cdot \mathbf{A}_n^{\frac{1}{2}}\, g = 0. \tag{114}$$

As this will be useful later, let us remark that for every $g \in H^n_{0,r}(0,1)$, one has

$$\int_0^1 \mathbf{A}_n^{\frac{1}{2}}\, \partial_x \widetilde{v}_n^{\text{aux},l}(t,\cdot) \cdot \mathbf{A}_n^{\frac{1}{2}}\, g = \sum_{i=0}^{n-1} \mathscr{B}_i(\partial_x \widetilde{v})(t,0)\, \mathscr{S}_i(g)(0). \tag{115}$$

Similarly to the case of the classical Camassa–Holm equation, the introduction of this auxiliary test function is for the purpose of an auxiliary inequality. The purpose of the auxiliary inequality (116) is to control the entering energy fluxes.

Proposition 38. *For almost every $t \in I$, we have the inequality*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(c^2_{H^n}) + \left|\partial_x^n \widetilde{v}(t,0)\right|^2 v_l(t) \le C\Big(\big\|\widetilde{v}_n^{\text{aux},l}\big\|^2_{H^n} + \|\widetilde{v}\|^2_{H^n}\Big) + \frac{1}{4}\left|\partial_x^n \widetilde{v}(t,1)\right|^2 |v_r| + |\widetilde{y}_l|^2. \tag{116}$$

Remark 39. If (107) is not verified, one can similarly introduce the function $\widetilde{v}_n^{\mathrm{aux},r}$ as

$$\mathrm{A}_n\,\widetilde{v}_n^{\mathrm{aux},r} = 0, \tag{117a}$$

$$\forall\, i \in [\![0, n-1]\!], \quad \mathscr{B}_i(\widetilde{v}_n^{\mathrm{aux},r})(\cdot,1) = -\mathscr{B}_i(\widetilde{v})(\cdot,1), \tag{117b}$$

$$\forall\, i \in [\![0, n-1]\!], \quad \mathscr{S}_i(\widetilde{v}_n^{\mathrm{aux},r})(\cdot,0) = 0, \tag{117c}$$

and get the inequality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\big\|\widetilde{v}_n^{\mathrm{aux},r}\big\|_{H^n}^2\Big) + \big|\partial_x^n\widetilde{v}(t,1)\big|^2 v_r^+(t) \le C\Big(\big\|\widetilde{v}_n^{\mathrm{aux},r}\big\|_{H^n}^2 + \|\widetilde{v}\|_{H^n}^2\Big) + \frac{1}{4}\big|\partial_x^n\widetilde{v}(t,0)\big|^2|v_l| + |\widetilde{y}_r|^2. \tag{118}$$

In order to prove Proposition 38, let us prove Lemma 40 and Proposition 41. Lemma 40 states that the auxiliary function $\widetilde{v}_n^{\mathrm{aux},l}$ is regular enough to be used as a test function in (91). Proposition 41 is an inequality similar to the classical Rellich estimate on the normal and tangential derivatives of harmonic functions, see for example [18]. We will use Proposition 41 to control one of the boundary terms on the outgoing boundaries.

Lemma 40. *The function $\widetilde{v}_n^{\mathrm{aux},l}$ lies in $W^{1,\infty}\big(I, H^n(0,1)\big)$.*

Proof. Let us prove the regularity in time of the function $\widetilde{v}_n^{\mathrm{aux},l}$. We call V the function

$$V := \widetilde{v} + \widetilde{v}_n^{\mathrm{aux},l}. \tag{119}$$

Since, by Lemma 34, $\widetilde{v}$ already belongs to $W^{1,\infty}\big(I, H^n([0,1])\big)$, it is sufficient to prove that V belongs to that space as well. Moreover, using Lemma 42 and Remark 33, we get that for all $g \in H_{0,r}^n$ and for almost every $t \in I$, one has

$$\int_0^1 \mathbf{A}_n^{\frac{1}{2}} V(t,\cdot)\cdot\mathbf{A}_n^{\frac{1}{2}} g = \int_0^1 \widetilde{y}(t,\cdot)\,g. \tag{120}$$

Moreover, by taking $g \in H_{0,r}^n(0,1)$ as a constant-in-time test function in (91), one gets for almost every $T_0 \le t_0 < t_1 \le T_1$:

$$\int_0^1 \big(\widetilde{y}(t_1,\cdot) - \widetilde{y}(t_0,\cdot)\big)g = \int_{t_0}^{t_1}\int_0^1 \Big(\big(\widetilde{y}\widehat{v} + \widehat{y}\widetilde{v}\big)\partial_x g - \big(\widetilde{y}\,\partial_x\widehat{v} + \widehat{y}\,\partial_x\widetilde{v}\big)g\Big). \tag{121}$$

Let us remark that there is no boundary term because $g(1) = 0$ and $\forall^{\text{a.e.}}\ t \in I$, $\widetilde{y}_l(t,0) = 0$. We apply (120) and (121) with $V(t_1,\cdot) - V(t_0,\cdot)$ instead of g and since $n \ge 1$ we get

$$\big\|V(t_1,\cdot) - V(t_0,\cdot)\big\|_{H^n} \le \big|t_1 - t_0\big|\big(\|\widehat{y}\|_{L^\infty}\|\widetilde{v}\|_{H^1} + \|\widetilde{y}\|_{L^\infty}\|\widehat{v}\|_{H^1}\big). \qquad \square$$

Proposition 41. *There exists a constant $C > 0$ such that for every $t \in I$*

$$\big|\partial_x^n\widetilde{v}_n^{\mathrm{aux},l}(t,1)\big| \le C\big\|\widetilde{v}_n^{\mathrm{aux},l}(t,\cdot)\big\|_{H^n(0,1)}. \tag{122}$$

Proof. Let $\chi \in C^\infty(0,1)$ be a function equal to zero in a neighborhood of 0 and equal to 1 in a neighborhood of 1. We use Lemma 44 with $\widetilde{v}_n^{\mathrm{aux},l}$ instead of f and of g, and χ instead of w.

$$\int_0^1 \Big[\partial_x(\chi\cdot),\mathbf{A}_n^{\frac{1}{2}}\Big](\widetilde{v}_n^{\mathrm{aux},l})\cdot\mathbf{A}_n^{\frac{1}{2}}(\widetilde{v}_n^{\mathrm{aux},l}) + \int_0^1 \mathbf{A}_n^{\frac{1}{2}}(\widetilde{v}_n^{\mathrm{aux},l})\cdot\Big[\chi\,\partial_x,\mathbf{A}_n^{\frac{1}{2}}\Big](\widetilde{v}_n^{\mathrm{aux},l})$$
$$= \Big[\chi\mathbf{A}_n^{\frac{1}{2}}(\widetilde{v}_n^{\mathrm{aux},l})\cdot\mathbf{A}_n^{\frac{1}{2}}(\widetilde{v}_n^{\mathrm{aux},l})\Big]_0^1 + \sum_{i=0}^{n-1}\Big[\mathscr{B}_i(\widetilde{v}_n^{\mathrm{aux},l})S_i\big(\partial_x(\chi\widetilde{v}_n^{\mathrm{aux},l})\big)\Big]_0^1 + \sum_{i=0}^{n-1}\Big[\mathscr{B}_i(\widetilde{v}_n^{\mathrm{aux},l})S_i(\chi\,\partial_x\widetilde{v}_n^{\mathrm{aux},l})\Big]_0^1.$$

Let us remind that the two terms involving $\mathrm{A}_n\,\widetilde{v}_n^{\mathrm{aux},l}$ are equal to zero because $\mathrm{A}_n\,\widetilde{v}_n^{\mathrm{aux},l} = 0$ (see (108a)).

Using the assumptions on χ, we get that

$$\Big[\chi\mathbf{A}_n^{\frac{1}{2}}(\widetilde{v}_n^{\mathrm{aux},l})\cdot\mathbf{A}_n^{\frac{1}{2}}(\widetilde{v}_n^{\mathrm{aux},l})\Big]_0^1 = \big|\partial_x^n\widetilde{v}_n^{\mathrm{aux},l}(t,1)\big|^2, \tag{123}$$

as well as

$$\sum_{i=0}^{n-1}\left[\mathscr{B}_i(\widetilde{v}_n^{\mathrm{aux},l})S_i\big(\partial_x(\chi\widetilde{v}_n^{\mathrm{aux},l})\big)\right]_0^1=-\left|\partial_x^n\widetilde{v}_n^{\mathrm{aux},l}(t,1)\right|^2, \tag{124}$$

$$\sum_{i=0}^{n-1}\left[\mathscr{B}_i(\widetilde{v}_n^{\mathrm{aux},l})S_i(\chi\partial_x\widetilde{v}_n^{\mathrm{aux},l})\right]_0^1=-\left|\partial_x^n\widetilde{v}_n^{\mathrm{aux},l}(t,1)\right|^2. \tag{125}$$

Therefore

$$\left|\partial_x^n\widetilde{v}_n^{\mathrm{aux},l}(t,1)\right|^2=-\int_0^1\left[\partial_x(\chi\cdot),\mathbf{A}_n^{\frac12}\right](\widetilde{v}_n^{\mathrm{aux},l})\cdot\mathbf{A}_n^{\frac12}(\widetilde{v}_n^{\mathrm{aux},l})-\int_0^1\mathbf{A}_n^{\frac12}(\widetilde{v}_n^{\mathrm{aux},l})\cdot\left[\chi\partial_x,\mathbf{A}_n^{\frac12}\right](\widetilde{v}_n^{\mathrm{aux},l}),$$

which allows us to conclude, using Lemma 43, that there exists a constant C such that

$$\left|\partial_x^n\widetilde{v}_n^{\mathrm{aux},l}(t,1)\right|^2\le C\|\chi\|_{W^{n+1,\infty}(0,1)}\left\|\widetilde{v}_n^{\mathrm{aux},l}(t,\cdot)\right\|_{H^n(0,1)}^2. \qquad \square$$

Proof of Proposition 38. By Lemma 40, we know that we can take $\widetilde{v}_n^{\mathrm{aux},l}$ as an auxiliary test function in (91), which we do, giving:

$$\int_{t_0}^{t_1}\int_0^1\Big(\widetilde{y}\partial_t\widetilde{v}_n^{\mathrm{aux},l}+\big(\widetilde{y}\widehat{v}+\widehat{y}\widetilde{v}\big)\partial_x\widetilde{v}_n^{\mathrm{aux},l}-\big(\widetilde{y}\partial_x\widehat{v}+\widehat{y}\partial_x\widetilde{v}\big)\widetilde{v}_n^{\mathrm{aux},l}\Big)$$
$$=\int_0^1\widetilde{v}_n^{\mathrm{aux},l}(t_1,\cdot)\widetilde{y}(t_1,\cdot)-\int_0^1\widetilde{v}_n^{\mathrm{aux},l}(t_0,\cdot)\widetilde{y}(t_0,\cdot). \tag{126}$$

As a reminder, the term $-\int_{t_0}^{t_1}\widetilde{v}_n^{\mathrm{aux},l}(\cdot,0)\widetilde{y}_l v_l$ is equal to zero because $\widetilde{y}_l=0$ on I and the term $\int_{t_0}^{t_1}\widetilde{v}_n^{\mathrm{aux},l}(\cdot,1)\widetilde{y}_r v_r$ is also equal to zero because $\widetilde{v}_n^{\mathrm{aux},l}(\cdot,1)=0$.

Let us now simplify each term of (126).

- First let us fix $t\in I$ and compute $\int_0^1\widetilde{v}_n^{\mathrm{aux},l}(t,\cdot)\widetilde{y}(t,\cdot)$:

$$\begin{aligned}\int_0^1\widetilde{y}(t,\cdot)\widetilde{v}_n^{\mathrm{aux},l}(t,\cdot)&=\int_0^1\mathrm{A}_n\,\widetilde{v}(t,\cdot)\widetilde{v}_n^{\mathrm{aux},l}(t,\cdot)\\&=\int_0^1\mathbf{A}_n^{\frac12}\,\widetilde{v}(t,\cdot)\cdot\mathbf{A}_n^{\frac12}\,\widetilde{v}_n^{\mathrm{aux},l}(t,\cdot)+\sum_{i=0}^{n-1}\left[\mathscr{B}_i(\widetilde{v})(t,\cdot)\mathscr{S}_i(\widetilde{v}_n^{\mathrm{aux},l})(t,\cdot)\right]_0^1\\&=\sum_{i=0}^{n-1}\mathscr{B}_i(\widetilde{v})(t,0)\mathscr{S}_i(\widetilde{v}_n^{\mathrm{aux},l})(t,0)\\&=-\int_0^1\left|\mathbf{A}_n^{\frac12}\,\widetilde{v}_n^{\mathrm{aux},l}(t,\cdot)\right|^2\\&=-\left\|\widetilde{v}_n^{\mathrm{aux},l}(t,\cdot)\right\|_{H^n}^2.\end{aligned} \tag{127}$$

  We used Lemma 42 and Remark 33 to get from the first line to the second. We used (114) with $\widetilde{v}$ instead of g and the fact that $\widetilde{v}_n^{\mathrm{aux},l}$ belongs to $H_{0,r}^k$ to get from the second to the third. Then we used (110) with $\widetilde{v}_n^{\mathrm{aux},l}$ instead of g to get the last line.
- The same computation allows us to simplify $\int_{t_0}^{t_1}\int_0^1\widetilde{y}\partial_t\widetilde{v}_n^{\mathrm{aux},l}$:

$$\begin{aligned}\int_{t_0}^{t_1}\int_0^1\widetilde{y}\partial_t\widetilde{v}_n^{\mathrm{aux},l}&=\int_{t_0}^{t_1}\int_0^1\mathrm{A}_n\,\widetilde{v}\partial_t\widetilde{v}_n^{\mathrm{aux},l}\\&=\int_{t_0}^{t_1}\sum_{i=0}^{n-1}\mathscr{B}_i(\widetilde{v})(t,0)\mathscr{S}_i(\partial_t\widetilde{v}_n^{\mathrm{aux},l})(t,0)\\&=-\int_{t_0}^{t_1}\int_0^1\mathbf{A}_n^{\frac12}\,\widetilde{v}_n^{\mathrm{aux},l}\cdot\partial_t\mathbf{A}_n^{\frac12}\,\widetilde{v}_n^{\mathrm{aux},l}.\end{aligned} \tag{128}$$

We can simplify (128) by integration by parts in time:

$$\begin{aligned}\int_{t_0}^{t_1}\int_0^1 \mathbf{A}_n^{\frac{1}{2}}\,\widetilde{v}_n^{\mathrm{aux},l}\cdot\partial_t\mathbf{A}_n^{\frac{1}{2}}\,\widetilde{v}_n^{\mathrm{aux},l} &= \frac{1}{2}\left(\int_0^1 \left|\mathbf{A}_n^{\frac{1}{2}}\,\widetilde{v}_n^{\mathrm{aux},l}(t_1,\cdot)\right|^2 - \int_0^1 \left|\mathbf{A}_n^{\frac{1}{2}}\,\widetilde{v}_n^{\mathrm{aux},l}(t_0,\cdot)\right|^2\right)\\ &= \frac{1}{2}\left\|\widetilde{v}_n^{\mathrm{aux},l}(t_1,\cdot)\right\|_{H^n}^2 - \frac{1}{2}\left\|\widetilde{v}_n^{\mathrm{aux},l}(t_0,\cdot)\right\|_{H^n}^2.\end{aligned} \tag{129}$$

- The terms $\int_0^1 \widehat{y}\widetilde{v}\partial_x\widetilde{v}_n^{\mathrm{aux},l}$ and $\int_0^1 \widehat{y}\partial_x\widetilde{v}\widetilde{v}_n^{\mathrm{aux},l}$ can be bounded by the Cauchy–Schwarz inequality:

$$\left|\int_0^1 \widehat{y}\widetilde{v}\partial_x\widetilde{v}_n^{\mathrm{aux},l}\right| \le \|\widehat{y}\|_{L^\infty}\left(\|\widetilde{v}\|_{H^1}^2 + \left\|\widetilde{v}_n^{\mathrm{aux},l}\right\|_{H^1}^2\right), \tag{130}$$

$$\left|\int_0^1 \widehat{y}\partial_x\widetilde{v}\widetilde{v}_n^{\mathrm{aux},l}\right| \le \|\widehat{y}\|_{L^\infty}\left(\|\widetilde{v}\|_{H^1}^2 + \left\|\widetilde{v}_n^{\mathrm{aux},l}\right\|_{H^1}^2\right). \tag{131}$$

- The term $\int_0^1 \widetilde{y}\partial_x\widehat{v}\widetilde{v}_n^{\mathrm{aux},l}$ can be simplified using Lemma 42 as well as (113):

$$\begin{aligned}\int_0^1 \widetilde{y}\partial_x\widehat{v}\widetilde{v}_n^{\mathrm{aux},l} &= \int_0^1 \mathrm{A}_n\,\widetilde{v}\partial_x\widehat{v}\widetilde{v}_n^{\mathrm{aux},l}\\ &= \int_0^1 \mathbf{A}_n^{\frac{1}{2}}\,\widetilde{v}\cdot\mathbf{A}_n^{\frac{1}{2}}(\partial_x\widehat{v}\widetilde{v}_n^{\mathrm{aux},l}) + \sum_{i=0}^{n-1}\mathscr{B}_i(\widetilde{v})(0)S_i(\partial_x\widehat{v}\widetilde{v}_n^{\mathrm{aux},l})(0)\\ &= \int_0^1 \mathbf{A}_n^{\frac{1}{2}}(\widetilde{v}-\widetilde{v}_n^{\mathrm{aux},l})\cdot\mathbf{A}_n^{\frac{1}{2}}(\partial_x\widehat{v}\widetilde{v}_n^{\mathrm{aux},l}).\end{aligned} \tag{132}$$

Then, using the Cauchy–Schwarz inequality once again as well as Lemma 43,

$$\left|\int_0^1 \widetilde{y}\partial_x\widehat{v}\widetilde{v}_n^{\mathrm{aux},l}\right| \le \|\widehat{v}\|_{W^{n+1,\infty}}\left(\|\widetilde{v}\|_{H^n}^2 + \left\|\widetilde{v}_n^{\mathrm{aux},l}\right\|_{H^n}^2\right). \tag{133}$$

- Let us simplify $\int_0^1 \widetilde{y}\widehat{v}\partial_x\widetilde{v}_n^{\mathrm{aux},l}$. We apply Lemma 44 with $\widetilde{v}$ instead of f, $\widetilde{v}_n^{\mathrm{aux},l}$ instead of g and $\widehat{v}$ instead of w. The two terms involving $\mathrm{A}_n\,\widetilde{v}_n^{\mathrm{aux},l}$ disappear thanks to Remark 33.

$$\begin{aligned}\int_0^1 \widehat{v}\widetilde{y}\partial_x\widetilde{v}_n^{\mathrm{aux},l} = &-\int_0^1 \left[\partial_x(\widehat{v}\cdot),\mathbf{A}_n^{\frac{1}{2}}\right](\widetilde{v})\cdot\mathbf{A}_n^{\frac{1}{2}}(\widetilde{v}_n^{\mathrm{aux},l}) - \int_0^1 \mathbf{A}_n^{\frac{1}{2}}(\widetilde{v})\cdot\left[\widehat{v}\partial_x,\mathbf{A}_n^{\frac{1}{2}}\right](\widetilde{v}_n^{\mathrm{aux},l})\\ &+\sum_{i=0}^{n-1}\left[\mathscr{B}_i(\widetilde{v}_n^{\mathrm{aux},l})S_i\big(\partial_x(\widehat{v}\widetilde{v})\big)\right]_0^1 + \sum_{i=0}^{n-1}\left[\mathscr{B}_i(\widetilde{v})S_i(\widehat{v}\partial_x\widetilde{v}_n^{\mathrm{aux},l})\right]_0^1 + \left[\widehat{v}\mathbf{A}_n^{\frac{1}{2}}(\widetilde{v}_n^{\mathrm{aux},l})\cdot\mathbf{A}_n^{\frac{1}{2}}(\widetilde{v})\right]_0^1.\end{aligned} \tag{134}$$

Both integrals can be bounded by using Lemma 43 as follows:

$$\left|\int_0^1 \left[\partial_x(\widehat{v}\cdot),\mathbf{A}_n^{\frac{1}{2}}\right](\widetilde{v})\cdot\mathbf{A}_n^{\frac{1}{2}}(\widetilde{v}_n^{\mathrm{aux},l})\right| \le C\|\widehat{v}\|_{W^{n+1,\infty}(0,1)}\left(\|\widetilde{v}\|_{H^n(0,1)}^2 + \left\|\widetilde{v}_n^{\mathrm{aux},l}\right\|_{H^n(0,1)}^2\right), \tag{135}$$

$$\left|\int_0^1 \mathbf{A}_n^{\frac{1}{2}}(\widetilde{v})\cdot\left[\widehat{v}\partial_x,\mathbf{A}_n^{\frac{1}{2}}\right](\widetilde{v}_n^{\mathrm{aux},l})\right| \le C\|\widehat{v}\|_{W^{n,\infty}(0,1)}\left(\|\widetilde{v}\|_{H^n(0,1)}^2 + \left\|\widetilde{v}_n^{\mathrm{aux},l}\right\|_{H^n(0,1)}^2\right). \tag{136}$$

For $i \le n-2$, one has $\mathscr{S}_i\big(\partial_x(\widehat{v}\widetilde{v})\big) = 0$, and $\mathscr{S}_{n-1}\big(\partial_x(\widehat{v}\widetilde{v})\big) = \widehat{v}\partial_x^n\widetilde{v}$. Moreover, for any function f, $\mathscr{B}_{n-1}(f) = -\partial_x^n f$. Therefore,

$$\sum_{i=0}^{n-1}\left[\mathscr{B}_i(\widetilde{v}_n^{\mathrm{aux},l})S_i\big(\partial_x(\widehat{v}\widetilde{v})\big)\right]_0^1 = -\partial_x^n\widetilde{v}(\cdot,1)\partial_x^n\widetilde{v}_n^{\mathrm{aux},l}(\cdot,1)v_r - \left|\partial_x^n\widetilde{v}(\cdot,0)\right|^2 v_l. \tag{137}$$

We also have

$$\left[\widehat{v}\mathbf{A}_n^{\frac{1}{2}}(\widetilde{v}_n^{\mathrm{aux},l})\cdot\mathbf{A}_n^{\frac{1}{2}}(\widetilde{v})\right]_0^1 = \partial_x^n\widetilde{v}(\cdot,1)\,\partial_x^n\widetilde{v}_n^{\mathrm{aux},l}(\cdot,1)\,v_r + \left|\partial_x^n\widetilde{v}(\cdot,0)\right|^2 v_l. \tag{138}$$

Moreover, using the variational formulation (113) for $\widetilde{v}_n^{\text{aux},l}$, one gets

$$\begin{aligned}\sum_{i=0}^{n-1}&\Big[\mathscr{B}_i(\widetilde{v})S_i(\widehat{v}\partial_x\widetilde{v}_n^{\text{aux},l})\Big]_0^1\\&=\sum_{i=0}^{n-1}\mathscr{B}_i(\widetilde{v})(\cdot,1)S_i(\widehat{v}\partial_x\widetilde{v}_n^{\text{aux},l})(\cdot,1)-\sum_{i=0}^{n-1}\mathscr{B}_i(\widetilde{v})(\cdot,0)S_i(\widehat{v}\partial_x\widetilde{v}_n^{\text{aux},l})(\cdot,0)\\&=\mathscr{B}_{n-1}(\widetilde{v})(\cdot,1)\mathscr{S}_{n-1}(\widehat{v}\partial_x\widetilde{v}_n^{\text{aux},l})(\cdot,1)+\int_0^1\mathbf{A}_n^{\frac12}\,\widetilde{v}_n^{\text{aux},l}\cdot\mathbf{A}_n^{\frac12}(\widehat{v}\partial_x\widetilde{v}_n^{\text{aux},l})\\&\qquad+\sum_{i=0}^{n-1}\mathscr{B}_i(\widetilde{v}_n^{\text{aux},l})(\cdot,1)\mathscr{S}_i(\widehat{v}\partial_x\widetilde{v}_n^{\text{aux},l})(1)\\&=-(\partial_x^n\widetilde{v}(\cdot,1)+\partial_x^n\widetilde{v}_n^{\text{aux},l}(\cdot,1))\,\partial_x^n\widetilde{v}_n^{\text{aux},l}(\cdot,1)\,v_r+\int_0^1\mathbf{A}_n^{\frac12}\,\widetilde{v}_n^{\text{aux},l}\cdot\mathbf{A}_n^{\frac12}(\widehat{v}\partial_x\widetilde{v}_n^{\text{aux},l}).\end{aligned}\tag{139}$$

We exchange $\widehat{v}$ and $\mathbf{A}_n^{\frac12}$ up to a commutator in $\int_0^1\mathbf{A}_n^{\frac12}\,\widetilde{v}_n^{\text{aux},l}\cdot\mathbf{A}_n^{\frac12}(\widehat{v}\partial_x\widetilde{v}_n^{\text{aux},l})$,

$$\int_0^1\mathbf{A}_n^{\frac12}\,\widetilde{v}_n^{\text{aux},l}\cdot\mathbf{A}_n^{\frac12}(\widehat{v}\partial_x\widetilde{v}_n^{\text{aux},l})=\int_0^1\mathbf{A}_n^{\frac12}\,\widetilde{v}_n^{\text{aux},l}\cdot\widehat{v}\partial_x\mathbf{A}_n^{\frac12}\,\widetilde{v}_n^{\text{aux},l}+\int_0^1\mathbf{A}_n^{\frac12}\,\widetilde{v}_n^{\text{aux},l}\cdot\Big[\mathbf{A}_n^{\frac12},\widehat{v}\partial_x\Big]\widetilde{v}_n^{\text{aux},l},\tag{140}$$

then we integrate by part

$$\begin{aligned}\int_0^1\mathbf{A}_n^{\frac12}\,\widetilde{v}_n^{\text{aux},l}\cdot\widehat{v}\partial_x\mathbf{A}_n^{\frac12}\,\widetilde{v}_n^{\text{aux},l}&=\frac12\int_0^1\widehat{v}\partial_x\Big(\big|\mathbf{A}_n^{\frac12}\,\widetilde{v}_n^{\text{aux},l}\big|^2\Big)\\&=-\frac12\int_0^1\partial_x\widehat{v}\big|\mathbf{A}_n^{\frac12}\,\widetilde{v}_n^{\text{aux},l}\big|^2+\Big[\big|\partial_x^n\widetilde{v}_n^{\text{aux},l}\big|^2\widehat{v}\Big]_0^1.\end{aligned}\tag{141}$$

Combining (139), (140) and (141), we get that

$$\begin{aligned}\sum_{i=0}^{n-1}\Big[\mathscr{B}_i(\widetilde{v})S_i(\widehat{v}\partial_x\widetilde{v}_n^{\text{aux},l})\Big]_0^1=-\partial_x^n\widetilde{v}(\cdot,1)\,\partial_x^n\widetilde{v}_n^{\text{aux},l}(\cdot,1)\,v_r+\int_0^1\mathbf{A}_n^{\frac12}\,\widetilde{v}_n^{\text{aux},l}\cdot\Big[\mathbf{A}_n^{\frac12},\widehat{v}\partial_x\Big]\widetilde{v}_n^{\text{aux},l}\\-\frac12\int_0^1\partial_x\widehat{v}\big|\mathbf{A}_n^{\frac12}\,\widetilde{v}_n^{\text{aux},l}\big|^2-\big|\partial_x^n\widetilde{v}(\cdot,0)\big|^2v_l.\end{aligned}\tag{142}$$

Once again, we bound the trilinear term as follows:

$$\left|\int_0^1\mathbf{A}_n^{\frac12}\,\widetilde{v}_n^{\text{aux},l}\cdot\Big[\mathbf{A}_n^{\frac12},\widehat{v}\partial_x\Big]\widetilde{v}_n^{\text{aux},l}\right|\le C\|\widehat{v}\|_{W^{n+1,\infty}}\big\|\widetilde{v}_n^{\text{aux},l}\big\|_{H^n}^2,\tag{143}$$

$$\left|\int_0^1\partial_x\widehat{v}\big|\mathbf{A}_n^{\frac12}\,\widetilde{v}_n^{\text{aux},l}\big|^2\right|\le\|\widehat{v}\|_{W^{1,\infty}}\big\|\widetilde{v}_n^{\text{aux},l}\big\|_{H^n}^2.\tag{144}$$

Using Proposition 41, we can bound $\partial_x^n\widetilde{v}(\cdot,1)\,\partial_x^n\widetilde{v}_n^{\text{aux},l}(\cdot,1)\,v_r$ as follows:

$$\begin{aligned}\partial_x^n\widetilde{v}(\cdot,1)\,\partial_x^n\widetilde{v}_n^{\text{aux},l}(\cdot,1)\,v_r&\le\frac14\big|\partial_x^n\widetilde{v}(\cdot,1)\big|^2v_r+\big|\partial_x^n\widetilde{v}_n^{\text{aux},l}(\cdot,1)\big|^2v_r\\&\le\frac14\big|\partial_x^n\widetilde{v}(\cdot,1)\big|^2v_r+C\big\|\widetilde{v}_n^{\text{aux},l}\big\|_{H^n}^2.\end{aligned}\tag{145}$$

Using (134)–(138) and (142)–(145), we get

$$\begin{aligned}&\left|\int_{t_0}^{t_1}\int_0^1\widehat{v}\widetilde{y}\partial_x\widetilde{v}_n^{\text{aux},l}+\int_{t_0}^{t_1}\big|\partial_x^n\widetilde{v}(\cdot,0)\big|^2v_l\right|\\&\qquad\le C\|\widehat{v}\|_{L^\infty([0,T],W^{n+1,\infty}(0,1))}\int_{t_0}^{t_1}\Big(\|\widetilde{v}\|_{H^n}^2+\big\|\widetilde{v}_n^{\text{aux},l}\big\|_{H^n}^2\Big)+\frac14\int_{t_0}^{t_1}\big|\partial_x^n\widetilde{v}(\cdot,1)\big|^2v_r.\end{aligned}\tag{146}$$

Now, let us assemble (127), (128), (129), and (146) to get

$$\frac12\Big[\big\|\widetilde{v}_n^{\text{aux},l}\big\|_{H^n}^2\Big]_{t_0}^{t_1}+\int_{t_0}^{t_1}\big|\partial_x^n\widetilde{v}(\cdot,0)\big|^2v_l\le C\int_{t_0}^{t_1}\Big(\|\widetilde{v}\|_{H^n}^2+\big\|\widetilde{v}_n^{\text{aux},l}\big\|_{H^n}^2\Big)+\frac14\int_{t_0}^{t_1}\big|\partial_x^n\widetilde{v}(\cdot,1)\big|^2v_r.\qquad\square$$

3.3. *Gronwall argument and end of the proof*

Let $I_1, \dots, I_K$ be the intervals on which neither v_l nor v_l changes sign. We assume that the initial and boundary conditions are the same. We prove by induction on k that $\widetilde{v}$ is equal to zero on I_k. First, for every $k \in [[1, K]]$, we construct $\widetilde{v}_n^{\text{aux},l}$ and/or $\widetilde{v}_n^{\text{aux},r}$ on I_k according to the signs of v_l and v_r on I_k.

Initialization step. By hypothesis, $\widetilde{v}$ is equal to 0 at time zero.

Induction step. Let us fix $k \in [[1, K]]$ and assume that $\widetilde{v}$ is equal to 0 at the beginning of I_k. Then, the auxiliary functions created on the interval I_k are equal to 0 at the beginning of the interval I_k. We denote by $E_{\text{rel},k}\colon I_k \to \mathbb{R}_+$ the quantity:

$$E_{\text{rel},k}(t) := \begin{cases} \|\widetilde{v}\|_{H^n}^2 + \left\|\widetilde{v}_n^{\text{aux},l}\right\|_{H^n}^2 + \left\|\widetilde{v}_n^{\text{aux},r}\right\|_{H^n}^2 & \text{if } I_k \subset \Gamma_l \cap \Gamma_r, \\ \|\widetilde{v}\|_{H^n}^2 + \left\|\widetilde{v}_n^{\text{aux},l}\right\|_{H^n}^2 & \text{if } I_k \subset \Gamma_l \setminus (\Gamma_l \cap \Gamma_r), \\ \|\widetilde{v}\|_{H^n}^2 + \left\|\widetilde{v}_n^{\text{aux},r}\right\|_{H^n}^2 & \text{if } I_k \subset \Gamma_r \setminus (\Gamma_l \cap \Gamma_r), \\ \|\widetilde{v}\|_{H^n}^2 & \text{if } I_k \subset [0,T] \setminus (\Gamma_l \cup \Gamma_r). \end{cases} \tag{147}$$

We sum inequality (97) with two times inequality (116) if $I_k \subset \Gamma_l$, and two times inequality (118) if $I_k \subset \Gamma_r$. One gets that there exists a constant $C > 0$ such that

$$E'_{\text{rel},k}(t) + \frac{1}{2}\left(\left|\partial_x^n \widetilde{v}(t,0)\right|^2 |v_l| + \left|\partial_x^n \widetilde{v}(t,1)\right|^2 |v_r|\right) \le C E_{\text{rel},k}(t). \tag{148}$$

Hence, by the Gronwall inequality, since $E_{\text{rel},k}$ is equal to zero at the beginning of I_k, it is equal to zero on I_k. In particular, $\widetilde{v}$ is equal to 0 on I_k. Since $\widetilde{v}$ belongs to $C^0\big([0,T], H^n(0,1)\big)$, we get that $\widetilde{v}$ is equal to 0 at the beginning of I_{k+1}, which concludes the induction as well as the proof of Theorem 19.

Appendix A. Integration by parts and commutator for A_n

Lemma 42. *Let $f \in H^{2n}(0,1)$ and $g \in H^n(0,1)$ be two functions. We have the equality*

$$\int_0^1 \mathrm{A}_n f\, g = \int_0^1 \mathbf{A}_n^{\frac{1}{2}} f \cdot \mathbf{A}_n^{\frac{1}{2}} g + \sum_{i=0}^{n-1} \left[\mathscr{B}_i(f)\mathscr{S}_i(g)\right]_0^1, \tag{149}$$

where $\cdot$ is the standard scalar product on $\mathbb{R}^{n+1}$ and the operators $\mathscr{B}_i$ and $\mathscr{S}_i$ are defined through

$$\forall\, x \in \{0,1\}, \quad \mathscr{B}_i(f)(x) := \sum_{k=i+1}^{n} (-1)^{k+i}\, \partial_x^{2k-1-i} f(x), \tag{150a}$$

$$\forall\, x \in \{0,1\}, \quad \mathscr{S}_i(g)(x) := \partial_x^i g(x). \tag{150b}$$

Let us remark that the operators $\mathscr{B}_i$ and $\mathscr{S}_i$ are boundary operators of respective order $2n-1-i$ and i.

Proof. By induction on $k \in \mathbb{N}$,

$$\forall\, f \in H^{2k}(0,1),\ \forall\, g \in H^k(0,1), \quad \int_0^1 (\partial_x^{2k} f) g = (-1)^k \int_0^1 (\partial_x^k f)(\partial_x^k g) + \sum_{i=0}^{k-1} (-1)^i \left[(\partial_x^{2k-1-i} f)(\partial_x^i g)\right]_0^1.$$

By summation on $k \in \{1, \dots, n\}$, we have

$$\int_0^1 \mathrm{A}_n f\, g = \int_0^1 \mathbf{A}_n^{\frac{1}{2}} f \cdot \mathbf{A}_n^{\frac{1}{2}} g + \sum_{k=0}^{n} \sum_{i=0}^{k-1} (-1)^{k+i} \left[\partial_x^{2k-1-i} f\, \partial_x^i g\right]_0^1, \tag{151}$$

which can be rewritten into (149), since

$$\sum_{k=0}^{n}\sum_{i=0}^{k-1}(-1)^{k+i}\left[\partial_x^{2k-1-i}f\,\partial_x^i g\right]_0^1=\sum_{i=0}^{n-1}\sum_{k=i+1}^{n}(-1)^{k+i}\left[\partial_x^{2k-1-i}f\,\partial_x^i g\right]_0^1$$
$$=\sum_{i=0}^{n-1}\left[\mathscr{B}_i(f)\mathscr{S}_i(g)\right]_0^1. \qquad \square$$

Let $k\in\mathbb{N}^*$, $f\in W^{k,\infty}(0,1)$, and $\mathscr{A}$ a differential operator of order k. We denote by $[\mathscr{A},f]$ the commutator operator

$$\forall\, g\in H^k(0,1),\quad [f,\mathscr{A}]g:=f\mathscr{A}g-\mathscr{A}(fg). \tag{152}$$

The two following lemmas are used in the proof of Theorem 19 to bound trilinear terms. Lemma 43 is a simple consequence of Leibniz formula. Lemma 44 is a consequence of a repeated use of Lemma 42.

Lemma 43. *There exists a constant $C>0$ depending only on k and $\mathscr{A}$ such that*

$$\forall\, g\in H^k(0,1),\quad \left\|[f,\mathscr{A}]g\right\|_{L^2}\le C\|f\|_{W^{k,\infty}}\|g\|_{H^{k-1}}. \tag{153}$$

Proof. Let $f\in W^{k\infty}(0,1)$, $g\in H^{k-1}(0,1)$ be two functions and $a_0,\dots,a_k\in\mathbb{R}$ be real numbers. We apply Leibniz formula in the sense of distribution over $(0,1)$:

$$\begin{aligned}[f,\mathscr{A}]g&=f\mathscr{A}(g)-\mathscr{A}(fg)\\&=\sum_{i=0}^{k}a_i\sum_{j=0}^{i-1}\binom{i}{j}g^{(j)}f^{i-j}\\&=\sum_{j=0}^{k-1}\left(\sum_{i=j+1}^{k}\binom{i}{j}a_i f^{i-j}\right)g^{(j)}\\&=\sum_{j=0}^{k-1}b_j g^{(j)},\end{aligned}$$

where $b_j:=\sum_{i=j+1}^{k}\binom{i}{j}a_i f^{i-j}$. Now, since $f\in W^{k,\infty}$, for all j, $b_j\in W^{k-j-1,\infty}\subset L^\infty$ and

$$\|b_j\|_{L^\infty}\le\sum_{i=1}^{k}|a_i|.$$

Then, since the product of an L^2-function and an L^∞-function is in L^2, $[f,\mathscr{A}]g$ is in L^2, and we have

$$\left\|[f,\mathscr{A}]g\right\|_{L^2}\le(k-1)\sum_{i=1}^{k}|a_i|\,\|f\|_{W^{k,\infty}}\|g\|_{H^{k-1}}. \qquad \square$$

Lemma 44. *Let $f,g,w\in H^{2n}(0,1)$ be functions. We have the following equality:*

$$\int_0^1 w\,\mathrm{A}_n(f)\,\partial_x g+\int_0^1\partial_x(wf)\,\mathrm{A}_n(g)+\int_0^1\left[\partial_x(w\cdot),\mathbf{A}_n^{\frac12}\right](f)\cdot\mathbf{A}_n^{\frac12}(g)-\int_0^1\mathbf{A}_n^{\frac12}(f)\cdot\left[\mathbf{A}_n^{\frac12},w\,\partial_x\right](g)$$
$$=\sum_{i=0}^{n-1}\left[\mathscr{B}_i(g)S_i\big(\partial_x(wf)\big)\right]_0^1+\sum_{i=0}^{n-1}\left[\mathscr{B}_i(f)S_i(w\,\partial_x g)\right]_0^1+\left[w\,\mathbf{A}_n^{\frac12}(f)\cdot\mathbf{A}_n^{\frac12}(g)\right]_0^1. \tag{154}$$

Proof. First of all, let us remark that for $n\ge1$, by Sobolev embedding, one has $W^{n,\infty}\subset H^{n+1}\subset H^{2n}$. Hence all the integrals make sense in L^1 as an integral of a function which is a L^∞-L^2-L^2 product.

Now let us apply Lemma 42 with f and $w\,\partial_x g$ instead of f and g:

$$\int_0^1\mathrm{A}_n(f)\,w\,\partial_x g=\int_0^1\mathbf{A}_n^{\frac12}(f)\cdot\mathbf{A}_n^{\frac12}(w\,\partial_x g)+\sum_{i=0}^{n-1}\left[\mathscr{B}_i(f)\mathscr{S}_i(w\,\partial_x g)\right]_0^1. \tag{155}$$

We exchange $\mathbf{A}_n^{\frac{1}{2}}$ and $w\partial_x$ up to a commutator:

$$\int_0^1 \mathbf{A}_n^{\frac{1}{2}}(f)\cdot \mathbf{A}_n^{\frac{1}{2}}(w\partial_x g) = \int_0^1 \mathbf{A}_n^{\frac{1}{2}}(f)\cdot w\partial_x \mathbf{A}_n^{\frac{1}{2}}(g) + \int_0^1 \mathbf{A}_n^{\frac{1}{2}}(f)\cdot \Big[\mathbf{A}_n^{\frac{1}{2}}, w\partial_x\Big](g). \tag{156}$$

We perform an integration by parts

$$\int_0^1 \mathbf{A}_n^{\frac{1}{2}}(f)\cdot w\partial_x \mathbf{A}_n^{\frac{1}{2}}(g) = -\int_0^1 \partial_x\big(w\mathbf{A}_n^{\frac{1}{2}}(f)\big)\cdot \mathbf{A}_n^{\frac{1}{2}}(g) + \Big[w\mathbf{A}_n^{\frac{1}{2}}(f)\cdot \mathbf{A}_n^{\frac{1}{2}}(g)\Big]_0^1. \tag{157}$$

We exchange $\partial_x(w\,\cdot)$ and $\mathbf{A}_n^{\frac{1}{2}}$ up to another commutator

$$\int_0^1 \partial_x\big(w\mathbf{A}_n^{\frac{1}{2}}(f)\big)\cdot \mathbf{A}_n^{\frac{1}{2}}(g) = \int_0^1 \mathbf{A}_n^{\frac{1}{2}}\big(\partial_x(wf)\big)\cdot \mathbf{A}_n^{\frac{1}{2}}(g) + \int_0^1 \Big[\partial_x(w\,\cdot), \mathbf{A}_n^{\frac{1}{2}}\Big](f)\cdot \mathbf{A}_n^{\frac{1}{2}}(g). \tag{158}$$

Finally we apply Lemma 42 once again, this time with g and $\partial_x(wf)$ instead of f and g:

$$\int_0^1 \mathbf{A}_n^{\frac{1}{2}}\big(\partial_x(wf)\big)\cdot \mathbf{A}_n^{\frac{1}{2}}(g) = \int_0^1 \partial_x(wf)\,\mathrm{A}_n(g) - \sum_{i=0}^{n-1} \mathscr{B}_i(g)\mathscr{S}_i\big(\partial_x(wf)\big). \tag{159}$$

Combining (155)–(159), we obtain (154). □

Appendix B. Sketch of the proof of Theorem 18

Let v be a function on $\Omega_T := [0,T]\times[0,1]$, which verifies the boundary conditions (21b) and (21c). We denote by ϕ the flow of v. It is defined as the unique maximal solution of the following ODE:

$$\partial_1\phi(s,t,x) = v\big(s,\phi(s,t,x)\big), \tag{160a}$$

$$\phi(t,t,x) = x. \tag{160b}$$

The quantity $\phi(s,t,x)$ is the position at time s of the particle which was in x at time t. The quantity $\phi(\cdot,t,x)$ is defined on an interval of time $\big[e(t,x),h(t,x)\big]$ where $e(t,x)$ and $h(t,x)$ are the time of entrance and exit of the domain for the particle going through x at time t.

We define the sets Ω_L, Ω_R, Ω_I and Ω_S as:

$$\begin{aligned}
\Omega_S :=& \Big\{(t,x)\in\Omega_T;\ \exists\, s\in\big[e(t,x),h(t,x)\big],\ \big(\phi(s,t,x)=0 \text{ and } v_l(s)=0\big)\\
&\quad \text{or } \big(\phi(s,t,x)=1 \text{ and } v_r(s)=0\big)\Big\}\\
&\quad \cup\Big\{\big(s,\phi(s,0,0)\big);\ s\in\big[0,h(0,0)\big]\Big\}\cup\Big\{\big(s,\phi(s,0,1)\big);\ s\in\big[0,h(0,1)\big]\Big\},\\
\Omega_I :=& \big\{(t,x)\in\Omega_T\setminus\Omega_S;\ e(t,x)=0\big\},\\
\Omega_L :=& \Big\{(t,x)\in\Omega_T;\ e(t,x)>0 \text{ and } \phi\big(e(t,x),t,x\big)=0\Big\},\\
\Omega_R :=& \Big\{(t,x)\in\Omega_T;\ e(t,x)>0 \text{ and } \phi\big(e(t,x),t,x\big)=1\Big\}.
\end{aligned}$$

The sets Ω_I, Ω_L and Ω_R contain the particles which enter the domain at time 0, from the left and from the right respectively. The set Ω_S is called the singular set, it contains the particles which were at time 0 at the boundary as well as the particles which were on the boundary with velocity zero at some point in time.

We define the function $y\in L^\infty(\Omega_T)$ by

$$y(t,x) := \begin{cases} y_0\big(\phi(0,t,x)\big)\exp\Big(-2\int_0^t \partial_x v\big(s,\phi(s,t,x)\big)\mathrm{d}s\Big) & \text{for } (x,t)\in\Omega_I,\\ y_l^c\big(e(t,x)\big)\exp\Big(-2\int_{e(t,x)}^t \partial_x v\big(s,\phi(s,t,x)\big)\mathrm{d}s\Big) & \text{for } (x,t)\in\Omega_L,\\ y_r^c\big(e(t,x)\big)\exp\Big(-2\int_{e(t,x)}^t \partial_x v\big(s,\phi(s,t,x)\big)\mathrm{d}s\Big) & \text{for } (x,t)\in\Omega_R.\end{cases}$$

We refer to [24] for the study of the transport equation with stretching (18a). The facts that we will use are:

- the function y is well-defined in $L^\infty(\Omega_T)$, together with the estimate

$$\|y\|_{L^\infty(\Omega_T)} \le \max\big(\|y_0\|_{L^\infty}, \|y_l^c\|_{L^\infty}, \|y_r^c\|_{L^\infty}\big)\exp\big(2T\|\partial_x v\|_{L^\infty}\big); \tag{161}$$

- the function y is the unique solution of (18a) with initial condition y_0 and boundary conditions y_r^c and y_l^c;
- the function y is in $W^{1,\infty}\big([0,T],H^{-1}(0,1)\big)$, together with the estimate

$$\|\partial_t y\|_{L^\infty([0,T],H^{-1}(0,1))} \le 3\|y\|_{L^\infty(\Omega_T)}\|v\|_{L^\infty([0,T],W^{1,\infty}(0,1))}. \tag{162}$$

To simplify the notation, we denote $L_t^\infty W_x^{2n,\infty}$ instead of $L^\infty\big([0,T],W^{2n,\infty}(0,1)\big)$ and similarly for $L_t^\infty W_x^{1,\infty}$, $W_t^{1,\infty}H_x^n$ as well as $W_t^{1,\infty}H_x^{-1}$.

We can then introduce the solution u to the system

$$\mathrm{A}_n u = y, \tag{163a}$$

$$\mathbf{v_l} = \big(\mathscr{S}_i(u)(0)\big)_{i\in[[0,n-1]]}, \tag{163b}$$

$$\mathbf{v_r} = \big(\mathscr{S}_i(u)(1)\big)_{i\in[[0,n-1]]}. \tag{163c}$$

We call $\mathscr{F}$ the operator which to $v \in L_t^\infty W_x^{2n,\infty}\cap W_t^{1,\infty}H_x^n$ associates $u\in L_t^\infty W_x^{2n,\infty}\cap W_t^{1,\infty}H_x^n$.

For B_0 and B_1 positive numbers, we introduce the space $C_{B_0,B_1,T}$ as

$$C_{B_0,B_1,T} := \big\{v \in L_t^\infty W_x^{2n,\infty}\cap W_t^{1,\infty}H_x^n;\ \|v\|_{L^\infty W^{2n,\infty}} \le B_0 \text{ and } \|v\|_{W^{1,\infty}H^n}\le B_1\big\}. \tag{164}$$

The end of the proof is threefold:

- find B_0 and B_1 such that $\mathscr{F}$ maps $C_{B_0,B_1,T}$ into itself;
- prove that $C_{B_0,B_1,T}$ is compact with respect to $\|\cdot\|_{L_t^\infty W_x^{1,\infty}}$;
- prove that $\mathscr{F}$ is continuous with respect to $\|\cdot\|_{L_t^\infty W_x^{1,\infty}}$.

Once all this is done one can conclude by applying Schauder Fixed Point Theorem.

Lemma 45. *There exists a time $T>0$ as well as B_0 and B_1 such that $\mathscr{F}$ maps $C_{B_0,B_1,T}$ into itself.*

Proof. Let us take $v\in L_t^\infty W_x^{2n,\infty}\cap W_t^{1,\infty}H_x^n$ and denote $u:=\mathscr{F}(v)$. We denote by c_1 and c_2

$$c_1 := \max\big(\|y_0\|_{L_x^\infty}, \|y_l^c\|_{L_t^\infty}, \|y_r^c\|_{L_t^\infty}\big)$$

$$c_2 := \|\mathbf{v_l}\|_{L_t^\infty} + \|\mathbf{v_r}\|_{L_t^\infty}$$

the two constants depending on the initial and boundary data. Combining the estimates (161) and (162) with the elliptic estimates from Lemma 13, we get that there exists a constant C depending only on n such that

$$\|u\|_{L_t^\infty W_x^{2n,\infty}} \le C\Big(c_1\exp\big(2T\|v\|_{L_t^\infty W_x^{2n,\infty}}\big)+c_2\Big) \tag{165}$$

and

$$\|\partial_t u\|_{L_t^\infty H_x^n} \le C\Big(c_1\exp\big(2T\|v\|_{L_t^\infty W_x^{2n,\infty}}\big)+c_2\Big)\|v\|_{L_t^\infty W_x^{1,\infty}}. \tag{166}$$

We choose $B_0 := 2C(c_1+c_2)$. For T small enough one has

$$C\big(c_1\exp(2TB_0)+c_2\big) < 2C(c_1+c_2) = B_0; \tag{167}$$

we choose such a T. Then we choose $B_1 := B_0^2$. □

Lemma 46. *For any B_0, B_1 and T, the space $C_{B_0,B_1,T}$ is compact with respect to the norm $\|\cdot\|_{L_t^\infty W_x^{1,\infty}}$.*

Proof. For $n=1$ this was done in [24].

For $n\ge 2$, it is easier.

First of all, thanks to Banach–Alaoglu, the closed balls of $L_t^\infty W_x^{2n,\infty}$ and $W_t^{1,\infty}H_x^n$ are compact for the weak-$*$ topology. Therefore, they are closed for this topology, and in particular, $C_{B_0,B_1,T}$ is closed in $L_t^\infty W_x^{1,\infty}$.

We have $W_t^{1,\infty}H_x^{n-1}\hookrightarrow W_t^{1,\infty}H_x^1$. Therefore for $(t,x),(t',x')\in\Omega_T$ and $u\in C_{B_0,B_1,T}$, one has

$$\big|\partial_x u(t,x)-\partial_x u(t',x')\big| \le |t-t'|\sqrt{|x-x'|}\,\|u\|_{W_t^{1,\infty}H_x^n}.$$

Therefore, thanks to Ascoli's theorem, any sequence $(U_n)_n$ in $C_{B_0,B_1,T}$ admits a subsequence $(U_{\sigma(n)})_n$ such that $(\partial_x U_{\sigma(n)})_n$ converges in $L_t^\infty L_x^\infty$. Up to a second extraction, we can assume also that $(U_{\mu(n)})_n$ and $(\partial_x U_{\mu(n)})_n$ converge in $L_t^\infty L_x^\infty$ and L_t^∞ respectively. In that case $(U_{\mu(n)})_n$ does converge in $L_t^\infty W_x^{1,\infty}$, as wanted. □

Lemma 47. *The operator* $\mathscr{F}$ *is continuous with respect to the norm* $\|\cdot\|_{L_t^\infty W_x^{1,\infty}}$.

The proof of this Lemma does not differ from Proposition 2.4 in [24].

Combining all the arguments above, we proved the existence of B_0, B_1 and of a function $u \in C_{B_0,B_1,T}$ which is a fixed point of $\mathscr{F}$. That is:

- the unique solution y of (18a) with initial condition y_0 and boundary conditions y_r^c and y_l^c is equal to $\mathrm{A}_n\, u$;
- the function u verifies the boundary condition (163b)–(163c).

As is, we created a weak solution in the sense of distribution of the Camassa–Holm equation. It is a weak solution in the sense of Definition 17 due to Theorem 3 in [2].

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References

[1] R. Bhatt and A. V. Mikhailov, "On the inconsistency of the Camassa–Holm equation with shallow water theory", 2010. Online at https://arxiv.org/abs/1010.1932.

[2] F. Boyer, "Trace theorems and spatial continuity properties for the solutions of the transport equation", *Differ. Integral Equ.* **18** (2005), no. 8, pp. 891–934.

[3] A. Bressan, G. Chen and Q. Zhang, "Uniqueness of Conservative Solutions to the Camassa–Holm Equation via Characteristics", 2014. Online at https://arxiv.org/abs/1401.0312.

[4] A. Bressan and A. Constantin, "Global conservative solutions of the Camassa–Holm equation", *Arch. Ration. Mech. Anal.* **183** (2007), no. 2, pp. 215–239.

[5] R. Camassa and D. D. Holm, "An integrable shallow water equation with peaked solitons", *Phys. Rev. Lett.* **71** (1993), no. 11, pp. 1661–1664.

[6] G. M. Coclite, H. Holden and K. H. Karlsen, "Global weak solutions to a generalized hyperelastic-rod wave equation", *SIAM J. Math. Anal.* **37** (2005), no. 4, pp. 1044–1069.

[7] G. M. Coclite, H. Holden and K. H. Karlsen, "Well-posedness of higher-order Camassa–Holm equations", *J. Differ. Equations* **246** (2009), no. 3, pp. 929–963.

[8] A. Constantin and J. Escher, "Global existence and blow-up for a shallow water equation", *Ann. Sc. Norm. Super. Pisa, Cl. Sci. (4)* **26** (1998), no. 2, pp. 303–328.

[9] A. Constantin and J. Escher, "Wave breaking for nonlinear nonlocal shallow water equations", *Acta Math.* **181** (1998), no. 2, pp. 229–243.

[10] A. Constantin and J. Escher, "On the blow-up rate and the blow-up set of breaking waves for a shallow water equation", *Math. Z.* **233** (2000), no. 1, pp. 75–91.

[11] A. Constantin, V. S. Gerdjikov and R. I. Ivanov, "Inverse scattering transform for the Camassa–Holm equation", *Inverse Probl.* **22** (2006), no. 6, pp. 2197–2207.

[12] A. Constantin and R. I. Ivanov, "The Camassa–Holm equation as a geodesic flow for the H^1 right-invariant metric", 2007. Online at https://arxiv.org/abs/0706.3810.

[13] A. Constantin and B. Kolev, "H^k metrics on the diffeomorphism group of the circle", *J. Nonlinear Math. Phys.* **10** (2003), no. 4, pp. 424–430.

[14] A. Constantin and D. Lannes, "The hydrodynamical relevance of the Camassa–Holm and Degasperis–Procesi equations", *Arch. Ration. Mech. Anal.* **192** (2009), no. 1, pp. 165–186.

[15] J. Escher and Z. Yin, "Initial boundary value problems of the Camassa–Holm equation", *Commun. Partial Differ. Equations* **33** (2008), no. 1-3, pp. 377–395.

[16] J. Escher and Z. Yin, "Initial boundary value problems for nonlinear dispersive wave equations", *J. Funct. Anal.* **256** (2009), no. 2, pp. 479–508.

[17] B. Fuchssteiner and A. S. Fokas, "Symplectic structures, their Bäcklund transformations and hereditary symmetries", *Phys. D: Nonlinear Phenom.* **4** (1981), no. 1, pp. 47–66.

[18] L. Hörmander, "Inequalities Between Normal And Tangential Derivatives Of Harmonic Functions", in *Unpublished Manuscripts—from 1951 to 2007*, Springer, 2018, pp. 37–41.

[19] D. Lannes, "Modeling shallow water waves", 2020. Online at https://arxiv.org/abs/2001.09655.

[20] C. de Lellis, T. Kappeler and P. Topalov, "Low-regularity solutions of the periodic Camassa–Holm equation", *Commun. Partial Differ. Equations* **32** (2007), no. 1-3, pp. 87–126.

[21] J. Lenells, "Traveling wave solutions of the Camassa–Holm equation", *J. Differ. Equations* **217** (2005), no. 2, pp. 393–430.

[22] A. J. Majda and A. L. Bertozzi, *Vorticity and incompressible flow*, Cambridge Texts in Applied Mathematics, Cambridge University Press, 2002, pp. xii+545.

[23] F. Noisette and F. Sueur, "Uniqueness of Yudovich's solutions to the 2D incompressible Euler equation despite the presence of sources and sinks", 2021. Online at https://arxiv.org/abs/2106.11556.

[24] V. Perrollaz, "Initial boundary value problem and asymptotic stabilization of the Camassa–Holm equation on an interval", *J. Funct. Anal.* **259** (2010), no. 9, pp. 2333–2365.

[25] W. A. Weigant and A. A. Papin, "On the uniqueness of the solution of the flow problem with a given vortex", *Math. Notes* **96** (2014), no. 5-6, pp. 871–877.

[26] V. I. Yudovich, "A two-dimensional non-stationary problem on the flow of an ideal incompressible fluid through a given region", *Mat. Sb., N. Ser.* **64(106)** (1964), pp. 562–588.

[27] G. Zhang, Z. Qiao and F. Liu, "Cusp and smooth solitons of the Camassa–Holm equation under an inhomogeneous boundary condition", *Pac. J. Appl. Math.* **1** (2008), no. 1, pp. 105–121.