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Volker Branding

Classification results for polyharmonic helices in space forms

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Classification results for polyharmonic helices in space forms

Résultats de classification des hélices polyharmoniques dans les espaces formes

Volker Branding [a]

[a] University of Vienna, Faculty of Mathematics Oskar-Morgenstern-Platz 1, 1090 Vienna, Austria

E-mail: volker.branding@univie.ac.at

Abstract. We derive various classification results for polyharmonic helices, which are polyharmonic curves whose geodesic curvatures are all constant, in space forms. We obtain a complete classification of triharmonic helices in spheres of arbitrary dimension. Moreover, we show that polyharmonic helices of arbitrary order with non-zero geodesic curvatures to space forms of negative curvature must be geodesics.

Résumé. Nous obtenons divers résultats de classification des hélices polyharmoniques, c'est à dire des courbes polyharmoniques dont les courbures géodésiques sont toutes constantes, dans les espaces formes. Nous obtenons une classification complète des hélices triharmoniques sur les sphères de dimension arbitraire. De plus, nous montrons que les hélices polyharmoniques d'ordre arbitraire à courbure géodésique non nulle dans des espaces formes de courbure négative sont des géodésiques.

Keywords. r-harmonic curves, helices, space form.

Mots-clés. Courbes r-harmonique, hélices, espaces formes.

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1. Introduction and Results

One of the most important objects on a given Riemannian manifold are *geodesics* as they are the curves with minimal distance between two given points. The most practical way to deal with geodesics is to employ a variational approach. Hence, we consider a curve $\gamma\colon I \to M$, where $I \subset \mathbb{R}$ represents an interval, (M, g) a Riemannian manifold and by s we represent the parameter of the curve γ. Moreover, we use the notation $\gamma' = \frac{\mathrm{d}\gamma}{\mathrm{d}s}$. Then, we define the energy of the curve γ by

$$E_1(\gamma) = E(\gamma) := \frac{1}{2} \int_I |\gamma'|^2 \mathrm{d}s. \tag{1.1}$$

The critical points of (1.1) are precisely *geodesics* and are characterized by the equation

$$\tau(\gamma) := \nabla_{\gamma'} \gamma' = 0,$$

which is a second order non-linear ordinary differential equation for the curve γ. The existence of geodesics on a Riemannian manifold is guaranteed due to the celebrated Theorem of *Hopf–Rinow.*

Another interesting class of curves can be obtained by extremizing the *bienergy* of a curve γ which is given by

$$E_2(\gamma) := \frac{1}{2} \int_I |\tau(\gamma)|^2 \mathrm{d}s. \tag{1.2}$$

The critical points of this energy are called *biharmonic curves* and are characterized by a non-linear ordinary differential equation of fourth order. An important aspect of biharmonic curves is the fact that they always have constant geodesic curvature, we refer to [2] for references and the current status of research on biharmonic curves.

A higher order generalization of both (1.1) and (1.2) is provided by the r-energy of a curve

$$E_r(\gamma) := \frac{1}{2} \int_I |\nabla_T^{r-1} T|^2 \mathrm{d}s, \tag{1.3}$$

where $T = \gamma'$ is the tangent vector of γ.

The critical points of (1.3) were calculated in [8, 14] and are characterized by the equation

$$0 = \tau_r(\gamma) = \nabla_T^{2r-1} T + \sum_{l=0}^{r-2} (-1)^l R^M (\nabla_T^{2r-3-l} T, \nabla_T^l T) T \tag{1.4}$$

with R^M being the curvature tensor of the manifold M.

Solutions of (1.4) are called *polyharmonic curves of order r* or shortly *r-harmonic curves*. In the case of $r = 1$ the energy (1.3) reduces to the usual energy of a curve (1.1) whose critical points are *geodesics*. Clearly, every geodesic is a solution of the equation for r-harmonic curves (1.4), hence we are interested in finding non-geodesic solutions of (1.4) which we will call *proper r-harmonic curves.*

For the current status of research on higher order variational problems we refer to [3], a collection of recent results on r-harmonic curves can be found in [1]. We want to point out that r-harmonic curves have important applications in template matching and in the modelling of robot motions, see for example [7].

Throughout this article we will use the following notation. By s we will denote the parameter of the curve γ, the first, second and third derivative of γ will be written as $T := \gamma', \gamma''$ and γ''', respectively. The l-th derivative of γ with respect to s will be denoted by $\gamma^{(l)}$ where $l = 4,\ldots,2r$.

Moreover, we use the terminology *helix* to represent a curve whose geodesic curvatures $k_i, i = 1,\ldots$ are all constant.

Our first result is the explicit form of the Euler–Lagrange equation for 4-harmonic curves on the Euclidean sphere with the round metric.

Theorem 1. *Let $\gamma\colon I \to \mathbb{S}^n \subset \mathbb{R}^{n+1}$ be a curve which is parametrized by arclength. Then γ is a proper 4-harmonic curve if it is a non-geodesic solution of*

$$0 = \gamma^{(8)} + 2\gamma^{(6)} + 3\gamma^{(4)} - \gamma^{(4)}|\gamma''|^2 - 6\gamma''|\gamma''|^2 + 4\gamma'' - 2\gamma''\langle\gamma^{(4)},\gamma''\rangle$$
$$+ 5\frac{d^2}{ds^2}\big(\gamma'\langle\gamma^{(4)},\gamma'\rangle\big) - \frac{d^4}{ds^4}\big(|\gamma''|^2\gamma\big) - 6\gamma'\frac{d}{ds}|\gamma''|^2 - 2\gamma'\frac{d}{ds}\langle\gamma^{(4)},\gamma''\rangle - 5\frac{d}{ds}\big(\gamma''\langle\gamma^{(4)},\gamma'\rangle\big)$$
$$- \gamma\big(\langle\gamma^{(8)},\gamma\rangle + 2\langle\gamma^{(6)},\gamma\rangle + 3|\gamma''|^2 - |\gamma''|^4 + 6|\gamma''|^2 - 4 + 2\langle\gamma^{(4)},\gamma''\rangle\big)$$
$$- \gamma\left(5\left\langle\gamma,\frac{d^2}{ds^2}\big(\gamma'\langle\gamma^{(4)},\gamma'\rangle\big)\right\rangle - \left\langle\gamma,\frac{d^4}{ds^4}\big(|\gamma''|^2\gamma\big)\right\rangle - 5\left\langle\gamma,\frac{d\,d}{ds}\big(\gamma''\langle\gamma^{(4)},\gamma'\rangle\big)\right\rangle\right). \tag{1.5}$$

Remark 2. One explicit solution of (1.5) can be obtained as follows: The curve $\gamma\colon I \to \mathbb{S}^n$ given by

$$\gamma(s) = \cos(\sqrt{4}s)e_1 + \sin(\sqrt{4}s)e_2 + e_3,$$

where $e_i, i = 1, 2, 3$ are mutually perpendicular and satisfy $|e_1|^2 = |e_2|^2 = \frac{1}{4}, |e_3|^2 = \frac{3}{4}$, is a proper 4-harmonic curve which is parametrized by arclength. The existence of this particular 4-harmonic curve was already established in [1, Theorem 1.5] without using the Euler–Lagrange equation (1.5).

Our next result provides a characterization of triharmonic helices in the sphere extending the analysis presented in [1].

Theorem 3. *Consider a curve* $\gamma \colon I \to \mathbb{S}^n \subset \mathbb{R}^{n+1}$ *of the form*

$$\gamma(s) = \cos(as)e_1 + \sin(as)e_2 + \cos(bs)e_3 + \sin(bs)e_4$$

with $|e_1|^2 = |e_2|^2, |e_3|^2 = |e_4|^2$ *and* $|e_1|^2 + |e_3|^2 = 1$. *Then,* γ *is a proper triharmonic curve parametrized by arclength if the following algebraic relations hold*

$$
\begin{aligned}
a^4 + b^4 - 4(a^2 + b^2) + 3a^2 b^2 + 3 &= 0, \\
|e_1|^2 a^2 + |e_3|^2 b^2 &= 1.
\end{aligned}
\tag{1.6}
$$

Remarks 4.

(1) The condition (1.6) has already been derived in [1, Equation 2.3] using a different approach as utilized in this manuscript.

(2) Setting $a^2 = x$ and $b^2 = y$ the equation (1.6) describes a particular conic section which turns out to be a hyperbola. Hence, it is obvious that there is a whole family of triharmonic helices in the sphere.

It is well-known that a polyharmonic curve of order r has $2r - 2$ non-zero geodesic curvatures and effectively lies on a target of dimension $2r - 1$. Exploiting this fact the next Theorem gives some further characterizations of r-harmonic helices in the cases $r = 3, 4$. From a computational point of view it turns out to be more effective to work with the geodesic curvatures of a curve instead of trying to explicitly solve the Euler–Lagrange equation.

Theorem 5. *Let* $\gamma \colon I \to M$ *be a proper* r-*harmonic curve parametrized by arclength where* M *is a space form of constant curvature* K. *Moreover, assume that the geodesic curvatures* $k_i, i = 1, \ldots, 2r - 2$ *are all constant.*

(1) *If* $r = 3$ *the geodesic curvatures* $k_j, j = 1, \ldots 4$ *satisfy*

$$
\begin{aligned}
(k_1^2 + k_2^2)^2 + k_2^2 k_3^2 &= K(2k_1^2 + k_2^2), \\
k_2 k_3 (k_1^2 + k_2^2 + k_3^2 + k_4^2) &= k_2 k_3 K.
\end{aligned}
\tag{1.7}
$$

(2) *If* $r = 4$ *the geodesic curvatures* $k_j, j = 1, \ldots 6$ *satisfy*

$$
\begin{aligned}
(k_1^2 + k_2^2)^3 + k_2^2 k_3^2 (2k_1^2 + 2k_2^2 + k_3^2 + k_4^2) & \\
&= K\big((k_1^2 + k_2^2)^2 + k_2^2 k_3^2\big) + 2K k_1^2 (k_1^2 + k_2^2), \\
k_2 k_3 \big((k_1^2 + k_2^2)^2 + (k_3^2 + k_4^2)^2 + k_1^2 k_3^2 + 2k_2^2 k_3^2 + k_4^2 (k_1^2 + k_2^2 + k_5^2)\big) & \\
&= k_2 k_3 K(2k_1^2 + k_2^2 + k_3^2 + k_4^2), \\
k_2 k_3 k_4 k_5 \big(k_1^2 + k_2^2 + k_3^2 + k_4^2 + k_5^2 + k_6^2\big) & \\
&= k_2 k_3 k_4 k_5 K.
\end{aligned}
\tag{1.8}
$$

Remarks 6.

(1) Note that in (1.7) and (1.8) there does not appear a factor of k_1. As we are considering proper tri- and 4-harmonic curves we have that $k_1 \neq 0$ such that this factor can be split off.

(2) In the case of $k_1, k_2 \neq 0$ and $k_j = 0$, $j \geq 3$ the first equation of (1.7) and the first equation of (1.8) reduce to the formula obtained in [1, Theorem 1.1].

(3) An immediate consequence of Theorem 5 is that triharmonic and 4-harmonic helices whose geodesic curvatures are all non-zero need to be geodesics if the target is a space form of non-positive curvature. More precisely, for triharmonic helices the first equality of (1.7) leads to a contradiction if K is non-positive, while for 4-harmonic helices (1.8) gives a contradiction whenenver K is non-positive.

Employing Theorem 5 we can deduce the following classification result:

Theorem 7. *A proper triharmonic helix* $\gamma \colon I \to \mathbb{S}^n \subset \mathbb{R}^{n+1}$ *parametrized by arclength must be one of the following:*

(1) *A circle of the form ($n \geq 2$)*

$$\gamma(s) = \cos(\sqrt{3}s)e_1 + \sin(\sqrt{3}s)e_2 + e_3,$$

where $e_i, i = 1, 2, 3$ *are mutually perpendicular and satisfy* $|e_1|^2 = |e_2|^2 = \frac{1}{3}, |e_3|^2 = \frac{2}{3}$.

(2) *A non-planar curve of the form ($n \geq 3$)*

$$\gamma(s) = \cos(as)e_1 + \sin(as)e_2 + \cos(bs)e_3 + \sin(bs)e_4$$

with $|e_1|^2 = |e_2|^2, |e_3|^2 = |e_4|^2, |e_1|^2 + |e_3|^2 = 1$ *and* $e_i, i = 1, \ldots, 4$ *mutually perpendicular satisfying*

$$a^4 + b^4 - 4(a^2 + b^2) + 3a^2b^2 + 3 = 0, \qquad |e_1|^2 a^2 + |e_3|^2 b^2 = 1.$$

Remark 8. It is quite remarkable that triharmonic helices on the sphere have the same structure as biharmonic curves on the sphere. In both cases there exist two families of the form detailed in the previous theorem, which consist of a planar family and a non-planar generalization, see Theorem 17 for the precise details on biharmonic curves. However, biharmonic curves necessarily have constant curvature while there may be triharmonic curves on the sphere of non-constant geodesic curvature. In order to obtain a complete classification of triharmonic curves on the sphere one would need to obtain a full understanding of the non-constant curvature case as well.

The last result of this manuscript provides a characterization of polyharmonic helices of arbitrary order whose geodesic curvatures are all different from zero.

Theorem 9. *Let* $\gamma \colon I \to M$ *be an* r-*harmonic curve parametrized by arclength whose geodesic curvatures* $k_i, i = 1, \ldots, 2r - 2$ *are all constant and non-zero. Moreover, suppose that* M *is a space form of constant curvature* K. *Then, the following equation holds*

$$\sum_{j=1}^{2r-2} k_j^2 = K.$$

Remark 10. The previous Theorem gives further insights into the structure of higher order variational problems. In particular, it gives further evidence to support the claim that polyharmonic maps to space forms of negative curvature must be harmonic, by showing that there do not exist proper r-harmonic helices when K is non-positive, while there may be additional solutions in the case of a spherical target. So far, this observations was mostly made in the case of codimension one, that is for polyharmonic hypersurfaces, see for example [11]. The analysis above suggests that this fact stays true in higher codimension.

Furthermore, we will collect a number of results giving rise to the following conjecture:

Conjecture 11. *The equation for r-harmonic curves (1.4) admits solutions with non-constant geodesic curvature*

$$k_1(s) = \frac{\alpha}{s^{r-2}}, \qquad \alpha \in \mathbb{R}, \alpha \neq 0, \tag{1.9}$$

where s represents the parameter of the curve γ which we assume to be parametrized by arclength.

This conjecture is based on

(1) the well-known fact that biharmonic curves ($r = 2$) necessarily have constant geodesic curvature,
(2) the results of the recent article on triharmonic curves [12],
(3) and the observations presented in Subsection 3.1.

This article is organized as follows: In Section 2 we provide some background material on the Euler–Lagrange method and use it to reprove a number of well-known results on biharmonic curves on the Euclidean sphere. Finally, in Section 3 we give the proofs of the main results of the article.

2. Some preliminary results

Throughout this article we consider a space form of constant curvature K in which case the Riemann curvature tensor acquires the simple form

$$R(X, Y)Z = K(\langle Y, Z \rangle X - \langle X, Z \rangle Y),$$

where X, Y, Z are vector fields and K represents the constant curvature of the space form.

In this case the equation for polyharmonic curves (1.4) simplifies to

$$\tau_r(\gamma) = \nabla_T^{2r-1} T + K \sum_{l=0}^{r-2} (-1)^l \big(\langle T, \nabla_T^l T \rangle \nabla_T^{2r-3-l} T - \langle T, \nabla_T^{2r-3-l} T \rangle \nabla_T^l T \big). \tag{2.1}$$

2.1. *The Euler–Lagrange method for polyharmonic curves*

Let us briefly recall the so-called *Euler–Lagrange method* which is a powerful tool in the analysis of one-dimensional variational problems. This method is the cornerstone of the Lagrangian formulation of classical mechanics in theoretical physics, see for example [9, Chapter 7]. Moreover, this method can also successfully be applied in order to study biharmonic curves [6], biharmonic maps [10, 13] and also higher order variational problems [3, Theorem 4.5].

The following theorem may be well-known in the mathematics community. However, for the sake of completeness we also provide a complete proof below.

Theorem 12. *Let $\gamma \colon I \to \mathbb{R}^q$ be a curve. Suppose we have an energy functional*

$$E_r(\gamma) = \int_I \mathscr{L}_r \, \mathrm{d}s,$$

where the Lagrangian

$$\mathscr{L}_r = \mathscr{L}_r(\gamma, \gamma', \dots, \gamma^{(r-1)}, \gamma^{(r)})$$

may depend on the derivatives of the curve γ up to order r.

Then, γ is a critical point of $E_r(\gamma)$ if the following ordinary differential equation holds

$$\sum_{l=1}^{r} (-1)^l \frac{\mathrm{d}^l}{\mathrm{d}s^l} \frac{\partial \mathscr{L}_r}{\partial \gamma^{(l)}} + \frac{\partial \mathscr{L}_r}{\partial \gamma} = 0. \tag{2.2}$$

Proof. We choose $\beta \in C_c^\infty(I, \mathbb{R}^q)$ and compute the first variation of $E_r(\gamma)$ as follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} E_r(\gamma + t\beta) = \frac{\mathrm{d}}{\mathrm{d}t} \int_I \mathscr{L}_r(\gamma + t\beta, \gamma' + t\beta', \gamma'' + t\beta'', \ldots, \gamma^{(r-1)} + t\beta^{(r-1)}, \gamma^{(r)} + t\beta^{(r)}) \, \mathrm{d}s\Big|_{t=0}$$

$$= \int_I \frac{\mathrm{d}}{\mathrm{d}t}\big(\mathscr{L}_r(\gamma + t\beta, \gamma' + t\beta', \gamma'' + t\beta'', \ldots, \gamma^{(r-1)} + t\beta^{(r-1)}, \gamma^{(r)} + t\beta^{(r)})\big) \, \mathrm{d}s\Big|_{t=0}$$

$$= \int_I \left(\frac{\partial \mathscr{L}_r}{\partial \gamma} \beta + \frac{\partial \mathscr{L}_r}{\partial \gamma'} \beta' + \frac{\partial \mathscr{L}_r}{\partial \gamma''} \beta'' + \ldots + \frac{\partial \mathscr{L}_r}{\partial \gamma^{(r-1)}} \beta^{(r-1)} + \frac{\partial \mathscr{L}_r}{\partial \gamma^{(r)}} \beta^{(r)}\right) \mathrm{d}s.$$

Now, we use integration by parts

$$\int_I \frac{\partial \mathscr{L}_r}{\partial \gamma^{(p)}} \beta^{(p)} \, \mathrm{d}s = \int_I (-1)^p \frac{\mathrm{d}^p}{\mathrm{d}s^p}\left(\frac{\partial \mathscr{L}_r}{\partial \gamma^{(p)}}\right) \beta \, \mathrm{d}s, \qquad 1 \le p \le r,$$

where we used that β is compactly supported. By combining both equations the proof is complete. $\qquad\square$

In the following we will often make use of the following lemma which follows from a direct calculation.

Lemma 13. *Let* $\gamma \colon I \to \mathbb{S}^n \subset \mathbb{R}^{n+1}$ *be a curve which is parametrized by arclength. Then the following identities hold*

$$\langle \gamma, \gamma' \rangle = 0, \quad \langle \gamma'', \gamma \rangle = -1, \quad \langle \gamma''', \gamma \rangle = 0, \quad \langle \gamma', \gamma'' \rangle = 0, \quad \langle \gamma^{(4)}, \gamma \rangle + \langle \gamma''', \gamma' \rangle = 0, \quad \langle \gamma^{(4)}, \gamma \rangle = |\gamma''|^2.$$

Throughout this section we will frequently make use of the inclusion map $\iota \colon \mathbb{S}^n \to \mathbb{R}^{n+1}$ and also exploit the special structure of the Levi–Civita connection on the sphere

$$\mathrm{d}\iota(\nabla_T X) = X' + \langle X, \gamma' \rangle \gamma,$$

where X is a vector field on $\mathbb{S}^n \subset \mathbb{R}^{n+1}$.

2.2. *Biharmonic curves on the sphere*

In order to highlight the power of the Euler–Lagrange method we will first investigate biharmonic curves on the Euclidean sphere and give a new-proof of some well-known results which serves as an inspiration for the classification results on triharmonic helices presented in this manuscript.

The intrinsic form of the equation for biharmonic curves on the sphere is given by

$$\tau_2(\gamma) = \nabla_T^3 T + |T|^2 \nabla_T T - \langle T, \nabla_T T \rangle T.$$

Assuming that $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ and considering a curve $\gamma \colon I \to \mathbb{S}^n \subset \mathbb{R}^{n+1}$ we obtain the following Lagrangian for biharmonic curves on the sphere

$$\mathscr{L}_2^{\mathbb{S}^n}(\gamma'', \gamma', \gamma) = |\gamma''|^2 - |\gamma'|^4 + \lambda(|\gamma|^2 - 1). \tag{2.3}$$

Note that we have to include the Lagrange multiplyer λ as we are constraining the curve γ to be on the unit sphere.

Then, employing Theorem 12, a direct calculation shows that the critical points of (2.3) are given by

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\left(\frac{\partial \mathscr{L}_2^{\mathbb{S}^n}}{\partial \gamma''}\right) - \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial \mathscr{L}_2^{\mathbb{S}^n}}{\partial \gamma'}\right) + \frac{\partial \mathscr{L}_2^{\mathbb{S}^n}}{\partial \gamma} = 2(\gamma^{(4)} + 2(|\gamma'|^2)' \gamma' + 2|\gamma'|^2 \gamma'' + \lambda \gamma).$$

Taking also into account the variation of $\mathscr{L}_2^{\mathbb{S}^n}$ with respect to the Lagrange multiplier λ we obtain the Euler–Lagrange equation

$$\gamma^{(4)} + 2(|\gamma'|^2)' \gamma' + 2|\gamma'|^2 \gamma'' + \lambda \gamma = 0 \tag{2.4}$$

together with the constraint $|\gamma|^2 = 1$.

From now on, we will assume that the curve γ is parametrized with respect to arclength, that is $|\gamma'|^2 = 1$, such that (2.4) simplifies to

$$\gamma^{(4)} + 2\gamma'' + \lambda\gamma = 0. \tag{2.5}$$

In order to determine λ we test (2.5) with γ and find

$$\lambda = -\langle\gamma^{(4)}, \gamma\rangle - 2\langle\gamma, \gamma''\rangle = -|\gamma''|^2 + 2,$$

where we used the identities provided by Lemma 13 and thus exploited the fact that γ is parametrized with respect to arclength.

It is well-known that biharmonic curves have constant geodesic curvature which, using our framework, can be seen as follows:

Remarks 14.

(1) It is easy to see that the Lagrange multiplyer λ and the geodesic curvature k_1 of the curve γ are related via the identity

$$\lambda = -|\gamma''|^2 + 2 = -k_1^2 + 1.$$

Hence, the inclusion of the Lagrange multiplyer λ in (2.3) has the effect that it forces the curve γ to have constant geodesic curvature.

This fact is well-known and is usually deduced by choosing a Frenet-frame for the curve γ and analyzing the associated Frenet equations.

(2) We will present another short argument why biharmonic curves always need to have constant geodesic curvature that holds for biharmonic curves on an arbitrary manifold.

Suppose we have a biharmonic curve on a Riemannian manifold, then it satisfies

$$\tau_2(\gamma) = \nabla_T^3 T + R(\nabla_T T, T)T = 0.$$

Multiplying this equation with T we obtain

$$\langle\nabla_T^3 T, T\rangle = 0$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}s}\langle\nabla_T^2 T, T\rangle - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}|\nabla_T T|^2 = 0.$$

Exploiting that the curve is parametrized with respect to arclength we can then deduce that

$$\frac{\mathrm{d}}{\mathrm{d}s}|\nabla_T T|^2 = 0,$$

which implies that $k_1^2 = \text{const}.$

Combining the previous observations we get the following well-known result:

Proposition 15. *Let* $\gamma\colon I \to \mathbb{S}^n \subset \mathbb{R}^{n+1}$ *be a curve parametrized by arclength. Then* γ *is biharmonic if*

$$\gamma^{(4)} + 2\gamma'' + \gamma(2 - |\gamma''|^2) = 0. \tag{2.6}$$

Remarks 16.

(1) The equation for biharmonic curves on spheres (2.6) was first derived in [5, Corollary 4.2] making use of geometric methods. In that reference the equation for biharmonic curves to spheres is given in the following form

$$\gamma^{(4)} + 2\gamma'' + \gamma(1 - k_1^2) = 0,$$

where k_1 represents the geodesic curvature of the curve γ. Noting that

$$k_1^2 = |\nabla_T T|^2 = |\gamma''|^2 - |\gamma'|^4 = |\gamma''|^2 - 1$$

it is obvious that this version is the same as (2.6).

(2) We would like to point out that it is necessary to include the Lagrange multiplyer λ in the Lagrangian (2.3) as we are dealing with a constraint variational problem. However, we do not have to include a second Lagrange multiplyer to justify that the curve is parametrized with respect to arclength. The fact that we are choosing an arclength parametrization can be considered as making a convenient choice in order to simplify our calculations but it is not a constraint required from the actual variational problem.

The following result was proved in [4] and [5, Proposition 4.4].

Theorem 17. *Let $\gamma\colon I \to \mathbb{S}^n \subset \mathbb{R}^{n+1}$ be a curve parametrized by arclength. Then there exist the following two classes of proper biharmonic curves on $\mathbb{S}^n$:*

(1) *When $k_1^2 = 1$ these are circles parametrized by*

$$\gamma(s) = \cos(\sqrt{2}s)e_1 + \sin(\sqrt{2}s)e_2 + e_3, \tag{2.7}$$

where $e_i, i = 1,2,3$ are constant orthogonal vectors satisfying $|e_1|^2 = |e_2|^2 = |e_3|^2 = \frac{1}{2}$.

(2) *When $0 < k_1^2 < 1$ they are non-planar curves parametrized as follows*

$$\gamma(s) = \cos(as)e_1 + \sin(as)e_2 + \cos(bs)e_3 + \sin(bs)e_4, \tag{2.8}$$

where $|e_i|^2 = \frac{1}{2}, i = 1,\dots,4$ and $a^2 + b^2 = 2$ with $a^2 \neq b^2$.

In the following we will give a different proof of Theorem 17 as was originally presented in [4, 5], making use of the Euler–Lagrange method, as we want to employ it frequently in the rest of this article.

Proof of Theorem 17. In order to find the first class of solutions (2.7) we make the ansatz

$$\gamma(s) = \cos(as)e_1 + \sin(as)e_2 + e_3,$$

where $e_i, i = 1,2,3$ are constant orthogonal vectors satisfying $|e_1|^2 = |e_2|^2$ and $|e_1|^2 + |e_3|^2 = 1$ as we require $|\gamma|^2 = 1$ and $a \in \mathbb{R}$. In the following we set $\alpha^2 := |e_1|^2$. Inserting this ansatz into the Lagrangian for biharmonic curves (2.3) we find

$$\mathcal{L}_2^{\mathbb{S}^n}(\alpha) = a^4(\alpha^2 - \alpha^4).$$

To determine the critical points of $\mathcal{L}_2^{\mathbb{S}^n}(\alpha)$ we calculate

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} \mathcal{L}_2^{\mathbb{S}^n}(\alpha) = 2a^4 \alpha(1 - 2\alpha^2)$$

and it is clear that this expression vanishes if $\alpha^2 = \frac{1}{2}$. Finally, we use the fact that γ is parametrized with respect to arclength which, given our ansatz, is expressed via $\alpha^2 a^2 = 1$. Hence, we get that $a^2 = 2$ completing the proof.

In order to obtain the second class of solutions (2.8) we make the ansatz

$$\gamma(s) = \cos(as)e_1 + \sin(as)e_2 + \cos(bs)e_3 + \sin(bs)e_4,$$

where $|e_1|^2 = |e_2|^2, |e_3|^2 = |e_4|^2$. We set $\alpha_j^2 := |e_j|^2, j = 1,3$. Inserting this ansatz into the Lagrangian for biharmonic curves (2.3) we get

$$\mathcal{L}_2^{\mathbb{S}^n}(\alpha_1, \alpha_3, \lambda) = a^4(\alpha_1^2 - \alpha_1^4) + b^4(\alpha_3^2 - \alpha_3^4) - 2a^2 b^2 \alpha_1^2 \alpha_3^2 + \lambda(\alpha_1^2 + \alpha_3^2 - 1).$$

In order to find the critical points of this Lagrangian we differentiate with respect to $\alpha_1, \alpha_3, \lambda$ and set the resulting equations equal to zero leading to the system

$$a^4(1 - 2\alpha_1^2) - 2a^2 b^2 \alpha_3^2 - \lambda = 0,$$
$$b^4(1 - 2\alpha_3^2) - 2a^2 b^2 \alpha_1^2 - \lambda = 0,$$
$$\alpha_1^2 + \alpha_3^2 = 1.$$

Combining this set of equations we find after some algebraic manipulations

$$(a^4 + b^4 - 2a^2 b^2)(1 - 2\alpha_1^2) = 0$$

from which we deduce that $\alpha_1^2 = \alpha_3^2 = \frac{1}{2}$. The requirement that γ is parametrized with respect to arclength is then given by the constraint $a^2 + b^2 = 2, a^2 \neq b^2$ completing the proof. $\qquad\square$

3. Proofs of the main results

In this section we provide the proofs of the main results of this article.

Proof of Theorem 1. The proof is based on the Euler–Lagrange method provided by Theorem 12. Recall that the 4-energy of a curve $\gamma\colon I \to \mathbb{S}^n$ is given by

$$E_4(\gamma) = \int_I |\nabla_T^3 T|^2 \, \mathrm{d}s.$$

We again make use of the embedding $\iota\colon \mathbb{S}^n \to \mathbb{R}^{n+1}$ which helps us rewrite

$$\mathrm{d}\iota(\nabla_T^3 T) = \gamma^{(4)} + 4\langle \gamma''', \gamma' \rangle \gamma + 3|\gamma''|^2 \gamma + 5\langle \gamma', \gamma'' \rangle \gamma' + |\gamma'|^2 \gamma'' + |\gamma'|^4 \gamma.$$

Consequently, the Lagrangian associated with the 4-energy for a curve on the sphere has the form

$$\begin{aligned}
\mathscr{L}_4^{\mathbb{S}^n}(\gamma^{(4)}, \gamma''', \gamma'', \gamma', \gamma) = {}& |\gamma^{(4)}|^2 + 16|\langle \gamma''', \gamma' \rangle|^2 + 9|\gamma''|^4 + 35|\langle \gamma'', \gamma' \rangle|^2 |\gamma'|^2 \\
& + |\gamma'|^4 |\gamma''|^2 - |\gamma'|^8 \\
& + 8\langle \gamma''', \gamma' \rangle \langle \gamma^{(4)}, \gamma \rangle + 6\langle \gamma^{(4)}, \gamma \rangle |\gamma''|^2 + 10\langle \gamma'', \gamma' \rangle \langle \gamma^{(4)}, \gamma' \rangle \\
& + 2|\gamma'|^2 \langle \gamma^{(4)}, \gamma'' \rangle + 2|\gamma'|^4 \langle \gamma^{(4)}, \gamma \rangle \\
& + 24\langle \gamma''', \gamma' \rangle |\gamma''|^2 \\
& + \lambda(|\gamma|^2 - 1),
\end{aligned} \tag{3.1}$$

where we again introduced the Lagrange multiplyer λ to constrain the curve γ to $\mathbb{S}^n$.

By a direct calculation we find taking into account that the curve γ is parametrized with respect to arclength

$$\frac{\mathrm{d}^4}{\mathrm{d}s^4} \frac{\partial \mathscr{L}_4^{\mathbb{S}^n}}{\partial \gamma^{(4)}} = 2\gamma^{(8)} - 2\frac{\mathrm{d}^4}{\mathrm{d}s^4}\big(|\gamma''|^2 \gamma\big) + 2\gamma^{(6)} + 2\gamma^{(4)},$$

$$\frac{\mathrm{d}^3}{\mathrm{d}s^3}\left(\frac{\partial \mathscr{L}_4^{\mathbb{S}^n}}{\partial \gamma'''}\right) = 0,$$

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\left(\frac{\partial \mathscr{L}_4^{\mathbb{S}^n}}{\partial \gamma''}\right) = 2\gamma^{(6)} + 2\gamma^{(4)} + 10\frac{\mathrm{d}^2}{\mathrm{d}s^2}\big(\gamma'\langle \gamma^{(4)}, \gamma' \rangle\big),$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial \mathscr{L}_4^{\mathbb{S}^n}}{\partial \gamma'}\right) = 12\gamma''|\gamma''|^2 - 8\gamma'' + 4\gamma''\langle \gamma^{(4)}, \gamma'' \rangle + 12\gamma'\frac{\mathrm{d}}{\mathrm{d}s}|\gamma''|^2 + 4\gamma'\frac{\mathrm{d}}{\mathrm{d}s}\langle \gamma^{(4)}, \gamma'' \rangle + 10\frac{\mathrm{d}}{\mathrm{d}s}\big(\gamma''\langle \gamma^{(4)}, \gamma' \rangle\big),$$

$$\frac{\partial \mathscr{L}_4^{\mathbb{S}^n}}{\partial \gamma} = -2\gamma^{(4)}|\gamma''|^2 + 2\gamma^{(4)} + 2\lambda\gamma.$$

Varying (3.1) with respect to the Lagrange multiplyer λ we obtain the constraint $|\gamma|^2 = 1$. Hence, from Theorem 12 we can deduce that

$$\begin{aligned}
0 = {}& \gamma^{(8)} + 2\gamma^{(6)} + 3\gamma^{(4)} - \gamma^{(4)}|\gamma''|^2 - 6\gamma''|\gamma''|^2 + 4\gamma'' - 2\gamma''\langle \gamma^{(4)}, \gamma'' \rangle \\
& + 5\frac{\mathrm{d}^2}{\mathrm{d}s^2}\big(\gamma'\langle \gamma^{(4)}, \gamma' \rangle\big) - \frac{\mathrm{d}^4}{\mathrm{d}s^4}\big(|\gamma''|^2 \gamma\big) - 6\gamma'\frac{\mathrm{d}}{\mathrm{d}s}|\gamma''|^2 - 2\gamma'\frac{\mathrm{d}}{\mathrm{d}s}\langle \gamma^{(4)}, \gamma'' \rangle - 5\frac{\mathrm{d}}{\mathrm{d}s}\big(\gamma''\langle \gamma^{(4)}, \gamma' \rangle\big) + \lambda\gamma.
\end{aligned}$$

In order to determine λ we form the scalar product with γ, using the identifies provided by Lemma 13 and inserting back into the above equation completes the proof. $\qquad\square$

As the proof of Theorem 3 is based on the Lagrangian for triharmonic curves we again use the embedding of $\mathbb{S}^n$ into $\mathbb{R}^{n+1}$ via the map ι and find

$$d\iota(\nabla_T^2 T) = \gamma''' + 3\langle\gamma'', \gamma'\rangle\gamma + |\gamma'|^2\gamma'.$$

Thus, we obtain the following Lagrangian

$$\mathscr{L}_3^{\mathbb{S}^n}(\gamma, \gamma', \gamma'', \gamma''') = |\gamma'''|^2 + 9|\langle\gamma'', \gamma'\rangle|^2 + |\gamma'|^6 + 6\langle\gamma'', \gamma'\rangle\langle\gamma''', \gamma\rangle + 2|\gamma'|^2\langle\gamma', \gamma'''\rangle + \lambda(|\gamma|^2 - 1) \quad (3.2)$$

with the Lagrange multiplyer $\lambda \in \mathbb{R}$.

In order to prove Theorem 3 we first establish the following

Proposition 18. *Consider a curve $\gamma\colon I \to \mathbb{S}^n$ of the form*

$$\gamma(s) = \cos(as)e_1 + \sin(as)e_2 + \cos(bs)e_3 + \sin(bs)e_4 \quad (3.3)$$

with $|e_1|^2 = |e_2|^2, |e_3|^2 = |e_4|^2$. Then, γ is a proper triharmonic curve parametrized by arclength if the following algebraic relations hold

$$a^6(1 - 2\alpha_1^2) - 2a^4 + 3a^2 - 2a^2 b^4 \alpha_3^2 + \lambda = 0, \quad (3.4)$$
$$b^6(1 - 2\alpha_3^2) - 2b^4 + 3b^2 - 2b^2 a^4 \alpha_1^2 + \lambda = 0,$$
$$a^2\alpha_1^2 + b^2\alpha_3^2 = 1,$$
$$\alpha_1^2 + \alpha_3^2 = 1,$$

whenever $a^2, b^2 \neq 1$. Here, $\lambda \in \mathbb{R}$ and we have set $\alpha_j^2 = |e_j|^2, j = 1, 3$.

Proof. From the ansatz (3.3) we get

$$|\gamma^{(l)}|^2 = \alpha_1^2 a^{2l} + \alpha_3^2 b^{2l}, \qquad l = 1, 2, 3.$$

Inserting into the Lagrangian for triharmonic curves (3.2) we obtain

$$\mathscr{L}_3^{\mathbb{S}^n}(\alpha_1, \alpha_3, \lambda) = \alpha_1^2 a^6 + \alpha_3^2 b^6 + (\alpha_1^2 a^2 + \alpha_3^2 b^2)^3 - 2(\alpha_1^2 a^2 + \alpha_3^2 b^2)(\alpha_1^2 a^4 + \alpha_3^2 b^4) + \lambda(\alpha_1^2 + \alpha_3^2 - 1)$$
$$= a^6(\alpha_1^2 - 2\alpha_1^4 + \alpha_1^6) + b^6(\alpha_3^2 - 2\alpha_3^4 + \alpha_3^6) + a^2 b^4(3\alpha_1^2\alpha_3^4 - 2\alpha_1^4\alpha_3^2)$$
$$+ a^4 b^2(3\alpha_1^4\alpha_3^2 - 2\alpha_1^2\alpha_3^2) + \lambda(\alpha_1^2 + \alpha_3^2 - 1).$$

The critical points of $\mathscr{L}_3^{\mathbb{S}^n}(\alpha_1, \alpha_3, \lambda)$ are given by the set of equations

$$0 = a^6(1 - 4\alpha_1^2 + 3\alpha_1^4) + a^2 b^4(3\alpha_3^4 - 2\alpha_3^2) + a^4 b^2(6\alpha_1^2\alpha_3^2 - 2\alpha_3^2) + \lambda, \quad (3.5)$$
$$0 = b^6(1 - 4\alpha_3^2 + 3\alpha_3^4) + a^2 b^4(6\alpha_1^2\alpha_3^2 - 2\alpha_1^2) + a^4 b^2(3\alpha_1^4 - 2\alpha_1^2) + \lambda,$$
$$1 = \alpha_1^2 + \alpha_3^2.$$

In addition, we have the following constraint due to the requirement that the curve γ is supposed to be parametrized by arclength

$$a^2\alpha_1^2 + b^2\alpha_3^2 = 1.$$

Using this constraint in the first equation of (3.5) we manipulate

$$0 = a^6(1 - 4\alpha_1^2 + 3\alpha_1^4) + a^4((6\alpha_1^2 - 2)(1 - a^2\alpha_1^2)) + a^2\big(3(1 - a^2\alpha_1^2)^2 - 2b^4\alpha_3^2\big) + \lambda$$
$$= a^6(1 - 2\alpha_1^2) - 2a^4 + 3a^2 - 2a^2 b^4\alpha_3^2 + \lambda.$$

This shows the validity of the first equation in (3.4), the second one can be derived by the same method. The last two equations in (3.4) represent the fact that $|\gamma|^2 = |\gamma'|^2 = 1$. $\qquad\square$

Remark 19. One can easily check that $a = b = 1$ solves the system (3.4) which corresponds to a geodesic solution.

Proof of Theorem 3. Using the first two equations of (3.4) we obtain

$$a^6 - b^6 - 2(a^4 - b^4) + 3(a^2 - b^2) - 2\alpha_1^2 a^4 (a^2 - b^2) - 2\alpha_3^2 b^4 (a^2 - b^2) = 0.$$

Employing the identity

$$a^6 - b^6 = (a^2 - b^2)(a^4 + b^4 + a^2 b^2)$$

and assuming that $a \neq b$ we can thus deduce

$$a^4 + b^4 + a^2 b^2 - 2(a^2 + b^2) + 3 - 2\alpha_1^2 a^4 - 2\alpha_3^2 b^4 = 0.$$

In order to manipulate the last two terms involving α_1^2, α_3^2 we make use of the last two equations of (3.4) as follows

$$
\begin{aligned}
\alpha_1^2 a^4 + \alpha_3^2 b^4 &= a^2(1 - \alpha_3^2 b^2) + b^2(1 - \alpha_1^2 a^2) \\
&= a^2 + b^2 - a^2 b^2 (\alpha_1^2 + \alpha_3^2) \\
&= a^2 + b^2 - a^2 b^2
\end{aligned}
$$

such that we obtain

$$a^4 + b^4 + a^2 b^2 - 2(a^2 + b^2) + 3 - 2\alpha_1^2 a^4 - 2\alpha_3^2 b^4 = a^4 + b^4 - 4(a^2 + b^2) + 3a^2 b^2 + 3$$

yielding the claim. $\qquad\square$

For the further analysis we recall the following

Definition 20 (Frenet-frame). *Let $\gamma\colon I \to M$ be a curve which is parametrized with respect to arclength. Then, its Frenet-frame is defined by*

$$
\begin{aligned}
F_1 &= T, \\
\nabla_T F_1 &= k_1 F_2, \\
\nabla_T F_i &= -k_{i-1} F_{i-1} + k_i F_{i+1}, \qquad i = 2, \dots, n-1, \\
&\ \ \vdots \\
\nabla_T F_n &= -k_{n-1} F_{n-1},
\end{aligned}
\tag{3.6}
$$

where $k_i, i = 1, \dots n-1$ represent the curvatures of the curve γ.

Proof of Theorem 5. In order to prove the first result concerning the classification of triharmonic helices in space forms we note that the equation for triharmonic curves in space forms reads as

$$\nabla_T^5 T + K\nabla_T^3 T - K\langle T, \nabla_T^3 T\rangle T - K\langle T, \nabla_T T\rangle \nabla_T^2 T + K\langle T, \nabla_T^2 T\rangle \nabla_T T = 0. \tag{3.7}$$

A direct calculation using (3.6), assuming that $k_i, i = 1, \dots, 4$ are constant and $k_i = 0, i \geq 5$, shows that

$$
\begin{aligned}
\nabla_T^2 T &= -k_1^2 T + k_1 k_2 F_3, \\
\nabla_T^3 T &= -k_1(k_1^2 + k_2^2) F_2 + k_1 k_2 k_3 F_4, \\
\nabla_T^4 T &= k_1^2(k_1^2 + k_2^2) T - k_1 k_2 (k_1^2 + k_2^2 + k_3^2) F_3 + k_1 k_2 k_3 k_4 F_5, \\
\nabla_T^5 T &= k_1\big((k_1^2 + k_2^2)^2 + k_2^2 k_3^2\big) F_2 - k_1 k_2 k_3 (k_1^2 + k_2^2 + k_3^2 + k_4^2) F_4.
\end{aligned}
$$

Inserting these identities into (3.7) then yields

$$k_1\big((k_1^2 + k_2^2)^2 + k_2^2 k_3^2 - K(2k_1^2 + k_2^2)\big) F_2 - k_1 k_2 k_3 (k_1^2 + k_2^2 + k_3^2 + k_4^2 - K) F_4 = 0.$$

Testing this equation with both F_2, F_4 then completes the first part of the proof.

Concerning the second claim of the theorem, which is the classification of 4-harmonic helices in space forms, we recall that in this case the equation for 4-harmonic curves acquires the form

$$\nabla_T^7 T + K\nabla_T^5 T - K\langle \nabla_T^5 T, T\rangle T - K\langle \nabla_T T, T\rangle \nabla_T^4 T + K\langle \nabla_T^4 T, T\rangle \nabla_T T$$
$$+ K\langle \nabla_T^2 T, T\rangle \nabla_T^3 T - K\langle T, \nabla_T^3 T\rangle \nabla_T^2 T = 0, \quad (3.8)$$

which is precisely (2.1) for $r = 4$. In order to characterize the solutions of (3.8) with constant curvatures we use the Frenet equations (3.6) and a direct calculation shows

$$\nabla_T^2 T = -k_1^2 T + k_1 k_2 F_3,$$
$$\nabla_T^3 T = -k_1(k_1^2 + k_2^2)F_2 + k_1 k_2 k_3 F_4,$$
$$\nabla_T^4 T = k_1^2(k_1^2 + k_2^2)T - k_1 k_2(k_1^2 + k_2^2 + k_3^2)F_3 + k_1 k_2 k_3 k_4 F_5,$$
$$\nabla_T^5 T = k_1\big((k_1^2 + k_2^2)^2 + k_2^2 k_3^2\big)F_2 - k_1 k_2 k_3(k_1^2 + k_2^2 + k_3^2 + k_4^2)F_4 + k_1 k_2 k_3 k_4 k_5 F_6,$$
$$\nabla_T^6 T = -k_1^2\big((k_1^2 + k_2^2)^2 + k_2^2 k_3^2\big)T + k_1 k_2\big((k_1^2 + k_2^2)^2 + k_3^2(k_1^2 + 2k_2^2 + k_3^2 + k_4^2)\big)F_3$$
$$- k_1 k_2 k_3 k_4(k_1^2 + k_2^2 + k_3^2 + k_4^2 + k_5^2)F_5 + k_1 k_2 k_3 k_4 k_5 k_6 F_7,$$
$$\nabla_T^7 T = -k_1\big((k_1^2 + k_2^2)^3 + k_2^2 k_3^2(2k_1^2 + 2k_2^2 + k_3^2 + k_4^2)\big)F_2$$
$$+ k_1 k_2 k_3\big((k_1^2 + k_2^2)^2 + (k_3^2 + k_4^2)^2 + k_1^2 k_3^2 + 2k_2^2 k_3^2 + k_4^2(k_1^2 + k_2^2 + k_5^2)\big)F_4$$
$$- k_1 k_2 k_3 k_4 k_5(k_1^2 + k_2^2 + k_3^2 + k_4^2 + k_5^2 + k_6^2)F_6.$$

Using the above expressions it is easy to see that a number of terms in (3.8) vanish and we obtain the following simplification

$$\nabla_T^7 T + K\nabla_T^5 T + K\langle \nabla_T^4 T, T\rangle \nabla_T T + K\langle \nabla_T^2 T, T\rangle \nabla_T^3 T = 0,$$

which, when expressed in terms of its Frenet frame, acquires the form

$$k_1\Big[-(k_1^2 + k_2^2)^3 - k_2^2 k_3^2(2k_1^2 + 2k_2^2 + k_3^2 + k_4^2) + K\big((k_1^2 + k_2^2)^2 + k_2^2 k_3^2\big) + 2K k_1^2(k_1^2 + k_2^2)\Big]F_2$$

$$+ k_1 k_2 k_3\Big[(k_1^2 + k_2^2)^2 + (k_3^2 + k_4^2)^2 + k_1^2 k_3^2 + 2k_2^2 k_3^2 + k_4^2(k_1^2 + k_2^2 + k_5^2) - K(2k_1^2 + k_2^2 + k_3^2 + k_4^2)\Big]F_4$$

$$+ k_1 k_2 k_3 k_4 k_5\Big[-(k_1^2 + k_2^2 + k_3^2 + k_4^2 + k_5^2 + k_6^2) + K\Big]F_6 = 0.$$

The claim now follows from testing this system with F_2, F_4, F_6 completing the proof. □

Proof of Theorem 7. The idea of the proof is to use the constraints (1.7) and to perform a case by case analysis. First of all we note that we have $k_1 \neq 0$ as we are considering a proper triharmonic curve.

(1) If $k_1 \neq 0$ and $k_i = 0, i = 1, 2, 3$ then we get $k_1^2 = 2$ leading to the first class of curves. It was shown in [1, Theorem 1.1] that it actually solves the equation for triharmonic curves.

(2) If $k_1, k_2 \neq 0$ and $k_3 = k_4 = 0$ we are in the situation detailed in Theorem 3 leading to the second case.

(3) If $k_1, k_2, k_3 \neq 0$ and $k_4 = 0$ the constraints (1.7) acquire the form

$$\sum_{i=1}^{3} k_i^2 = 1, \qquad (k_1^2 + k_2^2)^2 + k_2^2 k_3^2 = 2k_1^2 + k_2^2. \tag{3.9}$$

Using the first equation to eliminate k_3^2 from the second one we find

$$k_1^2 + k_2^2 = 2$$

exploiting that $k_1 \neq 0$. Reinserting this into the second equation of (3.9) we find

$$k_2^2 + k_2^2 k_3^2 = 0$$

leading to a contradiction such that this case cannot occur.

(4) If $k_j \neq 0, j = 1,\ldots 4$ the constraints (1.7) are given by

$$\sum_{i=1}^{4} k_i^2 = 1, \qquad (k_1^2 + k_2^2)^2 + k_2^2 k_3^2 = 2k_1^2 + k_2^2. \tag{3.10}$$

Again, eliminating k_3^2 from the second equation, making use of the first constraint, we find

$$k_1^2(k_1^2 + k_2^2) - k_2^2 k_4^2 = 2k_1^2.$$

Using once more the first equation of (3.10) to replace $k_1^2 + k_2^2$ we arrive at

$$k_1^2(1 + k_3^2 + k_4^2) + k_2^2 k_4^2 = 0$$

leading to a contradiction again.

(5) If $k_1 \neq 0, k_2 = 0, k_3 \neq 0, k_4 = 0$ the system (1.7) reduces to $k_1^2 = 2$ leading to the first claim of the theorem.

(6) If $k_1, k_2 \neq 0, k_3 = 0, k_4 \neq 0$ we get the condition $(k_1^2 + k_2^2)^2 = 2k_1^2 + k_2^2$ leading to the second case of the theorem while k_4 can be arbitrary. However, it is a direct consequence of the Frenet equations (3.6) that once $k_3 = 0$ any geodesic curvature $k_j, j \geq 4$ will no longer appear when expressing the equation for triharmonic curves in terms of its Frenet frame.

The proof is now complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

A careful inspection of the proof of Theorem 5 shows that it is enough to know the structure of the highest order derivatives of r-harmonic helices to obtain classification results if we assume that all geodesic curvatures of the curve are non-zero. Hence, as a first step towards the proof of Theorem 9 we establish an expression for the iterated derivatives appearing in the equation for r-harmonic curves (1.4) suited to our particular analysis.

Lemma 21. *Let $\gamma\colon I \to M$ be an r-harmonic curve parametrized by arclength whose geodesic curvature are all constant together with its Frenet frame $\{F_j\}, j = 1, 2r - 2$.*

(1) *For $2 \leq l \leq 2r - 3$ we have*

$$\nabla_T^{2l-1} T = \sum_{j=1}^{l-2} a_j F_{2j} - \left(\prod_{i=1}^{2l-3} k_i\right)\left(\sum_{j=1}^{2l-2} k_j^2\right) F_{2l-2} + \left(\prod_{i=1}^{2l-1} k_i\right) F_{2l}, \tag{3.11}$$

where a_j is a function of $k_p, p = 1, \ldots, l-2$.

(2) *The highest derivative appearing in the equation for r-harmonic curves has the form*

$$\nabla_T^{2r-1} T = \sum_{j=1}^{r-2} b_j F_{2j} - \left(\prod_{i=1}^{2r-3} k_i\right)\left(\sum_{j=1}^{2r-2} k_j^2\right) F_{2r-2}, \tag{3.12}$$

where b_j is a function of $k_p, p = 1, \ldots l-2$.

Proof. The proof uses induction. Choosing $l = 3$ in (3.11) we get precisely the formula derived in the proof of Theorem 5 confirming the base case.

For the induction step we differentiate (3.11) using the Frenet equations (3.6) and find

$$\nabla_T^{2l} T = \sum_{j=1}^{l-2} \tilde{a}_j F_{2j-1} + \sum_{j=1}^{l-2} a_j k_{2j} F_{2j+1} + \left(\prod_{i=1}^{2l-3} k_i\right)\left(\sum_{j=1}^{2l-2} k_j^2\right) k_{2l-3} F_{2l-3}$$

$$- \left(\prod_{i=1}^{2l-3} k_i\right)\left(\sum_{j=1}^{2l-2} k_j^2\right) k_{2l-2} F_{2l-1} - \left(\prod_{i=1}^{2l-1} k_i\right) k_{2l-1} F_{2l-1} + \left(\prod_{i=1}^{2l-1} k_i\right) k_{2l} F_{2l+1},$$

where $\widetilde{a}_j$ is again a function of the $k_p, p = 1, \ldots l - 1$. Differentiating again using (3.6) then yields

$$
\begin{aligned}
\nabla_T^{2l+1} T = &- \sum_{j=1}^{l-2} \widetilde{a}_j k_{2j-2} F_{2j-2} + \sum_{j=1}^{l-2} \widetilde{a}_j k_{2j-1} F_{2j} - \sum_{j=1}^{l-2} \widetilde{a}_j k_{2j}^2 F_{2j} + \sum_{j=1}^{l-2} \widetilde{a}_j k_{2j} k_{2j+1} F_{2j+2} \\
&- \left(\prod_{i=1}^{2l-3} k_i \right) \left(\sum_{j=1}^{2l-2} k_j^2 \right) k_{2l-3} k_{2l-4} F_{2l-4} + \left(\prod_{i=1}^{2l-3} k_i \right) \left(\sum_{j=1}^{2l-2} k_j^2 \right) k_{2l-3}^2 F_{2l-2} \\
&+ \left(\prod_{i=1}^{2l-3} k_i \right) \left(\sum_{j=1}^{2l-2} k_j^2 \right) k_{2l-2}^2 F_{2l-2} - \left(\prod_{i=1}^{2l-3} k_i \right) \left(\sum_{j=1}^{2l-2} k_j^2 \right) k_{2l-2} k_{2l-1} F_{2l} \\
&+ \left(\prod_{i=1}^{2l-1} k_i \right) k_{2l-1} k_{2l-2} F_{2l-2} - \left(\prod_{i=1}^{2l-1} k_i \right) k_{2l-1}^2 F_{2l} \\
&- \left(\prod_{i=1}^{2l-1} k_i \right) k_{2l}^2 F_{2l} + \left(\prod_{i=1}^{2l+1} k_i \right) F_{2l+2}.
\end{aligned}
$$

Now, it is straightforward to see that

$$
\left(\prod_{i=1}^{2l-3} k_i \right) \left(\sum_{j=1}^{2l-2} k_j^2 \right) k_{2l-2} k_{2l-1} F_{2l} + \left(\prod_{i=1}^{2l-1} k_i \right) k_{2l-1}^2 F_{2l} + \left(\prod_{i=1}^{2l-1} k_i \right) k_{2l}^2 F_{2l} = \left(\prod_{i=1}^{2l-1} k_i \right) \left(\sum_{j=1}^{2l} k_j^2 \right) F_{2l}.
$$

Hence, we may conclude that

$$
\nabla_T^{2l+1} T = \sum_{j=1}^{l-1} \widetilde{a}_j F_{2j} - \left(\prod_{i=1}^{2l-1} k_i \right) \left(\sum_{j=1}^{2l} k_j^2 \right) F_{2l} + \left(\prod_{i=1}^{2l+1} k_i \right) F_{2l+2}
$$

completing the induction step and thus establishing the first claim. The second formula follows from the first one taking into account that for an r-harmonic curve we have $k_{2r-1} = 0$. $\qquad\square$

We are now ready to give the proof of Theorem 9.

Proof of Theorem 9. First, we rewrite the equation for r-harmonic curves in space forms (2.1), extracting the two leading derivatives, as follows

$$
\nabla_T^{2r-1} T + K \nabla_T^{2r-3} T - K \langle T, \nabla_T^{2r-3} T \rangle T + \sum_{l=1}^{r-2} (-1)^l \left(\langle T, \nabla_T^l T \rangle \nabla_T^{2r-3-l} T - \langle T, \nabla_T^{2r-3-l} T \rangle \nabla_T^l T \right) = 0. \quad (3.13)
$$

Testing this equation with F_{2r-2} we obtain

$$
\begin{aligned}
\langle \nabla_T^{2r-1} T, F_{2r-2} \rangle &+ K \langle \nabla_T^{2r-3} T, F_{2r-2} \rangle \\
&+ \sum_{l=1}^{r-2} (-1)^l \left(\langle T, \nabla_T^l T \rangle \langle \nabla_T^{2r-3-l} T, F_{2r-2} \rangle - \langle T, \nabla_T^{2r-3-l} T \rangle \langle \nabla_T^l T, F_{2r-2} \rangle \right) = 0.
\end{aligned}
$$

Regarding the last two terms, we make the following splitting into even and odd addends

$$
\sum_{l=1}^{r-2} (-1)^l \langle \nabla_T^{2r-3-l} T, F_{2r-2} \rangle = \sum_{l=1}^{\frac{r-2}{2}} \left(\langle \nabla_T^{2r-3-2l} T, F_{2r-2} \rangle - \langle \nabla_T^{2r-3-(2l-1)} T, F_{2r-2} \rangle \right).
$$

As $2r - 3 - (2l-1) = 2r - 2 - 2l$ is clearly even it is a direct consequence of the Frenet equations (3.6) that $\nabla_T^{2r-3-(2l-1)} T$ can be written as

$$
\nabla_T^{2r-3-(2l-1)} T = \sum_j a_j F_{2j+1},
$$

where a_j are functions of the geodesic curvatures k_j, such that

$$
\langle \nabla_T^{2r-3-(2l+1)} T, F_{2r-2} \rangle = 0.
$$

Secondly, using (3.11), we find

$$\langle \nabla_T^{2r-3-2l} T, F_{2r-2} \rangle = \sum_{j=1}^{r-l-3} a_j \underbrace{\langle F_{2j}, F_{2r-2} \rangle}_{=0} - \left(\prod_{i=1}^{2r-2l-5} k_i \right) \left(\sum_{j=1}^{2r-2l-4} k_j^2 \right) \underbrace{\langle F_{2(r-l-2)}, F_{2r-2} \rangle}_{=0}$$

$$+ \left(\prod_{i=1}^{2r-2l-3} k_i \right) \underbrace{\langle F_{2(r-l-1)}, F_{2r-2} \rangle}_{=0}$$

for all $l \geq 1$. Hence, we may conclude that

$$\sum_{l=1}^{r-2} (-1)^l \langle \nabla_T^{2r-3-l} T, F_{2r-2} \rangle = 0$$

and by the same reasoning we can also deduce that

$$\langle \nabla_T^l T, F_{2r-2} \rangle = 0, \qquad 1 \leq l \leq r-2.$$

Now, using (3.11) and (3.12) we obtain from the equation for r-harmonic curves (3.13) that

$$0 = \langle \nabla_T^{2r-1} T, F_{2r-2} \rangle + K \langle \nabla_T^{2r-3} T, F_{2r-2} \rangle$$

$$= - \left(\prod_{i=1}^{2r-3} k_i \right) \left(\sum_{j=1}^{2r-2} k_j^2 \right) + K \left(\prod_{i=1}^{2r-3} k_i \right).$$

This completes the proof.   □

3.1. *Evidence in support of Conjecture 1*

In this subsection we will collect a number of results which support the statement of Conjecture 11.

First, recall that the equation for a triharmonic curve on a general Riemannian manifold is given by

$$0 = \tau_3(\gamma) = \nabla_T^5 T + R^M(\nabla_T^3 T, T) T - R^M(\nabla_T^2 T, \nabla_T T) T, \tag{3.14}$$

which is precisely (1.4) for $r = 3$.

Proposition 22. *Let $\gamma \colon I \to M$ be a triharmonic curve parametrized with respect to arclength. Then the following equation holds*

$$\frac{d^3}{ds^3} |\nabla_T T|^2 - \frac{d}{ds} |\nabla_T^2 T|^2 = 0. \tag{3.15}$$

Proof. To obtain the first identity we multiply (3.14) by T and calculate

$$0 = \langle \nabla_T^5 T, T \rangle$$

$$= \frac{d}{ds} \langle \nabla_T^4 T, T \rangle - \langle \nabla_T^4 T, \nabla_T T \rangle$$

$$= \frac{d^2}{ds^2} \langle \nabla_T^3 T, T \rangle - 2 \frac{d}{ds} \langle \nabla_T^3 T, \nabla_T T \rangle + \langle \nabla_T^3 T, \nabla_T^2 T \rangle$$

$$= \frac{d^3}{ds^3} \langle \nabla_T^2 T, T \rangle - 3 \frac{d^2}{ds^2} \langle \nabla_T^2 T, \nabla_T T \rangle + \frac{5}{2} \frac{d}{ds} |\nabla_T^2 T|^2.$$

To finish the proof we note that $\langle \nabla_T^2 T, T \rangle = -|\nabla_T T|^2$ which holds as γ is parametrized with respect to arclength.   □

Corollary 23. *Let $\gamma\colon I \to N$ be a triharmonic curve parametrized with respect to arclength. Then the following conservation law holds*

$$\frac{d^2}{ds^2}|\nabla_T T|^2 - |\nabla_T^2 T|^2 = c_1 \tag{3.16}$$

for some $c_1 \in \mathbb{R}$.

Proof. This is a direct consequence of the conservation law (3.15). $\qquad\square$

Choosing a Frenet frame along γ equation (3.16) implies

$$(k_1')^2 + 2k_1 k_1'' - k_1^4 - k_1^2 k_2^2 = c_1,$$

where $k_i, i = 1,2$ represent the curvatures of the curve γ. In the case of $c_1 = 0$ this equation is solved by

$$k_1 = \frac{\alpha}{s}, \qquad k_2 = \frac{\beta}{s}, \qquad \alpha^2 + \beta^2 = 5,$$

which gives rise to the triharmonic curve with non-constant geodesic curvature and torsion constructed in [12].

In the following, we extend the previous analysis to the case of 4-harmonic curves. These are solutions of

$$0 = \tau_4(\gamma) = \nabla_T^7 T + R^M(\nabla_T^5 T, T)T - R^M(\nabla_T^4 T, \nabla_T T)T + R^M(\nabla_T^3 T, \nabla_T^2 T)T, \tag{3.17}$$

which is precisely (1.4) in the case of $r = 4$.

Proposition 24. *Let $\gamma\colon I \to M$ be a 4-harmonic curve parametrized with respect to arclength. Then the following conservation law holds*

$$\frac{d^5}{ds^5}|\nabla_T T|^2 - 2\frac{d^3}{ds^3}|\nabla_T^2 T|^2 + \frac{d}{ds}|\nabla_T^3 T|^2 = 0. \tag{3.18}$$

Proof. Testing (3.17) with T a direct calculation yields the following identity

$$0 = \frac{d^2}{ds^2}\langle\nabla_T^5 T, T\rangle - 2\frac{d}{ds}\langle\nabla_T^5 T, \nabla_T T\rangle + \langle\nabla_T^5 T, \nabla_T^2 T\rangle.$$

From the proof of Proposition 22 we know that

$$\langle\nabla_T^5 T, T\rangle = -\frac{5}{2}\frac{d^3}{ds^3}|\nabla_T T|^2 + \frac{5}{2}\frac{d}{ds}|\nabla_T^2 T|^2.$$

Moreover, we have

$$\langle\nabla_T^5 T, \nabla_T T\rangle = \frac{d^4}{ds^4}\frac{1}{2}|\nabla_T T|^2 - \frac{d^2}{ds^2}2|\nabla_T^2 T|^2 + |\nabla_T^3 T|^2.$$

Finally, a direct calculation shows

$$\langle\nabla_T^5 T, \nabla_T^2 T\rangle = \frac{1}{2}\frac{d^3}{ds^3}|\nabla_T^2 T|^2 - \frac{3}{2}\frac{d}{ds}|\nabla_T^3 T|^2.$$

The claim follows by combing the different equations. $\qquad\square$

A dimensional analysis of the conservation law (3.18) suggests the following: Assume that we are looking for a 4-harmonic curve with non-constant geodesic curvature k_1. Inspecting the terms in (3.18) suggests that $k_1 = \frac{C}{s^2}$ as all three terms scale as $\frac{1}{s^9}$. Hence, one may expect that there exist 4-harmonic curves with non-constant geodesic curvature $k_1 = \frac{\alpha}{s^2}, \alpha \in \mathbb{R}$.

Finally, we note that the construction of conservation laws can be carried out for polyharmonic curves as well by multiplying (1.4) with T and manipulating the resulting equation as we have demonstrated for triharmonic and 4-harmonic curves. Again, a simple dimensional analysis then leads to the conclusion of Conjecture 11.

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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