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Quantitative spectral inequalities for the anisotropic Shubin operators and applications to null-controllability

Inégalités spectrales quantitatives pour les opérateurs de Shubin anisotropes et applications en contrôlabilité à zéro

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Abstract. We prove quantitative spectral inequalities for the (anisotropic) Shubin operators on the whole Euclidean space, thus relating for functions from spectral subspaces associated to finite energy intervals their L^2-norm on the whole space to the L^2-norm on a suitable subset. A particular feature of our estimates is that the constant relating these L^2-norms is very explicit in geometric parameters of the corresponding subset of the whole space, which may become sparse at infinity and may even have finite measure. This extends results obtained recently by J. Martin and, in the particular case of the harmonic oscillator, by A. Dicke, I. Veselić, and the second author. We apply our results towards null-controllability of the associated parabolic equations, as well as to the ones associated to the (degenerate) Baouendi-Grushin operators acting on $\mathbb{R}^d \times \mathbb{T}^d$.

Résumé. On démontre des inégalités spectrales quantitatives pour les opérateurs de Shubin (anisotropes) sur tout l'espace euclidien, reliant ainsi pour les fonctions des sous-espaces spectraux associés à des intervalles d'énergie finie leur norme L^2 sur l'espace entier à la norme L^2 sur un sous-ensemble approprié. Une caractéristique particulière de nos estimations est que la constante reliant ces normes L^2 est très explicite en les paramètres géométriques du sous-ensemble de l'espace entier correspondant, qui peut devenir clairsemé à l'infini et même avoir une mesure finie. On étend ainsi des résultats obtenus récemment par J. Martin et, dans le cas particulier de l'oscillateur harmonique, par A. Dicke, I. Veselić et le deuxième auteur. Nous appliquons nos résultats à la contrôlabilité à zéro des équations paraboliques associées, ainsi qu'à celles associées aux opérateurs (dégénérés) de Baouendi-Grushin agissant sur $\mathbb{R}^d \times \mathbb{T}^d$.

Keywords. Spectral inequalities, null-controllability, Agmon estimates, anisotropic Shubin operators, Baouendi–Grushin operator.

Mots-clés. Inégalités spectrales, contrôlabilité à zéro, estimées d'Agmon, opérateurs de Shubin anisotropes, opérateur de Baouendi–Grushin.

2020 Mathematics Subject Classification. 35P05, 93B05, 35P10.

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1. Introduction

Quantitative spectral inequalities are instances of so-called *uncertainty relations* that, in the context of the present paper, take the form

$$\|f\|^2_{L^2(\Omega)} \leq d_0\, \mathrm{e}^{d_1\lambda^\eta}\|f\|^2_{L^2(\omega)}, \quad f \in \mathscr{E}_\lambda(A),\ \lambda \geq 0,$$

where ω is a measurable subset of a domain $\Omega \subset \mathbb{R}^d$, $\mathscr{E}_\lambda(A) = \mathbb{1}_{(-\infty,\lambda]}(A)$ denotes the spectral subspace for a non-negative selfadjoint operator A in $L^2(\Omega)$ associated with the interval $(-\infty,\lambda]$, and $d_0, d_1, \eta > 0$ are constants. Such inequalities can be viewed as quantitative variants of an identity theorem (in the sense that $f = 0$ on ω implies $f = 0$ on Ω) and are often considered under different names, depending on the context, such as *(quantitative) unique continuation estimates*, see e.g. [31, 33], or *uncertainty principles*, see e.g. [44]. The notion of *spectral inequalities* we adopt is common in the context of control theory, see e.g., [30, 31]. There is also a close relation to the notions of *vanishing order*, see, e.g., [20, 30], and *annihilating pairs* in Fourier analysis, see e.g. [8, 24].

In the present work, we prove spectral inequalities from sparse sensor sets ω with an explicit form of the constants when A is the (anisotropic) Shubin operator in $L^2(\mathbb{R}^d)$,

$$H_{k,m} = (-\Delta)^m + |x|^{2k}, \quad x \in \mathbb{R}^d, \tag{1}$$

where $k, m \geq 1$ are positive integers. Our inequalities complement recent results from [35] and, in the particular case of the harmonic oscillator, from [18]. For instance, very general spectral inequalities have been obtained in [35, Theorem 2.1(ii)] for every measurable set $\omega \subset \mathbb{R}^d$ with merely positive measure. These inequalities take the form

$$\|f\|^2_{L^2(\mathbb{R}^d)} \leq K\mathrm{e}^{K\lambda^{\frac{1}{2k}+\frac{1}{2m}}|\log\lambda|}\|f\|^2_{L^2(\omega)}, \quad f \in \mathscr{E}_\lambda(H_{k,m}),\ \lambda > 0, \tag{2}$$

where $K > 0$ is a positive constant depending on k, m, the dimension d and the set ω. The dependence of K on the set ω, however, is not explicit, even if more information on ω is available. Our inequalities mainly address this dependence if ω is sparse in a sense made precise below. The technique of proof used in the present paper follows the approach by Kovrijkine [28, 29] and builds upon recent developments in this field of research [8, 18, 23, 35, 38]. We apply our results in the context of exact null-controllability for the abstract Cauchy problems associated to $H_{k,m}$, as well as to the Baouendi-Grushin operator in $L^2(\mathbb{R}^d \times \mathbb{T}^d)$,

$$\Delta_\gamma = \Delta_x + |x|^{2\gamma}\Delta_y, \quad (x,y) \in \mathbb{R}^d \times \mathbb{T}^d,$$

with $\gamma \geq 1$ a positive integer. Note that for the latter we use the more traditional parameter γ, rather than just k as for the Shubin operators.

Outline of the work

In Section 2, we present in detail the main results contained in this work. Section 3 is then devoted to the proof of the spectral inequalities for the anisotropic Shubin operators. These spectral inequalities are used in Section 4 to prove null-controllability results for the evolution equations associated with both the Shubin operators on $\mathbb{R}^d$ and the Baouendi-Grushin operators on $\mathbb{R}^d \times \mathbb{T}^d$. Finally, Appendix A provides a statement on the asymptotics of the smallest eigenvalue of the anisotropic Shubin operator $H_{k,1}$ as $k \to \infty$, which is used in Example 20.

Notations

The following notations and conventions will be used throughout this work:

1. $\mathbb{N}$ denotes the set of natural numbers starting from zero.
2. The canonical Euclidean scalar product of $\mathbb{R}^d$ is denoted by $\cdot$, and $|\cdot|$ stands for the associated canonical Euclidean norm. We will also use the Japanese bracket notation $\langle\cdot\rangle = (1+|\cdot|^2)^{1/2}$.
3. The length of any multi-index $\alpha = (\alpha_1,\dots,\alpha_d) \in \mathbb{N}^d$ is denoted $|\alpha|$ and defined by
$$|\alpha| = \alpha_1 + \cdots + \alpha_d.$$
4. The Lebesgue measure of a measurable set $\omega \subset \mathbb{R}^d$ is denoted $|\omega|$.
5. $\mathbb{1}_\omega$ denotes the characteristic function of any subset $\omega \subset \mathbb{R}^d$.
6. For all measurable subsets $\omega \subset \mathbb{R}^d$, the inner product of $L^2(\omega)$ is denoted $\langle\cdot,\cdot\rangle_{L^2(\omega)}$, while $\|\cdot\|_{L^2(\omega)}$ stands for the associated norm.
7. For a nonnegative selfadjoint operator A on $L^2(\mathbb{R}^d)$, $\mathcal{E}_\lambda(A) = \mathbb{1}_{(-\infty,\lambda]}(A)$ with $\lambda \geq 0$ denotes the spectral subspace for A associated with the interval $(-\infty,\lambda]$.

2. Statement of the main results

This section is devoted to present in detail the main results contained in this work.

2.1. *Spectral inequalities for the Shubin operators*

Given two positive integers $k,m \geq 1$, we consider in $L^2(\mathbb{R}^d)$ the (anisotropic) Shubin operator as in (1), which is a non-negative and selfadjoint operator with purely discrete spectrum when equipped with its maximal domain

$$D(H_{k,m}) = \big\{g \in L^2(\mathbb{R}^d) : H_{k,m}g \in L^2(\mathbb{R}^d)\big\}.$$

Moreover, for $\lambda \geq 0$, let $\mathcal{E}_{\lambda,k,m} = \mathcal{E}_\lambda(H_{k,m})$ denote the spectral subspace for the operator $H_{k,m}$ associated with the interval $(-\infty,\lambda]$, cf. the notations at the end of Section 1.

For easier comparison, let us first state a result for the harmonic oscillator, corresponding to the case where $k = m = 1$, which covers and extends previous results from [8, 18, 23, 38], see Remark 2 below.

Theorem 1. *Let $\rho\colon \mathbb{R}^d \to (0,+\infty)$ and $\sigma\colon \mathbb{R}^d \to (0,1]$ be functions such that ρ and $1/\sigma$ are locally bounded, and let $\omega \subset \mathbb{R}^d$ be a measurable set satisfying*

$$\forall x \in \mathbb{R}^d, \quad |\omega \cap B(x,\rho(x))| \geq \sigma(x)|B(x,\rho(x))|. \tag{3}$$

Then, there exists a positive constant $K > 0$, depending only on the dimension d, such that for all $\lambda \geq 0$ and $f \in \mathcal{E}_{\lambda,1,1}$ we have

$$\|f\|^2_{L^2(\mathbb{R}^d)} \leq \left(\frac{K}{\theta_\lambda}\right)^{K(1+(L_\lambda)^2+L_\lambda\sqrt{\lambda})} \|f\|^2_{L^2(\omega)}, \tag{4}$$

where

$$\theta_\lambda := \inf_{|x|<\sqrt{2\lambda}} \sigma(x) \quad \textit{and} \quad L_\lambda := \sup_{|x|<\sqrt{2\lambda}} \rho(x).$$

Remark 2. Suppose that the functions σ and ρ satisfy the bounds

$$\forall x \in \mathbb{R}^d, \quad \sigma(x) \geq \theta^{\langle x\rangle^a} \quad \text{and} \quad \rho(x) \leq L\langle x\rangle^\delta \tag{5}$$

with some fixed $\theta \in (0,1]$, $a \geq 0$, $L > 0$, and $\delta \geq 0$. In this case, we have

$$\theta_\lambda \geq \theta^{(1+2\lambda)^{a/2}} \quad \text{and} \quad L_\lambda \leq L(1+2\lambda)^{\delta/2}.$$

It is then straightforward to verify that (4) takes the form

$$\|f\|^2_{L^2(\mathbb{R}^d)} \le \left(\frac{K}{\theta}\right)^{K^{1+a+\delta}(1+L^2\lambda^{\delta+a/2}+L\lambda^{(1+a+\delta)/2})} \|f\|^2_{L^2(\omega)}, \quad f \in \mathscr{E}_{\lambda,1,1}, \tag{6}$$

with a possibly different constant $K \ge 1$. This covers [18, Theorem 2.7], while the particular case of $a = 0$ has also previously been considered in [38, Theorem 2.1] under the additional assumption that ρ is 1/2-Lipschitz continuous.

The case where the functions σ and ρ are constant, and thus the parameters a and δ above can be chosen equal to zero, that is,

$$\forall x \in \mathbb{R}^d, \quad |\omega \cap B(x,L)| \ge \theta |B(x,L)|, \tag{7}$$

corresponds to so-called (θ, L)*-thick* sets. Such sets have been getting considerable attention in the past and have been previously discussed in this context in [8, Theorem 2.1 (iii)] and [23, Corollary 1.9]. In fact, [23, Corollary 1.9] also makes in this case the dependence on the dimension in (6) explicit. This could have been done in (4) with our technique as well, but we refrained from doing so for the sake of simplicity.

The spectral inequality in (4) is very explicit in terms of σ and ρ. The fact that only the uniform bounds of σ and ρ on the ball $B(0,\sqrt{2\lambda})$ enter the estimate (4) is due to the strong decay that the potential enforces on the eigenfunctions of the harmonic oscillator (and finite linear combinations thereof). This is an instance of a much more general phenomenon that also takes place in case of general (anisotropic) Shubin operators and eventually leads to a variant of Theorem 1 for these operators that, in particular, gives a positive answer to [19, Conjecture 1.6]. Our corresponding main result considers exactly the same geometry for $\omega \subset \mathbb{R}^d$ as in Theorem 1 and reads as follows.

Theorem 3. *There exists a constant $K > 0$, depending only on k, m, and the dimension d, such that for all measurable sets $\omega \subset \mathbb{R}^d$ satisfying the geometric condition* (3)*, and all $\lambda \ge 0$ and $f \in \mathscr{E}_{\lambda,k,m}$ we have*

$$\|f\|^2_{L^2(\mathbb{R}^d)} \le \left(\frac{K}{\theta_{\lambda,k}}\right)^{K(1+(L_{\lambda,k})^{1+\frac{k}{m}}+L_{\lambda,k}\lambda^{\frac{1}{2m}}+\log(1+\lambda))} \|f\|^2_{L^2(\omega)}, \tag{8}$$

where

$$\theta_{\lambda,k} := \inf_{|x|<(2\lambda)^{1/2k}} \sigma(x) \quad \textit{and} \quad L_{\lambda,k} := \sup_{|x|<(2\lambda)^{1/2k}} \rho(x). \tag{9}$$

Remark 4. Similarly as for the harmonic oscillator, the potential $|x|^{2k}$ enforces a strong decay of (finite linear combinations of) eigenfunctions of the operator $H_{k,m}$, so that such functions are localized around the origin. More precisely, Corollary 28 below states that for all $\lambda \ge 0$ and $f \in \mathscr{E}_{\lambda,k,m}$,

$$\|f\|^2_{L^2(\mathbb{R}^d)} \le 2\|f\|^2_{L^2(B(0,(2\lambda)^{1/2k}))}.$$

It is therefore sufficient to prove for functions in $\mathscr{E}_{\lambda,k,m}$ estimates on the ball $B(0,(2\lambda)^{1/2k})$ in order to obtain similar estimates on the whole space $\mathbb{R}^d$. This also explains why in (8) only the bounds of σ and ρ on the ball $B(0,(2\lambda)^{1/2k})$ enter.

While the just mentioned localization behaviour is completely consistent with the case of the harmonic oscillator in Theorem 1, it is worth to note that the term $\log(1+\lambda)$ on the right-hand side of (8) does not appear in (4). This term turns out to be quite unfavourable (see Remark 5 below), and we conjecture that it can indeed be just skipped. The reason why it comes into play within our framework is related to obtaining Agmon estimates for spectral subspaces as explained in Remark 25 in Section 3.2 below. Nevertheless, since $\log(1+\lambda)$ is dominated by every power of λ, it should be emphasized that our bound from Theorem 3 still gives a proper quantitative spectral inequality that is strong enough to be applied in the context of null-controllability and thus obtain results in Corollaries 9 and 11 and Theorems 15 and 17(ii) below that were otherwise not accessible before.

Remark 5. Suppose again that the functions σ and ρ satisfy (5), so that

$$\theta_{\lambda,k} \geq \theta^{(1+(2\lambda)^{1/k})^{a/2}} \quad \text{and} \quad L_{\lambda,k} \leq L(1+(2\lambda)^{1/k})^{\delta/2}.$$

In this case, it is easy to check that the spectral inequality (8) can be written as

$$\|f\|^2_{L^2(\mathbb{R}^d)} \leq \left(\frac{K}{\theta}\right)^{K^{1+a+\delta}(1+\lambda^{\frac{a}{2k}})(1+L^{1+\frac{k}{m}}\lambda^{\delta(\frac{1}{2k}+\frac{1}{2m})}+L\lambda^{\frac{\delta}{2k}+\frac{1}{2m}}+\log(1+\lambda))} \|f\|^2_{L^2(\omega)} \tag{10}$$

with a possibly different constant $K \geq 1$. This extends [35, Theorem 2.1(i)], where only the case $a = 0$ and $\delta \in [0,1]$ is considered. At the same time, our bound in (10) is much more explicit in the model parameters, which is very useful in the context of control theory, see Section 2.2 below. It should be mentioned, however, that in (10) with $a = 0$ the formal homogenization limit as $L \to 0$ results in a right-hand side where the constant still depends on λ. This is due to the $\log(1+\lambda)$-term in (10) (resp. (8)) but is highly unintuitive and not consistent with the known behaviour for the free Laplacian and the harmonic oscillator. This is one reason why this term is considered unfavourable and should be removed in future research if possible, cf. Remark 25 below.

It is also worth to note that for $a = 0$ (for simplicity) and $\delta \in [0,1]$ the estimate (10) can for $\lambda \geq 1$ be written as

$$\|f\|^2_{L^2(\mathbb{R}^d)} \leq K\mathrm{e}^{K\lambda^{\frac{\delta}{2k}+\frac{1}{2m}}} \|f\|^2_{L^2(\omega)}$$

with yet another constant $K > 0$, now also depending on L, θ, and δ. This is stronger than the general estimate (2). By contrast, if $a = 0$ and $\delta > 1$, estimate (10) writes for $\lambda \geq 1$ as

$$\|f\|^2_{L^2(\mathbb{R}^d)} \leq K\mathrm{e}^{K\lambda^{\delta\left(\frac{1}{2k}+\frac{1}{2m}\right)}} \|f\|^2_{L^2(\omega)}$$

and is therefore worse than the general estimate (2), although the latter only uses that ω has positive measure. It is not yet clear how to reconcile this different behavior in the two regimes $\delta \leq 1$ and $\delta > 1$.

In the end of this subsection, let us present examples of measurable sets satisfying the geometric condition (3).

Example 6. Suppose that the local scale $\rho \equiv L > 0$ is constant and that $\sigma = w/(\sqrt{d}+1)^d$ with a radially symmetric function $w\colon \mathbb{R}^d \to (0,1]$ that is non-increasing with respect to the modulus and for which $1/w$ is locally bounded. Inspired by [18, Example 2.3] and [16, Example 4.17], with $l = L/(\sqrt{d}+1)$ and $r_j = lw(j)^{1/d}$ consider the set

$$\omega = \bigcup_{j \in l\mathbb{Z}^d} B(j, r_j).$$

This set ω satisfies the geometric condition (3). Indeed, given $x \in \mathbb{R}^d$, there is $j \in l\mathbb{Z}^d$ with $|j| \leq |x|$ and $|x - j| < l\sqrt{d}$, so that $|x-j| + r_j < l(\sqrt{d}+1) = L$. Hence, the ball $B(x,L)$ contains the ball $B(j, r_j)$, so that

$$\frac{|\omega \cap B(x,L)|}{|B(x,L)|} \geq \frac{|B(j,r_j)|}{|B(x,L)|} = \left(\frac{r_j}{L}\right)^d = \sigma(j) \geq \sigma(x).$$

It is worth to note that under the condition $\sum_{j \in l\mathbb{Z}^d} w(j) < \infty$, the above set ω has finite measure.

Example 7. Suppose that $d \geq 2$. Inspired by [34, p. 32], let us consider a non-decreasing continuous function $R\colon [0,+\infty) \to (0,+\infty)$, a non-increasing continuous function $r\colon [0,+\infty) \to (0,1)$, and the associated set

$$\omega_{r,R} = \{(x,y) \in \mathbb{R}^{d-1} \times \mathbb{R} \colon |y| > R(|x|)(1 - r(|x|))\}.$$

It is then easy to see that the intersection $\omega_{r,R} \cap B((x,0), R(|x|))$ is always non-empty (and open). Hence, the set $\omega_{r,R}$ satisfies the geometric condition (3) with the functions ρ and σ given by $\rho(x, y) = R(|x|)$ and

$$\sigma(x, y) := \frac{|\omega_{r,R} \cap B((x, y), \rho(x, y))|}{|B((x, y), \rho(x, y))|} \geq \frac{|\omega_{r,R} \cap B((x,0), R(|x|))|}{|B((x,0), R(|x|))|} > 0,$$

respectively.

2.2. *Exact null-controllability*

As application of the spectral inequalities from Theorems 1 and 3, we study the exact null-controllability for two classes of diffusive equations, being elliptic and hypoelliptic, respectively.

Definition 8 (Exact null-controllability). *Let $\Omega \subset \mathbb{R}^d$ be a domain, and let P be a non-negative selfadjoint operator in $L^2(\Omega)$. Given a measurable set $\omega \subset \Omega$, the evolution equation*

$$\begin{cases} \partial_t f(t,x) + Pf(t,x) = h(t,x)\mathbb{1}_\omega(x), & t > 0,\ x \in \Omega, \\ f(0,\cdot) = f_0 \in L^2(\Omega), \end{cases} \tag{11}$$

is said to be exactly null-controllable from the control support ω in time $T > 0$ *if for every initial datum $f_0 \in L^2(\Omega)$ there exists a control function $h \in L^2((0,T)\times\Omega)$ such that the mild solution to* (11) *satisfies $f(T,\cdot) = 0$.*

2.2.1. *The fractional anisotropic Shubin evolution equations*

Let us first consider the evolution equations of the form (11) associated to the elliptic operators $P = H^s_{k,m}$ with $s > 0$, that is,

$$\begin{cases} \partial_t f(t,x) + H^s_{k,m} f(t,x) = h(t,x)\mathbb{1}_\omega(x), & t > 0,\ x \in \mathbb{R}^d, \\ f(0,\cdot) = f_0 \in L^2(\mathbb{R}^d). \end{cases} \tag{$E_{s,k,m}$}$$

Here, the fractional powers of the operator $H_{k,m}$ are understood via standard functional calculus.

The spectral inequalities in Theorems 1 and 3 allow us to derive many exact null-controllability results for the equation ($E_{s,k,m}$), and we choose to present only three statements. We first give two general results closely related to Remark 5.

Corollary 9. *Let $\omega \subset \mathbb{R}^d$ be a measurable set as in* (3)*, and suppose that the two functions $\sigma\colon \mathbb{R}^d \to (0,1]$ and $\rho\colon \mathbb{R}^d \to (0,+\infty)$ satisfy*

$$\sigma(x) \geq \theta^{\langle x\rangle^a} \quad \text{and} \quad \rho(x) \leq L\langle x\rangle^\delta, \quad x \in \mathbb{R}^d,$$

with some fixed $L > 0$, $\delta \in [0,1]$, $\theta \in (0,1]$, and $a \geq 0$. Then, for all $s > 0$ satisfying

$$\frac{\delta + a}{2k} + \frac{1}{2m} < s,$$

the equation ($E_{s,k,m}$) *is exactly null-controllable from ω in every positive time $T > 0$.*

Remark 10. Corollary 9 extends [36, Corollary 2.12] (cf. also [17, Corollary 1.2]), which only deals with the case $a = 0$. Moreover, recall from [35, Theorem 2.5] (whose proof is based on the general spectral inequalities (2)) that whenever $s > 1/(2k) + 1/(2m)$, the equation ($E_{s,k,m}$) is exactly null-controllable from every measurable control support $\omega \subset \mathbb{R}^d$ with positive measure and in every positive time $T > 0$. Corollary 9 therefore provides a new result only in the case $0 \leq \delta + a < 1$.

Corollary 11. *Let $\omega \subset \mathbb{R}^d$ be a measurable set as in* (3)*, where the function σ satisfies*

$$\sigma(x) \geq \theta^{\langle x\rangle^a}, \quad x \in \mathbb{R}^d,$$

with some fixed $\theta \in (0,1]$ and $a \geq 0$, and the function ρ exhibits a growth at infinity that is slower than any power, that is,

$$\forall \delta > 0, \quad \rho(x) = o(|x|^\delta) \quad \text{as } |x| \longrightarrow +\infty.$$

Then, for all $s > a/2k + 1/2m$, the equation $(E_{s,k,m})$ is exactly null-controllable from the control support ω in every positive time $T > 0$.

Remark 12. Corollary 11 is, in fact, a quite straightforward consequence of Corollary 9, see Section 4.1 below. Nevertheless, it should be mentioned that the particular case of $a = 0$, although not explicitly stated in the literature, could have been proven also by using the results from [34, Chapter 6, Section 3].

It is well known from [40, Theorem 1.10] that the equation $(E_{1,1,1})$ is not null-controllable in any positive time whenever the control support $\omega \subset \mathbb{R}^d$ is contained in a half space. In fact, it can be readily checked that a half space satisfies a geometric condition of the form (3) with a constant function σ and a function ρ taking the form

$$\rho(x) = L\langle x\rangle, \quad x \in \mathbb{R}^d,$$

with some $L > 0$. Note that the latter exhibits a linear growth and is thus indeed excluded in Corollaries 9 and 11 above. This, however, raises the question whether local scales ρ can be allowed that exhibit an arbitrary sublinear growth. A first step in this direction is taken by the following last result of this subsection.

Corollary 13. *Let $\omega \subset \mathbb{R}^d$ be a measurable set as in* (3)*, and suppose that the function σ is constant and that ρ satisfies*

$$\rho(x) \leq \frac{L\langle x\rangle}{(g\circ g)^\alpha(|x|)g(|x|)} \quad \textit{where} \quad g(r) = \log(e+r), \quad r \geq 0, \tag{12}$$

with some $L > 0$ and $\alpha > 2$. Then, the equation $(E_{1,1,1})$ is exactly null-controllable from the control support ω in every positive time $T > 0$.

2.2.2. *The Baouendi–Grushin heat equation*

Let us now consider the fractional heat-like hypoelliptic evolution equation associated with the Baouendi–Grushin operator,

$$\begin{cases} \partial_t f(t,x,y) + (-\Delta_\gamma)^s f(t,x,y) = h(t,x,y)\mathbb{1}_\omega(x,y), \quad t>0,\ (x,y) \in \mathbb{R}^d \times \mathbb{T}^d, \\ f(0,\cdot,\cdot) = f_0 \in L^2(\mathbb{R}^d \times \mathbb{T}^d), \end{cases} \tag{$E_{\gamma,s}$}$$

where $s > 0$ and $\gamma \geq 1$ is a positive integer. Here, the Baouendi–Grushin operator Δ_γ acting on $\mathbb{R}^d \times \mathbb{T}^d$,

$$\Delta_\gamma = \Delta_x + |x|^{2\gamma}\Delta_y, \quad (x,y) \in \mathbb{R}^d \times \mathbb{T}^d,$$

is equipped with its maximal domain, which makes it a positive selfadjoint operator. Note that the hypothesis that $\mathbb{R}^d$ and $\mathbb{T}^d$ have the same spacial dimension d is just for simplicity, and nothing substantial would change if different dimensions would be allowed.

Our first result regarding the equation $(E_{\gamma,s})$ gives a necessary geometric condition on the control support ω for $(E_{\gamma,s})$ to be exactly null-controllable. It holds for all dissipation parameters $s > 0$.

Proposition 14. *If the equation $(E_{\gamma,s})$ is exactly null-controllable from the control support $\omega \subset \mathbb{R}^d \times \mathbb{T}^d$, then there exist $L > 0$ and $\theta \in (0,1]$ such that*

$$\forall x \in \mathbb{R}^d, \quad |\omega \cap (B(x,L) \times \mathbb{T}^d)| \geq \theta |B(x,L)|. \tag{13}$$

Positive null-controllability results for the equation $(E_{\gamma,s})$ strongly depend on how the dissipation parameter s relates to the critical hypoelliptic parameter $(1+\gamma)/2$. Let us first state a precise characterisation of null-controllability for a particular class of control supports in the strong dissipation regime $s > (1+\gamma)/2$.

Theorem 15. *Suppose that $s > (1+\gamma)/2$, and let $T > 0$ and $\omega \subset \mathbb{R}^d$ be measurable. The following assertions are equivalent:*

(i) *The equation $(E_{\gamma,s})$ is exactly null-controllable from the control support $\omega \times \mathbb{T}^d$ in time T.*
(ii) *The set ω is thick in $\mathbb{R}^d$.*

Remark 16. The techniques presented in the current work only allow to consider in the above result control supports that are strips of the form $\omega \times \mathbb{T}^d$, but not more general control supports satisfying the condition (13). The latter require a more sophisticated approach, which we postpone to a follow-up paper [5]. In particular, we prove there that Theorem 15 also holds for such more general control supports.

In the critical dissipation regime $s = (1+\gamma)/2$, we state a positive null-controllability result from strips, and also a negative one for control supports avoiding the degeneracy line $\{x = 0\}$.

Theorem 17. *Suppose that $s = (1+\gamma)/2$, and denote by $\lambda_\gamma > 0$ the smallest eigenvalue of the anharmonic oscillator $H_{\gamma,1}$.*

(i) *For every measurable set $\omega \subset \mathbb{R}^d \times \mathbb{T}^d$ satisfying the condition $\overline{\omega} \cap \{x = 0\} = \emptyset$, the equation $(E_{\gamma,s})$ is never exactly null-controllable from ω in time $T > 0$ when $0 < T < T_*$, where the time $T_* > 0$ is given by*
$$T_* = \frac{1}{1+\gamma}\left(\frac{\operatorname{dist}(0,\omega)}{\sqrt{\lambda_\gamma}}\right)^{1+\gamma}.$$
(ii) *There exists a positive constant $c_\gamma > 0$ such that for every (θ, L)-thick set $\omega \subset \mathbb{R}^d$, the equation $(E_{\gamma,s})$ is exactly null-controllable from the control support $\omega \times \mathbb{T}^d$ in every positive time $T \geq T^*$, where $T^* > 0$ is given by*
$$T^* = c_\gamma \left(\frac{L}{\sqrt{\lambda_\gamma}}\right)^{1+\gamma} \log\Big(\frac{c_\gamma}{\theta}\Big).$$

Remarks 18.

(1) Recall from [27, Theorem 4.12] that when $d = 1$ and $\gamma = s = 1$, the equation $(E_{1,1})$ is never exactly null-controllable from any control support of the form $\mathbb{R} \times \omega$ where $\omega = \mathbb{T} \setminus [a, b]$. Therefore, one does not expect positive null-controllability results to hold for the equation $(E_{\gamma,s})$ in the regime $s = (1+\gamma)/2$ from more general control supports $\omega \subset \mathbb{R}^d \times \mathbb{T}^d$ satisfying the condition (13).
(2) Part (i) of the above statement is consistent with known results from the literature for the particular case $\gamma = 1$. Indeed, the time T_* then reduces to
$$T_* = \frac{\operatorname{dist}(0,\omega)^2}{2d},$$
and therefore takes the very same form as the (minimal) times appearing in the study of Grushin-type models, see, e.g., [1, Theorem 1.1], [6, Theorem 1], [9, Theorem 1.3], or [7, Theorem 1.1].

Our last result considers control supports $\omega \subset \mathbb{R}^d \times \mathbb{T}^d$ avoiding the degeneracy line $\{x = 0\}$ in the weak dissipation regime.

Theorem 19. *Whenever* $0 < s < (\gamma+1)/2$, *the equation* $(E_{\gamma,s})$ *is never exactly null-controllable from any control support* $\omega \subset \mathbb{R}^d \times \mathbb{T}^d$ *satisfying* $\overline{\omega} \cap \{x = 0\} = \emptyset$.

Let us finish this section with an example.

Example 20. For some fixed length $L > 0$, we consider the control support

$$\omega_L = B(0,L)^c \times \mathbb{T}^d \subset \mathbb{R}^d \times \mathbb{T}^d,$$

and the associated control time

$$T_{\gamma,s,L} = \inf\{T > 0 \colon (E_{\gamma,s}) \text{ is exactly null-controllable from } \omega_L \text{ at time } T\}.$$

It is easy to see that for all $\varepsilon > 0$ the set $B(0,L)^c$ is $(\gamma_\varepsilon, L_\varepsilon)$-thick in $\mathbb{R}^d$ with

$$L_\varepsilon = L + \varepsilon \quad \text{and} \quad \gamma_\varepsilon = 1 - \frac{L^d}{(L+\varepsilon)^d}.$$

Since the control support ω_L also satisfies the geometric condition $\overline{\omega}_L \cap \{x = 0\} = \emptyset$, it follows from Theorems 15, 17, and 19 that

$$\begin{cases} T_{\gamma,s,L} = 0 & \text{when } s > (1+\gamma)/2, \\ 0 < T_{\gamma,s,L} < +\infty & \text{when } s = (1+\gamma)/2, \\ T_{\gamma,s,L} = +\infty & \text{when } s < (1+\gamma)/2. \end{cases} \tag{14}$$

In the critical dissipation regime $s = (1+\gamma)/2$, we actually have from Theorem 17 for all $\varepsilon > 0$ the more precise two-sided estimate

$$\frac{1}{1+\gamma}\left(\frac{L}{\sqrt{\lambda_\gamma}}\right)^{1+\gamma} \leq T_{\gamma,(1+\gamma)/2,L} \leq c_\gamma \left(\frac{L+\varepsilon}{\sqrt{\lambda_\gamma}}\right)^{1+\gamma} \log\left(\frac{c_\gamma (L+\varepsilon)^d}{(L+\varepsilon)^d - L^d}\right), \tag{15}$$

where $\lambda_\gamma > 0$ denotes again the smallest eigenvalue of the anharmonic oscillator $H_{\gamma,1}$. Moreover, as stated in Corollary 37 below, λ_γ converges to λ_D as γ goes to $+\infty$, where $\lambda_D > 0$ stands for the smallest eigenvalue of the Dirichlet Laplacian on the canonical Euclidean unit ball $B(0,1)$ in $\mathbb{R}^d$ (this is a quite straightforward consequence of the theory of large coupling limit). Since then

$$\frac{L}{\sqrt{\lambda_\gamma}} \longrightarrow \frac{L}{\sqrt{\lambda_D}},$$

we immediately infer that

$$\frac{1}{1+\gamma}\left(\frac{L}{\sqrt{\lambda_\gamma}}\right)^{1+\gamma} \longrightarrow \begin{cases} +\infty & \text{when } L > \sqrt{\lambda_D}, \\ 0 & \text{when } L < \sqrt{\lambda_D}. \end{cases} \tag{16}$$

Together with (15), the latter implies, in particular, that, as $\gamma \to +\infty$,

$$T_{\gamma,(1+\gamma)/2,L} \longrightarrow +\infty \quad \text{when } L > \sqrt{\lambda_D}.$$

Moreover, further calculations suggest that the first instance of the constant c_γ in (15) can be replaced by c^γ with some constant $c > 1$ that does not depend on the dimension, and that the second instance can be replaced by a constant not depending on γ. As a consequence, we have $T_{\gamma,(1+\gamma)/2,L} \to 0$ as $\gamma \to +\infty$ for $L < \sqrt{\lambda_D}/c$; the regime $\sqrt{\lambda_D}/c \leq L \leq \sqrt{\lambda_D}$ is still unclear at the moment. In any case, since $\sqrt{\lambda_D}$ approaches $+\infty$ as the dimension d goes to $+\infty$, the asymptotic behaviour of $T_{\gamma,(1+\gamma)/2,L}$ depending on L in this fashion, and not, as one might expect a priori, on $L > 1$ and $L < 1$, respectively, is quite surprising. Moreover, as mentioned in Remark 18(2), the quantity (16) is consistent with minimal times appearing in the study of Grushin-type models. Motivated by this, we conjecture that the lower bound in (15) is actually an equality, that is,

$$T_{\gamma,(1+\gamma)/2,L} = \frac{1}{1+\gamma}\left(\frac{L}{\sqrt{\lambda_\gamma}}\right)^{1+\gamma}.$$

The relevant regimes of L for the asymptotic behaviour of $T_{\gamma,(1+\gamma)/2,L}$ as $\gamma \to +\infty$ would then be $L > \sqrt{\lambda_D}$ and $L < \sqrt{\lambda_D}$, that is,

$$T_{\gamma,(1+\gamma)/2,L} \longrightarrow \begin{cases} +\infty & \text{when } L > \sqrt{\lambda_D}, \\ 0 & \text{when } L < \sqrt{\lambda_D}. \end{cases}$$

Remark 21. Incidentally, as explained in Remark 34 below, the analogous proof as the one for Theorem 17(ii) yields that the fractional Schrödinger–Baouendi–Grushin equation

$$\begin{cases} i\partial_t f(t,x,y) + (-\Delta_\gamma)^s f(t,x,y) = h(t,x,y)\mathbb{1}_\omega(x,y), \ t\in\mathbb{R}, (x,y)\in\mathbb{R}^d\times\mathbb{T}^d, \\ f(0,\cdot,\cdot) = f_0 \in L^2(\mathbb{R}^d\times\mathbb{T}^d), \end{cases} \tag{$SE_{\gamma,s}$}$$

which is the oscillatory counterpart of the equation $(E_{\gamma,s})$, is never exactly null-controllable from any control support $\omega \subset \mathbb{R}^d \times \mathbb{T}^d$ satisfying the condition $\overline{\omega} \cap \{x = 0\} = \emptyset$. This difference in behavior between the equations $(E_{\gamma,s})$ and $(SE_{\gamma,s})$ contrasts with what is known for the heat and the corresponding Schrödinger equation, see e.g. [37, Section 2.2].

Remark 22. The results presented in this subsection are in line with articles devoted to the study of the null-controllability of Grushin-type heat equations. A pioneering article in this theory is [6], which paved the way for a numerous series of articles of which we can cite [1, 7, 9, 14, 21, 26]. All these works illustrate the fact that the null-controllability of Grushin-type heat equations is governed by minimal times as in Example 20, and some of these works are even devoted to the computation of these times. Let us also mention that the null-controllability of the Schrödinger-Grushin equation is studied in the papers [12, 32].

3. Spectral inequalities for the anisotropic Shubin operators

The objective of this section is to prove Theorems 1 and 3. To this end, we mainly focus on proving the latter result and then explain briefly how its proof can be adapted in order to obtain the stronger spectral inequality for the harmonic oscillator in Theorem 1.

3.1. *An abstract uncertainty relation*

Let us begin with recalling from [23] the abstract result that plays an essential role in obtaining our spectral inequalities. In order to give its statement, we need to introduce the following definition: given a domain $\Omega \subset \mathbb{R}^d$, a constant $\kappa \geq 1$, and a length $l > 0$, we call a finite or countably infinite family $\{Q_j\}_j$ of non-empty bounded convex open subsets $Q_j \subset \Omega$ a (κ, l)-*covering of* Ω if

(i) the set $\Omega \setminus \bigcup_j Q_j$ has Lebesgue measure zero;
(ii) each Q_j is contained in a hypercube with sides of length l parallel to coordinate axes;
(iii) the estimate $\sum_j \|g\|^2_{L^2(Q_j)} \leq \kappa \|g\|^2_{L^2(\Omega)}$ holds for all $g \in L^2(\Omega)$.

We now have the following particular case of an uncertainty relation from [23].

Proposition 23 ([23, Proposition 3.1]). *Let* $\{Q_j\}_j$ *be a* (κ, l)*-covering of a given domain* $\Omega \subset \mathbb{R}^d$, *and suppose that* $f \in \bigcap_{n\in\mathbb{N}} W^{n,2}(\Omega)$ *satisfies*

$$\forall n \in \mathbb{N}, \quad \sum_{|\alpha|=n} \frac{1}{\alpha!}\|\partial_x^\alpha f\|^2_{L^2(\Omega)} \leq \frac{C_B(n)}{n!}\|f\|^2_{L^2(\Omega)},$$

with constants $C_B(n) > 0$ *such that*

$$h := \sum_{n\in\mathbb{N}} \sqrt{C_B(n)}\,\frac{(10dl)^n}{n!} < \infty.$$

Then, for every measurable subset $\omega \subset \Omega$ *satisfying* $\tau := \inf_j |Q_j \cap \omega|/\operatorname{diam}(Q_j)^d > 0$*, we have*

$$\|f\|^2_{L^2(\Omega)} \le \frac{\kappa}{6}\left(\frac{24d|B(0,1)|}{\tau}\right)^{2\frac{\log\kappa}{\log 2}+4\frac{\log h}{\log 2}+5} \|f\|^2_{L^2(\omega)}.$$

In view of Proposition 23, we therefore need in the following to prove so-called Bernstein inequalities of the form

$$\sum_{|\alpha|=n} \frac{1}{\alpha!}\|\partial_x^\alpha f\|^2_{L^2(\Omega)} \le \frac{C_B(n,\lambda)}{n!}\|f\|^2_{L^2(\Omega)}, \quad n\in\mathbb{N},\ f\in\mathcal{E}_{\lambda,k,m}, \tag{17}$$

with a properly chosen domain $\Omega \subset \mathbb{R}^d$. In order to alleviate the writing, we use throughout this section the abbreviations

$$\mu := \frac{k}{k+m}, \quad \nu := \frac{m}{k+m}, \quad \zeta := \frac{1}{2k} + \frac{1}{2m}.$$

3.2. *Agmon estimates for spectral subspaces*

A key ingredient in obtaining inequalities of the form (17) is given by the following variant of Agmon estimates from [2] for spectral subspaces associated with the (anisotropic) Shubin operators $H_{k,m}$.

Proposition 24. *There exist positive constants* $c_1, c_2, c_3 > 0$ *and* $t_0 \in (0,1]$*, depending only on* k*,* m*, and the dimension* d*, such that for all* $t \in [0, t_0)$*,* $\lambda \ge 0$*, and* $f \in \mathcal{E}_{\lambda,k,m}$ *we have*

$$\|e^{c_1 t\langle x\rangle^{1/\nu}} f\|^2_{L^2(\mathbb{R}^d)} + \|e^{c_1 t\langle D_x\rangle^{1/\mu}} f\|^2_{L^2(\mathbb{R}^d)} \le c_2\lambda^{d\zeta} e^{c_3 t\lambda^\zeta}\|f\|^2_{L^2(\mathbb{R}^d)}. \tag{18}$$

Proof. We know from [2, Theorem 2.1] that there exist some positive constants $c_1, \tilde{c} > 0$, and $t_0 \in (0,1]$ such that for every normalized eigenfunction $\psi \in L^2(\mathbb{R}^d)$ of the operator $H_{k,m}$ and all $t \in [0, t_0)$ we have

$$\|e^{c_1 t\langle x\rangle^{1/\nu}}\psi\|_{L^2(\mathbb{R}^d)} + \|e^{c_1 t\langle D_x\rangle^{1/\mu}}\psi\|_{L^2(\mathbb{R}^d)} \le \tilde{c}e^{\tilde{c}t\lambda^\zeta},$$

where $\lambda > 0$ is the eigenvalue associated with the eigenfunction ψ; recall that $H_{k,m}$ has purely discrete spectrum. Expanding $f \in \mathcal{E}_{\lambda,k,m}$ for $\lambda \ge 0$ as a linear combination of eigenfunctions, we therefore deduce that for all $t \in [0, t_0)$ we have

$$\|e^{c_1 t\langle x\rangle^{1/\nu}} f\|^2_{L^2(\mathbb{R}^d)} + \|e^{c_1 t\langle D_x\rangle^{1/\mu}} f\|^2_{L^2(\mathbb{R}^d)} \le N(\lambda)\tilde{c}^2 e^{2\tilde{c}t\lambda^\zeta}\|f\|^2_{L^2(\mathbb{R}^d)},$$

where $N(\lambda)$ is chosen as the number of distinct eigenvalues of $H_{k,m}$ less or equal to λ. Using the Weyl law asymptotics from [13, Remark 5.7] for the eigenvalue counting function associated to $H_{k,m}$, cf. also [10, Theorem 2.3.2], we then observe that

$$N(\lambda) \le c'\lambda^{d\zeta},$$

with some constant $c' > 0$ depending only on k, m, and d. The proof is then ended upon choosing $c_2 = \tilde{c}^2 c'$ and $c_3 = 2\tilde{c}$. □

Remarks 25.

(1) The term $\lambda^{d\zeta}$ on the right-hand side of (18) is unexpected, and we indeed conjecture that (18) holds without this term, that is,

$$\|e^{c_1 t\langle x\rangle^{1/\nu}} f\|^2_{L^2(\mathbb{R}^d)} + \|e^{c_1 t\langle D_x\rangle^{1/\mu}} f\|^2_{L^2(\mathbb{R}^d)} \le c_2 e^{c_3 t\lambda^\zeta}\|f\|^2_{L^2(\mathbb{R}^d)}. \tag{19}$$

The reason the term $\lambda^{d\zeta}$ appears in (18) lies in the way we carry quantitative sharp Agmon estimates for single eigenfunctions of the operator $H_{m,k}$ over to finite linear combinations of eigenfunctions. To the best of our knowledge, there are very few results in the literature stating Agmon estimates for spectral subspaces which are sharp with respect to possible parameters involved ($t \in [0, t_0)$ in this case for us), the rare exception being the case of the harmonic oscillator, see [8, Proposition 3.3]. Proving the stronger estimates (19) would immediately allow us to remove the unfavorable term $\log(1+\lambda)$ in the spectral inequalities (8).

(2) In the particular case of $m = 1$, one may take $c_1 = \nu = 1/(k+1)$ and $t_0 = 1$ in Proposition 24. This follows from the above reasoning by simply replacing the Agmon estimates for single eigenfunctions from [2, Theorem 2.1] by more explicit ones for $m = 1$ with the mentioned values of c_1 and t_0, which can be obtained, for instance, by suitably adapting the proof in [2]. These more precise Agmon estimates are also consistent with classical ones from the literature, see, e.g., [25, Theorem 3.4].

3.3. *Bernstein inequalities*

Proposition 24 now allows us to prove a global Bernstein inequality, that is, an inequality of the form (17) with $\Omega = \mathbb{R}^d$.

Proposition 26. *There exist positive constants $c, C > 0$, depending only on k, m, and the dimension d, such that for all $n \ge 0$, $\delta > 0$, $\lambda \ge 0$ and $f \in \mathscr{E}_{\lambda,k,m}$ we have*

$$\sum_{|\alpha|=n} \frac{1}{\alpha!} \|\partial_x^\alpha f\|^2_{L^2(\mathbb{R}^d)} \le \frac{C_B(n,\lambda,\delta)/2}{n!} \|f\|^2_{L^2(\mathbb{R}^d)}$$

with

$$C_B(n,\lambda,\delta) = 2C^{2(1+n)}\,\delta^{2n}\,(n!)^2(1+\lambda^{d\zeta})\,\mathrm{e}^{(c+d)\delta^{-1/\nu}}\,\mathrm{e}^{c\delta^{-1}\lambda^{\frac{1}{2m}}}. \tag{20}$$

Proof. Using integration by parts (see also Lemma 2.1 and Remark 2.2 in [23]) and Plancherel's theorem, we have

$$\sum_{|\alpha|=n} \frac{1}{\alpha!} \|\partial_x^\alpha f\|^2_{L^2(\mathbb{R}^d)} = \frac{1}{n!}\langle(-\Delta)^n f, f\rangle_{L^2(\mathbb{R}^d)} = \frac{1}{n!}\langle|\xi|^{2n}\widehat{f}, \widehat{f}\rangle_{L^2(\mathbb{R}^d)} = \frac{1}{n!}\||\xi|^n\widehat{f}\|^2_{L^2(\mathbb{R}^d)},$$

where $\widehat{f}$ denotes the Fourier transform of the function f. We therefore have to estimate the quantity $\||\xi|^n\widehat{f}\|_{L^2(\mathbb{R}^d)}$. Note here that $\widehat{f}$ belongs to $\mathscr{E}_{\lambda,m,k}$ since $H_{k,m}$ is similar to $H_{m,k}$ by Fourier transform.

With $c_1, c_2, c_3 > 0$ and $t_0 \in (0,1]$ as in Proposition 24 and $t \in (0, t_0)$, we write

$$|\xi|^n = |\xi|^n\,\mathrm{e}^{-c_1 t\langle\xi\rangle^{1/\mu}}\,\mathrm{e}^{c_1 t\langle\xi\rangle^{1/\mu}},$$

and estimate

$$\||\xi|^n\widehat{f}\|_{L^2(\mathbb{R}^d)} \le \|\langle\xi\rangle^n\,\mathrm{e}^{-c_1 t\langle\xi\rangle^{1/\mu}}\|_{L^\infty(\mathbb{R}^d)}\|\mathrm{e}^{c_1 t\langle\xi\rangle^{1/\mu}}\,\widehat{f}\|_{L^2(\mathbb{R}^d)},$$

with, moreover,

$$\|\langle\xi\rangle^n\,\mathrm{e}^{-c_1 t\langle\xi\rangle^{1/\mu}}\|_{L^\infty(\mathbb{R}^d)} = \sup_{r\ge1} r^n\,\mathrm{e}^{-c_1 t r^{1/\mu}} \le \Big(\frac{\mu}{c_1\,\mathrm{e}\,t}\Big)^{n\mu} n^{n\mu} \le \Big(\frac{\mu}{c_1 t}\Big)^{n\mu}(n!)^\mu.$$

Applying Proposition 24 to $\widehat{f} \in \mathscr{E}_{\lambda,m,k}$ and taking into account that $\|\widehat{f}\|_{L^2(\mathbb{R}^d)} = \|f\|_{L^2(\mathbb{R}^d)}$, we thus obtain from the above that

$$\sum_{|\alpha|=n} \frac{1}{\alpha!} \|\partial_x^\alpha f\|^2_{L^2(\mathbb{R}^d)} \le \frac{c_2}{n!}\Big(\frac{\mu}{c_1 t}\Big)^{2n\mu}(n!)^{2\mu}\,\lambda^{d\zeta}\,\mathrm{e}^{c_3 t\lambda^\zeta}\,\|f\|^2_{L^2(\mathbb{R}^d)}.$$

Suppose that $\lambda > (1/\delta)^{2k}$. With the particular choice $t = t_0\mu\delta^{-1}\lambda^{-1/(2k)} < t_0\mu \le t_0$, we then have

$$\sum_{|\alpha|=n} \frac{1}{\alpha!}\|\partial_x^\alpha f\|^2_{L^2(\mathbb{R}^d)} \le \frac{c_2}{n!}\left(\frac{1}{c_1 t_0}\right)^{2n\mu} \delta^{2n\mu}\lambda^{\frac{n\mu}{k}}(n!)^{2\mu}\lambda^{d\zeta}\,\mathrm{e}^{c_3\mu\delta^{-1}\lambda^{\frac{1}{2m}}}\|f\|^2_{L^2(\mathbb{R}^d)}.$$

We further estimate

$$(\lambda^{\frac{1}{2k}})^{\mu n} = (\lambda^{\frac{1}{2m}})^{\nu n} = \delta^{\nu n}\left(\frac{\lambda^{\frac{1}{2m}}}{\delta}\right)^{\nu n} \le \delta^{\nu n}(n!)^\nu\,\mathrm{e}^{\nu\delta^{-1}\lambda^{\frac{1}{2m}}}.$$

Combining the last two inequalities, and taking into account that $\mu+\nu = 1$, we conclude that

$$\sum_{|\alpha|=n} \frac{1}{\alpha!}\|\partial_x^\alpha f\|^2_{L^2(\mathbb{R}^d)} \le \frac{C^{2(1+n)}}{n!}\delta^{2n}(n!)^2\lambda^{d\zeta}\,\mathrm{e}^{c\delta^{-1}\lambda^{\frac{1}{2m}}}\|f\|^2_{L^2(\mathbb{R}^d)} \tag{21}$$

with $C^2 = \max\{c_2, (c_1 t_0)^{-\mu}\}$ and $c = \max\{2, c_3\}$.

It remains to consider the case $\lambda \le (1/\delta)^{2k}$. Since then $\mathscr{E}_{\lambda,k,m} \subset \mathscr{E}_{(1/\delta)^{2k},k,m}$, we obtain from (21) with λ replaced by $(1/\delta)^{2k}$ that

$$\begin{aligned}\sum_{|\alpha|=n} \frac{1}{\alpha!}\|\partial_x^\alpha f\|^2_{L^2(\mathbb{R}^d)} &\le \frac{C^{2(1+n)}}{n!}\delta^{2n}(n!)^2\delta^{-2kd\zeta}\,\mathrm{e}^{c\delta^{-1}\delta^{-\frac{k}{m}}}\|f\|^2_{L^2(\mathbb{R}^d)}\\ &\le \frac{C^{2(1+n)}}{n!}\delta^{2n}(n!)^2\,\mathrm{e}^{(c+d)\delta^{-1/\nu}}\|f\|^2_{L^2(\mathbb{R}^d)},\end{aligned} \tag{22}$$

where for the last inequality we used $\delta^{-2kd\zeta} = \delta^{-d/\nu} \le \mathrm{e}^{d\delta^{-1/\nu}}$. In light of $\mathrm{e}^{(c+d)\delta^{-1/\nu}} \ge 1$ and $\mathrm{e}^{\delta^{-1}\lambda^{\frac{1}{2m}}} \ge 1$, the claim now follows from (21) and (22). □

Remarks 27.

(1) Bernstein inequalities closely related to Proposition 26 have recently been obtained in [35, (4.5)] using smoothing properties of the semigroup associated to (fractional powers of) $H_{k,m}$ established in [2]. These smoothing properties also rely on the Agmon estimates for eigenfunctions, so that our proof above is more direct. Moreover, our constant in (20) incorporates the parameter δ, which may be used to force convergence of an associated series, see (24) below, and thus makes our inequality more suitable for our purposes.

(2) In the particular case of the harmonic oscillator, that is, $k = m = 1$, Bernstein inequalities without the unfavorable term $1+\lambda^{d\zeta}$ have already been obtained in the literature. More precisely, [23, Proposition B.1] (cf. also [8, Proposition 3.3(i)]) states that for all $n \ge 0$, $\delta > 0$, $\lambda \ge 0$, and $f \in \mathscr{E}_{\lambda,1,1}$ we have

$$\sum_{|\alpha|=n} \frac{1}{\alpha!}\|\partial_x^\alpha f\|^2_{L^2(\mathbb{R}^d)} \le \frac{C_B(n,\lambda,\delta)/2}{n!}\|f\|^2_{L^2(\mathbb{R}^d)}$$

with

$$C_B(n,\lambda,\delta) = 2(2\delta)^{2n}(n!)^2\,\mathrm{e}^{e\delta^{-2}}\,\mathrm{e}^{2\delta^{-1}\sqrt{\lambda}}. \tag{23}$$

We are finally able to derive the local Bernstein inequalities of the desired form.

Corollary 28. *Let $\lambda > 0$, and let $\Omega \subset \mathbb{R}^d$ be an open set containing the ball $B(0,(2\lambda)^{1/2k})$. Then, for all $n \ge 0$, $\delta > 0$, and $f \in \mathscr{E}_{\lambda,k,m}$ we have*

$$\|f\|^2_{L^2(\mathbb{R}^d)} \le 2\|f\|^2_{L^2(\Omega)},$$

and

$$\sum_{|\alpha|=n} \frac{1}{\alpha!}\|\partial_x^\alpha f\|^2_{L^2(\Omega)} \le \frac{C_B(n,\lambda,\delta)}{n!}\|f\|^2_{L^2(\Omega)},$$

with $C_B(n,\lambda,\delta)$ as in (20).

Proof. We have

$$\||x|^k f\|^2_{L^2(\mathbb{R}^d)} = \langle |x|^{2k} f, f\rangle_{L^2(\mathbb{R}^d)} \le \langle H_{k,m} f, f\rangle_{L^2(\mathbb{R}^d)} \le \lambda \|f\|^2_{L^2(\mathbb{R}^d)},$$

where the last inequality follows by functional calculus. Hence,

$$\|f\|^2_{L^2(\mathbb{R}^d\setminus B(0,(2\lambda)^{1/2k}))} = \||x|^{-k}|x|^k f\|^2_{L^2(\mathbb{R}^d\setminus B(0,(2\lambda)^{1/2k}))} \le \frac{1}{2\lambda}\||x|^k f\|^2_{L^2(\mathbb{R}^d)} \le \frac{1}{2}\|f\|^2_{L^2(\mathbb{R}^d)}.$$

Since Ω contains the ball $B(0,(2\lambda)^{1/2k})$ by hypothesis, this implies that

$$\|f\|^2_{L^2(\mathbb{R}^d)} \le 2\|f\|^2_{L^2(B(0,(2\lambda)^{1/2k}))} \le 2\|f\|^2_{L^2(\Omega)}.$$

Moreover, we deduce from Proposition 26 that

$$\sum_{|\alpha|=n} \frac{1}{\alpha!}\|\partial_x^\alpha f\|^2_{L^2(\Omega)} \le \sum_{|\alpha|=n} \frac{1}{\alpha!}\|\partial_x^\alpha f\|^2_{L^2(\mathbb{R}^d)} \le \frac{C_B(n,\lambda,\delta)/2}{n!}\|f\|^2_{L^2(\mathbb{R}^d)},$$

which, together with the former inequality, proves the claim. □

For future reference and in light of Proposition 23, we now consider for $\delta, l > 0$ and $\lambda > 0$ the quantity

$$\begin{aligned} h(l,\lambda,\delta) &:= \sum_{n\ge 0} \sqrt{C_B(n,\lambda,\delta)}\,\frac{(10dl)^n}{n!} \\ &= \sqrt{2}C\,(1+\lambda^{d\zeta})^{1/2}\,\mathrm{e}^{(c+d)2^{-1}\delta^{-1/\nu}}\,\mathrm{e}^{c(2\delta)^{-1}\lambda^{\frac{1}{2m}}} \sum_{n\ge 0}(10dlC\delta)^n. \end{aligned}$$

With the particular choice $\delta^{-1} = 20dlC$, we deduce that there is a constant $C' > 0$, depending only on k, m, and the dimension d, such that

$$h(l,\lambda) := h(l,\lambda,(20dlC)^{-1}) \le C'(1+\lambda^{d\zeta})^{1/2}\,\mathrm{e}^{C'l^{1/\nu}}\,\mathrm{e}^{C'l\lambda^{\frac{1}{2m}}}. \tag{24}$$

3.4. *Conclusion of Theorem 3*

Let $\omega \subset \mathbb{R}^d$ be a measurable set as in (3), and let $f \in \mathscr{E}_{\lambda,k,m}$ with $\lambda \ge 0$. Consider $\lambda_{k,m} := \min\operatorname{spec}(H_{k,m}) > 0$. Then, if $\lambda \in [0,\lambda_{k,m})$, we have $\mathscr{E}_{\lambda,k,m} = \{0\}$ and there is nothing to prove. It therefore suffices to consider $\lambda \ge \lambda_{k,m} > 0$.

The key step is to use the well-known Besicovitch covering theorem in the following formulation taken from [18, Proposition 7.1]; see also [39, Theorem 2.7].

Proposition 29 (Besicovitch). *Let $A \subset \mathbb{R}^d$ be bounded, and let $\mathscr{B}$ be a family of open balls such that each point in A is the center of some ball from $\mathscr{B}$. Then there are at most countably many balls $(B_j)_j \subset \mathscr{B}$ such that*

$$\mathbb{1}_A \le \sum_j \mathbb{1}_{\overline{B}_j} \le K^d_{\mathrm{Bes}}, \tag{25}$$

where $K_{\mathrm{Bes}} \ge 1$ is a universal constant.

We are finally in position to prove Theorem 3.

Proof of Theorem 3. Suppose that $\lambda \ge \lambda_{k,m} > 0$, and let

$$A := B(0,(2\lambda)^{1/2k}) \quad \text{and} \quad \mathscr{B} := \{B(x,\rho(x)) : x \in A\}.$$

Besicovitch's covering theorem then implies that there is a finite or countably infinite collection of points $x_j \in A$ such that (25) holds with $B_j = B(x_j,\rho(x_j))$. In particular, A is contained in the union $\bigcup_j \overline{B}_j$. Let Ω be the interior of $\bigcup_j \overline{B}_j$. Then, Ω is open and contains the open set A by definition. Moreover, it is easy to see that Ω is a domain.

With $\theta_{\lambda,k}$ and $L_{\lambda,k}$ from (9), for each j we clearly have $\sigma(x_j) \geq \theta_{\lambda,k}$ and $\rho(x_j) \leq L_{\lambda,k}$. Hence, the family $\{B_j\}_j$ gives a $(K_{\mathrm{Bes}}^d, L_{\lambda,k})$-covering of Ω in the sense of Section 3.1, and from (3) we have

$$\inf_j \frac{|\omega \cap B_j|}{\operatorname{diam}(B_j)^d} = \frac{|B(0,1)|}{2^d} \inf_j \frac{|\omega \cap B(x_j, \rho(x_j))|}{|B(x_j,\rho(x_j))|} \geq \frac{|B(0,1)|}{2^d} \inf_j \sigma(x_j) \geq \frac{|B(0,1)|}{2^d}\theta_{\lambda,k}.$$

Taking into account Corollary 28 and (24), applying Proposition 23 with $\{Q_j\}_j = \{B_j\}_j$, $l = L_{\lambda,k}$, and $h(\lambda) = h(L_{\lambda,k}, \lambda)$ therefore yields

$$\|f\|^2_{L^2(\mathbb{R}^d)} \leq 2\|f\|^2_{L^2(\Omega)} \leq \frac{K_{\mathrm{Bes}}^d}{3}\left(\frac{24d2^d}{\theta_{\lambda,k}}\right)^{2\frac{\log K_{\mathrm{Bes}}^d}{\log 2}+4\frac{\log h(\lambda)}{\log 2}+5} \|f\|^2_{L^2(\omega\cap\Omega)}.$$

Here, we observe that for all $r \geq 0$, we have

$$1 + \lambda^r \leq (1+\lambda_{k,m}^{-r})\lambda^r \leq (1+\lambda_{k,m}^{-r})(1+\lambda)^r,$$

so that

$$\log h(\lambda) \leq \frac{1}{2}\log\big((C')^2(1+\lambda_{k,m}^{-d\zeta})\big) + \frac{d\zeta}{2}\log(1+\lambda) + C'(L_{\lambda,k})^{1/\nu} + C' L_{\lambda,k}\lambda^{\frac{1}{2m}}.$$

The claim therefore follows from the above upon an appropriate choice of the constant K, depending on d, C', $\lambda_{k,m}$, ν, ζ, and K_{Bes}, that is, effectively only on d, k, and m. □

We close this section by briefly discussing how the proof of Theorem 3 can be adapted to obtain Theorem 1.

Proof of Theorem 1. As mentioned in Remark 27(2), in the particular case of the harmonic oscillator, that is, $k = m = 1$, there are Bernstein inequalities available that do not contain the unfavorable term $1+\lambda^{d\zeta}$. Upon replacing the constant (20) by (23), one can then follow the proof of Theorem 3 verbatim towards a proof of Theorem 1, thereby avoiding the term $\log(1+\lambda)$ in the final estimate. □

4. Proof of the exact null-controllability results

In this last main section we use the spectral inequalities given by Theorems 1 and 3 in order to prove the exact null-controllability results from Section 2.2 for the evolution equations $(E_{s,k,m})$ and $(E_{\gamma,s})$.

Since the operators $H_{k,m}^s$ and $(-\Delta_\gamma)^s$ are selfadjoint in $L^2(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d \times \mathbb{T}^d)$, respectively, the Hilbert Uniqueness Method implies that the exact null-controllability of these equations is equivalent to the exact observability of the associated semigroups $(\mathrm{e}^{-tH_{k,m}^s})_{t\geq 0}$ and $(\mathrm{e}^{-t(-\Delta_\gamma)^s})_{t\geq 0}$. The latter is defined as follows.

Definition 30 (Exact observability). *Let $\tau > 0$, and let $\Omega \subset \mathbb{R}^d$ and $\omega \subset \Omega$ be measurable. A strongly continuous semigroup $(T(t))_{t\geq 0}$ on $L^2(\Omega)$ is said to be exactly observable from the set ω in time τ if there exists a positive constant $C_{\omega,\tau} > 0$ such that for all $g \in L^2(\Omega)$, we have*

$$\|T(\tau)g\|^2_{L^2(\Omega)} \leq C_{\omega,\tau}\int_0^\tau \|T(t)g\|^2_{L^2(\omega)}\,\mathrm{d}t.$$

In order to prove exact observability estimates, with an explicit observability constant $C_{\omega,\tau}$, we use the following quantitative result that is based on the well-known Lebeau–Robbiano strategy and is particularly well adapted to the equations we are studying.

Theorem 31 ([41, Theorem 2.8]). *Let A be a non-negative selfadjoint operator in $L^2(\mathbb{R}^d)$, and let $\omega \subset \mathbb{R}^d$ be measurable. Suppose that there are $d_0 > 0$, $d_1 \geq 0$, and $\eta \in (0,1)$ such that for all $\lambda \geq 0$ and $f \in \mathcal{E}_\lambda(A)$,*

$$\|f\|^2_{L^2(\mathbb{R}^d)} \leq d_0\,\mathrm{e}^{d_1\lambda^\eta}\|f\|^2_{L^2(\omega)}.$$

Then, there exist positive constants $c_1, c_2, c_3 > 0$*, only depending on* η*, such that for all* $T > 0$ *and* $g \in L^2(\mathbb{R}^d)$ *we have the observability estimate*

$$\|\mathrm{e}^{-TA} g\|_{L^2(\mathbb{R}^d)}^2 \leq \frac{C_{\mathrm{obs}}}{T} \int_0^T \|\mathrm{e}^{-tA} g\|_{L^2(\omega)}^2 \,\mathrm{d}t,$$

where the positive constant $C_{\mathrm{obs}} > 0$ *is given by*

$$C_{\mathrm{obs}} = c_1 d_0 (2d_0 + 1)^{c_2} \exp\left(c_3 \left(\frac{d_1}{T^\eta} \right)^{\frac{1}{1-\eta}} \right).$$

While the Lebeau–Robbiano strategy in Theorem 31 requires that the constant in the spectral inequality exhibits a sublinear power growth in the exponent in terms of the spectral parameter λ, the following statement allows a more general subexponential growth in λ, but does not provide a quantitative observability estimate.

Theorem 32 ([22, Theorem 5]). *Let* A *be a non-negative selfadjoint operator on* $L^2(\mathbb{R}^d)$*, and let* $\omega \subset \mathbb{R}^d$ *be measurable. Suppose that the spectral inequality*

$$\|f\|_{L^2(\mathbb{R}^d)}^2 \leq c\,\mathrm{e}^{c\lambda/((\log\log\lambda)^\alpha \log\lambda)} \|f\|_{L^2(\omega)}^2, \quad f \in \mathcal{E}_\lambda(A),\ \lambda > e,$$

holds with some $\alpha > 2$ *and* $c > 0$*. Then, for all* $T > 0$*, there exists a positive constant* $C_T > 0$ *such that for all* $g \in L^2(\mathbb{R}^d)$ *we have*

$$\|\mathrm{e}^{-TA} g\|_{L^2(\mathbb{R}^d)}^2 \leq C_T \int_0^T \|\mathrm{e}^{-tA} g\|_{L^2(\omega)}^2 \,\mathrm{d}t.$$

4.1. *Null-controllability of the Shubin evolution equations*

Let us first focus on the results regarding the equation $(E_{s,k,m})$. Here, in order to deal with the fractional powers of $H_{k,m}$, we use the fact that by the transformation formula for spectral measures, see, e.g., [42, Proposition 4.24], for all $s > 0$ and $\lambda \geq 0$ we have

$$\mathcal{E}_{\lambda,s,k,m} := \mathbb{1}_{(-\infty,\lambda]}(H_{k,m}^s) = \mathbb{1}_{(-\infty,\lambda^{1/s}]}(H_{k,m}) = \mathcal{E}_{\lambda^{1/s},k,m}. \tag{26}$$

In essence, this implies that a spectral inequality for $H_{k,m}$ yields a spectral inequality for $H_{k,m}^s$ by just replacing λ by $\lambda^{1/s}$ in the corresponding constant.

We are now in position to prove Corollaries 9 and 11.

Proof of Corollary 9. Under the hypotheses on σ and ρ, we are in the situation of Remark 5 with $\delta \leq 1$. It therefore immediately follows from (10) and (26) that for some constants $d_0 > 0$ and $d_1 \geq 0$ we have

$$\|f\|_{L^2(\mathbb{R}^d)}^2 \leq d_0\,\mathrm{e}^{d_1\lambda^\eta} \|f\|_{L^2(\omega)}^2, \quad f \in \mathcal{E}_{\lambda,s,k,m} = \mathcal{E}_{\lambda^{1/s},k,m},$$

with

$$\eta = \frac{\delta + a}{2sk} + \frac{1}{2sm} < 1.$$

The claim then immediately follows by applying Theorem 31. □

Proof of Corollary 11. Given $s > a/2k + 1/2m$, we pick a $\delta \in (0,1)$ such that

$$\frac{a}{2k} + \frac{1}{2m} < \frac{\delta + a}{2k} + \frac{1}{2m} < s.$$

The hypothesis on ρ, namely $\rho(x) = o(|x|^\delta)$ as $|x| \to +\infty$, then implies that there is $L > 0$ such that

$$\rho(x) \leq L\langle x\rangle^\delta, \quad x \in \mathbb{R}^d.$$

We are thus in the situation of Corollary 9 and the claim is just an instance of that result. □

While the two corollaries above rely on the Lebeau–Robbiano strategy from Theorem 31, Corollary 13 has to revert to the more general statement in Theorem 32.

Proof of Corollary 13. Under the hypothesis (12), it is easy to see that for, say, $\lambda \geq e+1$ we have

$$L_\lambda = \sup_{|x|<\sqrt{2\lambda}} \rho(x) \leq c' \frac{L\sqrt{\lambda}}{(\log\log\lambda)^\alpha \log\lambda}$$

with a suitably chosen constant $c' > 0$ depending on α, but not on L or λ. It then follows from Theorem 1 with a constant function σ that

$$\|f\|^2_{L^2(\mathbb{R}^d)} \leq c\, \mathrm{e}^{c\lambda/((\log\log\lambda)^\alpha \log\lambda)} \|f\|^2_{L^2(\omega)}, \quad f \in \mathscr{E}_{\lambda,1,1},\ \lambda \geq e+1,$$

where $c > 0$ is another constant, depending on L, c', and the dimension d. Taking into account that $\mathscr{E}_{\lambda,1,1} \subset \mathscr{E}_{e+1,1,1}$ for $e < \lambda < e+1$, the latter even holds for all $\lambda > e$ after suitably adapting the constant c. The claim then immediately follows from Theorem 32. ☐

4.2. *Null-controllability of the Baouendi–Grushin evolution equation*

Let us now turn to the null-controllability results for the degenerate parabolic equation $(E_{\gamma,s})$. We first observe that after passing to the Fourier side with respect to $\mathbb{T}^d$-variable, the Baouendi–Grushin operator is transformed as

$$\Delta_x + |x|^{2\gamma}\Delta_y \rightsquigarrow \Delta_x - |n|^2|x|^{2\gamma},$$

where $n \in \mathbb{Z}^d$ is the dual variable of $y \in \mathbb{T}^d$. This motivates to introduce the anharmonic oscillator $H_{\gamma;r}$ in $L^2(\mathbb{R}^d)$ with variably scaled potential[1],

$$H_{\gamma;r} := -\Delta_x + r^2|x|^{2\gamma}, \quad r \geq 0.$$

Consequently, for all $g \in L^2(\mathbb{R}^d \times \mathbb{T}^d)$ and $(x,y) \in \mathbb{R}^d \times \mathbb{T}^d$ we have

$$(\mathrm{e}^{-t(-\Delta_\gamma)^s} g)(x,y) = \sum_{n\in\mathbb{Z}^d} \mathrm{e}^{iy\cdot n} (\mathrm{e}^{-tH^s_{\gamma;|n|}}\, \widehat{g}_n)(x), \tag{27}$$

where

$$\widehat{g}_n = \int_{\mathbb{T}^d} \mathrm{e}^{-iy\cdot n} g(\cdot,y)\,\mathrm{d}y.$$

We first prove that the thickness condition is necessary to obtain a null-controllability result for the equation $(E_{\gamma,s})$.

Proof of Proposition 14. Suppose that the equation $(E_{\gamma,s})$ is exactly null-controllable from a given measurable set $\omega \subset \mathbb{R}^d \times \mathbb{T}^d$ in some positive time $T > 0$. This is equivalent to the existence of a positive constant $C_{\omega,T} > 0$ such that for all $g \in L^2(\mathbb{R}^d \times \mathbb{T}^d)$,

$$\|\mathrm{e}^{-T(-\Delta_\gamma)^s} g\|^2_{L^2(\mathbb{R}^d\times\mathbb{T}^d)} \leq C_{\omega,T} \int_0^T \|\mathrm{e}^{-t(-\Delta_\gamma)^s} g\|^2_{L^2(\omega)}\,\mathrm{d}t. \tag{28}$$

Now, every function $g \in L^2(\mathbb{R}^d)$ can be treated as a function in $L^2(\mathbb{R}^d \times \mathbb{T}^d)$ that is constant with respect to the $\mathbb{T}^d$-variable. As such, $\widehat{g}_n$ in (27) then satisfies $\widehat{g}_n = 0$ for $n \neq 0$ and $\widehat{g}_0 = g$, so that from (27) we obtain for all $t \geq 0$,

$$\mathrm{e}^{-t(-\Delta_\gamma)^s} g = \mathrm{e}^{-t(-\Delta_x)^s} g.$$

Inserting the latter into the observability estimate (28), we deduce that for all $g \in L^2(\mathbb{R}^d)$,

$$\|\mathrm{e}^{-T(-\Delta_x)^s} g\|^2_{L^2(\mathbb{R}^d)} \leq C_{\omega,T} \int_0^T \|\mathrm{e}^{-t(-\Delta_x)^s} g\|^2_{L^2(\omega)}\,\mathrm{d}t. \tag{29}$$

Moreover, by Fubini's theorem, the right-hand side of the latter inequality can for every $g \in L^2(\mathbb{R}^d)$ be rewritten as

$$\int_0^T \|\mathrm{e}^{-t(-\Delta_x)^s} g\|^2_{L^2(\omega)}\,\mathrm{d}t = \int_{\mathbb{T}^d}\int_0^T \|\mathrm{e}^{-t(-\Delta_x)^s} g\|^2_{L^2(\omega_y)}\,\mathrm{d}t\,\mathrm{d}y \tag{30}$$

[1] This notation is to be distinguished from the anisotropic Shubin operator $H_{k,m}$.

with

$$\omega_y = \{x \in \mathbb{R}^d : (x,y) \in \omega\}, \quad y \in \mathbb{T}^d.$$

We now proceed similarly as in the proof of [4, Theorem 2.1(i)]: Given $x_0 \in \mathbb{R}^d$, consider the particular (Gaussian) function $g = g_{x_0} : \mathbb{R}^d \to \mathbb{R}$ with

$$g(x) = g_{x_0}(x) = \exp\left(-\frac{|x-x_0|^2}{2}\right), \quad x \in \mathbb{R}^d,$$

the unitary Fourier transform of which is given by $(\mathscr{F}g)(\xi) = \widehat{g}(\xi) = \mathrm{e}^{-ix_0\cdot\xi} g_0(\xi)$. With $h_t(\xi) = \mathrm{e}^{-t|\xi|^{2s}} g_0(\xi)$, $\xi \in \mathbb{R}^d$, $t > 0$, we may choose $L > 0$ so large that

$$C_{\omega,T} \int_0^T \|\mathscr{F}^{-1} h_t\|^2_{L^2(B(0,L)^c)} \,\mathrm{d}t < \frac{1}{2} \|\mathrm{e}^{-T(-\Delta_x)^s} g\|^2_{L^2(\mathbb{R}^d)}.$$

In light of

$$\mathrm{e}^{-t(-\Delta_x)^s} g = \mathscr{F}^{-1}(\xi \longmapsto \mathrm{e}^{-t|\xi|^{2s}} \widehat{g}(\xi)) = (\mathscr{F}^{-1} h_t)(\cdot - x_0),$$

inserting $g = g_{x_0}$ into (29) and (30) and a change of variables then yield that

$$\begin{aligned}\frac{1}{2} \|\mathrm{e}^{-T(-\Delta_x)^s} g\|^2_{L^2(\mathbb{R}^d)} &\le C_{\omega,T} \int_{\mathbb{T}^d} \int_0^T \|\mathscr{F}^{-1} h_t\|^2_{L^2((\omega_y - x_0)\cap B(0,L))} \,\mathrm{d}t\,\mathrm{d}y \\ &\le C_{\omega,T} \int_0^T \|\mathscr{F}^{-1} h_t\|^2_{L^\infty(\mathbb{R}^d)} \,\mathrm{d}t \int_{\mathbb{T}^d} |\omega_y \cap B(x_0,L)| \,\mathrm{d}y\end{aligned}$$

with

$$\int_0^T \|\mathscr{F}^{-1} h_t\|^2_{L^\infty(\mathbb{R}^d)} \,\mathrm{d}t \le \int_0^T \|h_t\|^2_{L^1(\mathbb{R}^d)} \,\mathrm{d}t \le T \|g_0\|^2_{L^1(\mathbb{R}^d)} < \infty.$$

Hence, for some $\theta \in (0,1]$ independent of x_0, we have

$$|\omega \cap (B(x_0,L) \times \mathbb{T}^d)| = \int_{\mathbb{T}^d} |\omega_y \cap B(x_0,L)| \,\mathrm{d}y \ge \theta |B(x_0,L)|,$$

which proves the claim. □

Parts of the statements of Theorems 15, 17 and 19 can be proved simultaneously. Here, we first focus on the positive results in Theorem 15(ii) ⇒ (i) and Theorem 17(ii), which require some preparation. Consider for $r > 0$ the unitary transformation $M_{\gamma,r}$ in $L^2(\mathbb{R}^d)$ defined by

$$M_{\gamma,r} g = r^{\frac{d}{2(\gamma+1)}} g\left(r^{\frac{1}{\gamma+1}} \cdot\right), \quad g \in L^2(\mathbb{R}^d). \tag{31}$$

With $H_\gamma = H_{\gamma;1}$, a straightforward computation shows that

$$(M_{\gamma,r})^* (H_{\gamma;r})^s M_{\gamma,r} = r^{\frac{2s}{\gamma+1}} (H_\gamma)^s, \quad r, s > 0. \tag{32}$$

The latter allows, in particular, to obtain observability estimates for the operators $H^s_{\gamma;r}$, $r \ge 1$, $s > 1/2$, simultaneously:

Proposition 33. *Let $s > 1/2$. Then, there exists a constant $K > 0$, depending only on γ, s, and the dimension d, such that for all (θ, L)-thick sets $\omega \subset \mathbb{R}^d$, $r \ge 1$, $T > 0$, and $g \in L^2(\mathbb{R}^d)$, we have*

$$\|\mathrm{e}^{-TH^s_{\gamma;r}} g\|^2_{L^2(\mathbb{R}^d)} \le \frac{C_{\mathrm{obs}}}{T} \int_0^T \|\mathrm{e}^{-tH^s_{\gamma;r}} g\|^2_{L^2(\omega)} \,\mathrm{d}t,$$

where the positive constant $C_{\mathrm{obs}} > 0$ is given by

$$C_{\mathrm{obs}} = K\left(\frac{K}{\theta}\right)^{K(1+rL^{1+\gamma})} \exp\left(\frac{K((1+L)\log(K/\theta))^{\frac{2s}{2s-1}}}{T^{\frac{1}{2s-1}}}\right). \tag{33}$$

Proof. It follows from (10) with $a = 0$ and $\delta = 0$ that for every (θ, L)-thick set $\omega \subset \mathbb{R}^d$, we have

$$\|f\|^2_{L^2(\mathbb{R}^d)} \le \left(\frac{C}{\theta}\right)^{C(1+L^{1+\gamma}+L\sqrt{\lambda}+\log(1+\lambda))} \|f\|^2_{L^2(\omega)}, \quad f \in \mathscr{E}_{\lambda,\gamma,1},\ \lambda \ge 0, \tag{34}$$

with a constant $C > 0$ depending only on γ and the dimension d.

Let us now fix some $r \ge 1$ and a (θ, L)-thick set $\omega \subset \mathbb{R}^d$. In light of the similarity relation (32), we clearly have

$$(M_{\gamma,r})^* \mathscr{E}_\lambda(H^s_{\gamma;r}) \subset \mathscr{E}_\lambda\left(r^{\frac{2s}{\gamma+1}} H^s_\gamma\right) = \mathscr{E}_{\lambda/r^{2s/(\gamma+1)}}(H^s_\gamma) = \mathscr{E}_{\lambda^{1/s}/r^{2/(\gamma+1)},\gamma,1}.$$

Moreover, one easily checks that the set $\widetilde{\omega} := r^{1/(\gamma+1)}\omega$ is $(\theta, r^{1/(\gamma+1)}L)$-thick. We therefore deduce from (34) that for all $\lambda \ge 0$ and $f \in \mathscr{E}_\lambda(H^s_{\gamma;r})$, we have

$$\begin{aligned}\|f\|^2_{L^2(\mathbb{R}^d)} = \|(M_{\gamma,r})^* f\|^2_{L^2(\mathbb{R}^d)} &\le \left(\frac{C}{\theta}\right)^{C\left(1+rL^{1+\gamma}+L\lambda^{\frac{1}{2s}}+\log\left(1+r^{-\frac{2}{\gamma+1}}\lambda^{\frac{1}{s}}\right)\right)} \|(M_{\gamma,r})^* f\|^2_{L^2(\widetilde{\omega})} \\ &\le \left(\frac{C}{\theta}\right)^{C\left(1+rL^{1+\gamma}+(1+L)\lambda^{\frac{1}{2s}}\right)} \|f\|^2_{L^2(\omega)},\end{aligned}$$

since $r \ge 1$ and, thus, $\log(1 + r^{-\frac{2}{\gamma+1}}\lambda^{\frac{1}{s}}) \le \log(1+\lambda^{\frac{1}{s}}) \le \lambda^{\frac{1}{2s}}$. The latter can be rewritten as

$$\|f\|^2_{L^2(\mathbb{R}^d)} \le d_0\, e^{d_1\lambda^{\frac{1}{2s}}} \|f\|^2_{L^2(\omega)}, \quad f \in \mathscr{E}_\lambda(H^s_{\gamma;r}),$$

with

$$d_0 = \left(\frac{C}{\theta}\right)^{C(1+rL^{1+\gamma})} \quad \text{and} \quad d_1 = C(1+L)\log\left(\frac{C}{\theta}\right).$$

Theorem 31 then implies that there exist universal positive constants $c_1, c_2, c_3 > 0$ such that for all $T > 0$ and $g \in L^2(\mathbb{R}^d)$, we have

$$\|e^{-TH^s_{\gamma;r}} g\|^2_{L^2(\mathbb{R}^d)} \le \frac{C_{\mathrm{obs}}}{T}\int_0^T \|e^{-tH^s_{\gamma;r}} g\|^2_{L^2(\omega)}\,dt,$$

with $C_{\mathrm{obs}} = C_{\mathrm{obs}}(\omega, T, r)$ given by

$$C_{\mathrm{obs}} = c_1 d_0 (2d_0+1)^{c_2} \exp\left(\frac{c_3 d_1^{\frac{2s}{2s-1}}}{T^{\frac{1}{2s-1}}}\right).$$

It only remains to observe that there exists another positive constant $c_4 > 0$, depending only on the dimension d, such that

$$d_0(2d_0+1)^{c_2} \le \left(\frac{c_4}{\theta}\right)^{c_4(1+rL^{1+\gamma})}.$$

This ends the proof of Proposition 33 upon a suitable choice of the constant K. □

Proof of Theorem 15(ii) ⇒ (i) and Theorem 17(ii). Let $\omega \subset \mathbb{R}^d$ be a (θ, L)-thick set. We have to show that whenever $T \ge T^*$, with some time $T^* \ge 0$ depending on θ and L that is to be determined, there exists a constant $C_{\omega,T} > 0$ such that for all $g \in L^2(\mathbb{R}^d \times \mathbb{T}^d)$ we have

$$\|e^{-T(-\Delta_\gamma)^s} g\|^2_{L^2(\mathbb{R}^d\times\mathbb{T}^d)} \le C_{\omega,T}\int_0^T \|e^{-t(-\Delta_\gamma)^s} g\|^2_{L^2(\omega\times\mathbb{T}^d)}\,dt. \tag{35}$$

To this end, we first observe from (27), Fubini's theorem, and Parseval's identity that for every measurable set $\Omega \subset \mathbb{R}^d$ and all $t > 0$ and $g \in L^2(\mathbb{R}^d \times \mathbb{T}^d)$ we have

$$\|e^{-t(-\Delta_\gamma)^s} g\|^2_{L^2(\Omega\times\mathbb{T}^d)} = \sum_{n\in\mathbb{Z}^d} \|e^{-tH^s_{\gamma;n}}\widehat{g}_n\|^2_{L^2(\Omega)},$$

where we write $H_{\gamma;n} = H_{\gamma;|n|}$ for every $n \in \mathbb{Z}^d$. Inserting the latter into both sides of (35), once with $\Omega = \mathbb{R}^d$ and $t = T$ and once with $\Omega = \omega$, we immediately infer by Fubini's theorem that it suffices to show that

$$\|\mathrm{e}^{-TH^s_{\gamma;n}} g\|^2_{L^2(\mathbb{R}^d)} \leq C_{\omega,T} \int_0^T \|\mathrm{e}^{-tH^s_{\gamma;n}} g\|^2_{L^2(\omega)} \,\mathrm{d}t, \quad g \in L^2(\mathbb{R}^d),\ n \in \mathbb{Z}^d, \tag{36}$$

with a constant $C_{\omega,T} > 0$ not depending on n. Here, for $n = 0$, the operator $H^s_{\gamma;0}$ reduces to the fractional Laplacian $(-\Delta_x)^s$ on $\mathbb{R}^d$. Corresponding observability estimates from thick sets are well known in the literature, see, e.g., [41, Theorem 4.10] or [3, Theorem 1.12]. It is therefore sufficient to focus on the case $|n| \geq 1$. Here, on the one hand, we deduce from (32) that

$$\|\mathrm{e}^{-tH^s_{\gamma;n}} g\|_{L^2(\mathbb{R}^d)} \leq \mathrm{e}^{-t\lambda^s_\gamma |n|^{\frac{2s}{1+\gamma}}} \|g\|_{L^2(\mathbb{R}^d)}, \quad g \in L^2(\mathbb{R}^d),\ t \geq 0,$$

where $\lambda_\gamma > 0$ again denotes the smallest eigenvalue of the anharmonic oscillator H_γ. This implies, in particular, that

$$\|\mathrm{e}^{-TH^s_{\gamma;n}} g\|^2_{L^2(\mathbb{R}^d)} \leq \mathrm{e}^{-T\lambda^s_\gamma |n|^{\frac{2s}{1+\gamma}}} \|\mathrm{e}^{-(T/2)H^s_{\gamma;n}} g\|^2_{L^2(\mathbb{R}^d)}, \quad g \in L^2(\mathbb{R}^d).$$

On the other hand, it follows from Proposition 33 that for all $n \in \mathbb{Z}^d \setminus \{0\}$ and $g \in L^2(\mathbb{R}^d)$, we have

$$\|\mathrm{e}^{-(T/2)H^s_{\gamma;n}} g\|^2_{L^2(\mathbb{R}^d)} \leq \frac{2C_{\mathrm{obs}}}{T} \int_0^{T/2} \|\mathrm{e}^{-tH^s_{\gamma;n}} g\|^2_{L^2(\omega)} \,\mathrm{d}t,$$

where $C_{\mathrm{obs}} = C_{\mathrm{obs}}(\omega, T/2, |n|)$ is given by (33) with T replaced by $T/2$. Combining these two estimates, we therefore obtain that for all $n \in \mathbb{Z}^d \setminus \{0\}$ and $g \in L^2(\mathbb{R}^d)$,

$$\begin{aligned}\|\mathrm{e}^{-TH^s_{\gamma;n}} g\|^2_{L^2(\mathbb{R}^d)} &\leq \exp\big(K|n|L^{1+\gamma}\log(K/\theta) - |n|^{\frac{2s}{1+\gamma}}\lambda^s_\gamma T\big) \\ &\qquad \times K\left(\frac{K}{\theta}\right)^K \exp\left(\frac{K((1+L)\log(K/\theta))^{\frac{2s}{2s-1}}}{T^{\frac{1}{2s-1}}}\right) \frac{2}{T} \int_0^{T/2} \|\mathrm{e}^{-tH^s_{\gamma;n}} g\|^2_{L^2(\omega)} \,\mathrm{d}t.\end{aligned}$$

This shows (36), provided that

$$\sup_{|n|\geq 1} \exp\big(K|n|L^{1+\gamma}\log(K/\theta) - |n|^{\frac{2s}{1+\gamma}}\lambda^s_\gamma T\big) < +\infty.$$

The latter is the case for every $T > 0$ if $s > (1+\gamma)/2$, which proves the implication (ii) ⇒ (i) in Theorem 15, and if $s = (1+\gamma)/2$, it requires

$$T \geq T^* := K\lambda_\gamma^{-s} L^{1+\gamma} \log(K/\theta),$$

as claimed in Theorem 17(ii). □

We now finally turn to the negative null-controllability results for the equation $(E_{\gamma,s})$.

Proof of Theorem 17(i) and Theorem 19. Let $\omega \subset \mathbb{R}^d \times \mathbb{T}^d$ be a measurable set satisfying the geometric condition $\overline{\omega} \cap \{x = 0\} = \emptyset$. We assume that for some positive time $T > 0$ there exists a positive constant $C_{\omega,T} > 0$ such that for all functions $g \in L^2(\mathbb{R}^d \times \mathbb{T}^d)$ we have the observability estimate

$$\|\mathrm{e}^{-T(-\Delta_\gamma)^s} g\|^2_{L^2(\mathbb{R}^d\times\mathbb{T}^d)} \leq C_{\omega,T} \int_0^T \|\mathrm{e}^{-t(-\Delta_\gamma)^s} g\|^2_{L^2(\omega)} \,\mathrm{d}t. \tag{37}$$

Let $\psi_\gamma \in L^2(\mathbb{R}^d)$ be a normalized eigenfunction for the anharmonic oscillator H_γ corresponding to the smallest eigenvalue $\lambda_\gamma > 0$. For each $n \in \mathbb{Z}^d \setminus \{0\}$, consider the function $g_n \in L^2(\mathbb{R}^d \times \mathbb{T}^d)$ given by

$$g_n(x, y) = \mathrm{e}^{in\cdot y}(M_{\gamma,n}\psi_\gamma)(x), \quad (x, y) \in \mathbb{R}^d \times \mathbb{T}^d, \tag{38}$$

where the isometry $M_{\gamma,n} = M_{\gamma,|n|}$ in $L^2(\mathbb{R}^d)$ is defined as in (31). In light of the similarity relation (32), it is then clear that $(-\Delta_\gamma)g_n = |n|^{\frac{2}{1+\gamma}}\lambda_\gamma g_n$ as well as

$$\|g_n\|_{L^2(\mathbb{R}^d\times\mathbb{T}^d)} = 1 \quad \text{and} \quad \|g_n\|_{L^2(\omega)} = \|\psi_\gamma\|_{L^2(\omega_n)},$$

where

$$\omega_n = \{(|n|^{1/(1+\gamma)}x, y)\colon (x, y) \in \omega\},$$

and where ψ_γ is interpreted as a function in $L^2(\mathbb{R}^d \times \mathbb{T}^d)$ that is constant with respect to the $\mathbb{T}^d$-variable. The observability estimate (37) applied to $g = g_n$ therefore implies that

$$e^{-2|n|^{\frac{2s}{1+\gamma}}\lambda_\gamma^s T} \le C_{\omega,T}\int_0^T e^{-2|n|^{\frac{2s}{1+\gamma}}\lambda_\gamma^s t}\|\psi_\gamma\|^2_{L^2(\omega_n)}\,dt \le TC_{\omega,T}\|\psi_\gamma\|^2_{L^2(\omega_n)}. \tag{39}$$

Using the classical, more precise Agmon estimate for the anharmonic oscillator H_γ mentioned in part (2) of Remark 25, for every $\varepsilon \in (0,1)$ we have

$$\left\| e^{\frac{\varepsilon|x|^{1+\gamma}}{1+\gamma}}\psi_\gamma\right\|_{L^2(\mathbb{R}^d)} \le c_{\varepsilon,\gamma},$$

where $c_{\varepsilon,\gamma} > 0$ is a positive constant depending only on ε, γ, and the dimension d. Thus, with $L := \operatorname{dist}(0,\omega)$, for each $n \in \mathbb{Z}^d \setminus \{0\}$ we have

$$\|\psi_\gamma\|_{L^2(\omega_n)} = \left\| e^{-\frac{\varepsilon|x|^{1+\gamma}}{1+\gamma}} e^{\frac{\varepsilon|x|^{1+\gamma}}{1+\gamma}}\psi_\gamma\right\|_{L^2(\omega_n)} \le c_{\varepsilon,\gamma}\, e^{-\frac{\varepsilon|n|L^{1+\gamma}}{1+\gamma}}. \tag{40}$$

Inserting the latter into (39), we deduce for each $n \in \mathbb{Z}^d \setminus \{0\}$ that

$$1 \le TC_{\omega,T}c^2_{\varepsilon,\gamma}\exp\left(2|n|^{\frac{2s}{1+\gamma}}\lambda_\gamma^s T - \frac{2\varepsilon|n|L^{1+\gamma}}{1+\gamma}\right). \tag{41}$$

Now, if $0 < s < (1+\gamma)/2$ or if $s = (1+\gamma)/2$ and $0 < T < (\varepsilon/(1+\gamma))(L/\sqrt{\lambda_\gamma})^{\gamma+1}$, then

$$\exp\left(2|n|^{\frac{2s}{1+\gamma}}\lambda_\gamma^s T - \frac{2\varepsilon|n|L^{1+\gamma}}{1+\gamma}\right) \underset{|n|\to+\infty}{\longrightarrow} 0,$$

which contradicts the estimate (41). This ends the proof of Theorem 19 and, after letting $\varepsilon \to 1^-$, also of the one of Theorem 17(i). □

Remark 34. It is worth to note that the Schrödinger-type equation corresponding to the fractional Baouendi–Grushin operator, that is, the equation $(SE_{\gamma,s})$, is for no $s > 0$ and at no time $T > 0$ null-controllable from a control support ω satisfying the geometric condition $\overline{\omega} \cap \{x = 0\} = \emptyset$. Indeed, assume to the contrary that there exists a positive constant $C_{\omega,T} > 0$ such that for all $g \in L^2(\mathbb{R}^d \times \mathbb{T}^d)$ we have the observability estimate

$$\|g\|^2_{L^2(\mathbb{R}^d\times\mathbb{T}^d)} \le C_{\omega,T}\int_0^T \|e^{it(-\Delta_\gamma)^s}g\|^2_{L^2(\omega)}\,dt. \tag{42}$$

Inserting again the function g_n defined in (38) and using the estimate (40), we deduce that for all $n \in \mathbb{Z}^d \setminus \{0\}$ we have

$$1 \le C_{\omega,T}Tc^2_{\varepsilon,\gamma}\, e^{-\frac{2\varepsilon|n|L^{1+\gamma}}{1+\gamma}} \underset{|n|\to+\infty}{\longrightarrow} 0.$$

Hence, the estimate (42) can never hold for all $g \in L^2(\mathbb{R}^d \times \mathbb{T}^d)$ simultaneously.

Appendix A. Asymptotic bounds on the smallest eigenvalue of anharmonic oscillators

In this appendix, we prove a two-sided asymptotics as $k \to +\infty$ for the smallest eigenvalue λ_k of the anharmonic oscillator $H_k = H_{k,1} = -\Delta + |x|^{2k}$, $k \in \mathbb{N}\setminus\{0\}$, in $L^2(\mathbb{R}^d)$ equipped with its maximal domain. This is a key ingredient for Example 20 in the main part of the manuscript.

Lemma 35. *For fixed $\varepsilon > 0$, the two-sided bound*

$$\frac{\lambda_D}{(1+\varepsilon)^2} + o(1) \le \lambda_k \le \lambda_D + \int_{B(0,1)} |x|^{2k}|\psi_D(x)|^2\,dx \tag{43}$$

holds, where the lower bound is to be understood as $k \to +\infty$, and where $\lambda_D > 0$ denotes the smallest eigenvalue of the Dirichlet Laplacian on the canonical Euclidean unit ball $B(0,1)$ in $\mathbb{R}^d$ and ψ_D is an associated normalised eigenfunction.

Proof. The upper bound in (43) follows immediately from the standard min-max principle. Let us therefore focus on the lower bound. To this end, fix $\varepsilon > 0$ and observe that for all $x \notin B(0,1+\varepsilon)$ we have $|x|^{2k} \geq (1+\varepsilon)^{2k}$. This gives

$$-\Delta + (1+\varepsilon)^{2k} \mathbb{1}_{B(0,1+\varepsilon)^c} \leq H_k,$$

in the sense of quadratic forms, and it follows from the min-max principle that

$$\min \operatorname{spec}(-\Delta + (1+\varepsilon)^{2k} \mathbb{1}_{B(0,1+\varepsilon)^c}) \leq \lambda_k.$$

By a standard scaling argument, the operator $-\Delta + (1+\varepsilon)^{2k} \mathbb{1}_{B(0,1+\varepsilon)^c}$ is unitarily equivalent to $(1+\varepsilon)^{-2}(-\Delta + (1+\varepsilon)^{2(1+k)} \mathbb{1}_{B(0,1)^c})$. Moreover, by the theory of the large coupling limit [11, 15, 43], the spectrum of $-\Delta + M\mathbb{1}_{B(0,1)^c}$ converges to the one of the Dirichlet Laplacian on $B(0,1)$ as M goes to infinity. More specifically, if follows from [15] that $-\Delta + M\mathbb{1}_{B(0,1)^c}$ converges to the Dirichlet Laplacian on $B(0,1)$ in norm resolvent sense as $M \to +\infty$, so that indeed

$$\min \operatorname{spec}(-\Delta + M\mathbb{1}_{B(0,1)^c}) = \lambda_D + o(1) \quad \text{as } M \longrightarrow +\infty.$$

Applying this result with $M = (1+\varepsilon)^{2(1+k)}$ together with the unitary equivalence mentioned above then proves the lower bound in (43). This completes the proof. □

Remark 36. Since the eigenfunction ψ_D is radially symmetric, one may introduce a new function $\varphi_D \in C^\infty([0,1])$ with $|\psi_D(x)|^2 = \varphi_D(|x|)$. Using polar coordinates, we then obtain

$$\int_{B(0,1)} |x|^{2k} |\psi_D(x)|^2 \, \mathrm{d}x = |\mathbb{S}^{d-1}| \int_0^1 r^{2k+d-1} \varphi_D(r) \, \mathrm{d}r.$$

Now, successive integration by parts in the last integral gives for all $N \geq 1$ that, as $k \to +\infty$,

$$\int_0^1 r^{2k+d-1} \varphi_D(r) \, \mathrm{d}r = \sum_{j=1}^{N-1} \frac{\varphi_D^{(j-1)}(1)}{(2k+d-1+j)^j} + \mathcal{O}\left(\frac{1}{k^N}\right).$$

From Lemma 35 and Remark 36 and considering $\varepsilon \to 0^+$ in (43), we immediately obtain the following result.

Corollary 37. *We have $\lambda_k \to \lambda_D$ as $k \to +\infty$.*

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