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Comptes Rendus
Mathématique

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Volume 362 (2024), p. 1661-1665

Online since: 25 November 2024

https://doi.org/10.5802/crmath.671

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www.centre-mersenne.org — e-ISSN : 1778-3569

Comptes Rendus. Mathématique
2024, Vol. 362, p. 1661-1665
https://doi.org/10.5802/crmath.671

Research article / *Article de recherche*
Group theory, Representation theory / *Théorie des groupes, Théorie des représentations*

On the group pseudo-algebra of finite groups

Sur la pseudo-algèbre de groupe des groupes finis

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Abstract. Let G be a finite group. The group pseudo-algebra of G is defined as the multi-set $C(G) = \{(d, m_G(d)) \mid d \in \mathrm{Cod}(G)\}$, where $m_G(d)$ is the number of irreducible characters of G with codegree $d \in \mathrm{Cod}(G)$. We show that there exist two finite p-groups with distinct orders that have the same group pseudo-algebra, providing an answer to Question 3.2 in [7]. In addition, we also discuss under what hypothesis two p-groups with the same group pseudo-algebra will be isomorphic.

Résumé. Soit G un groupe fini. La pseudo-algèbre de groupe de G est définie comme le multi-ensemble $C(G) = \{(d, m_G(d)) \mid d \in \mathrm{Cod}(G)\}$, où $m_G(d)$ est le nombre de caractères irréductibles de G de codegré $d \in \mathrm{Cod}(G)$. Nous montrons qu'il existe deux p-groupes finis avec des ordres distincts qui ont la même pseudo-algèbre de groupe, ce qui fournit une réponse à la question 3.2 de [7]. De plus, nous discutons également sous quelles hypothèses deux p-groupes ayant la même pseudo-algèbre de groupe sont forcément isomorphes.

Keywords. Finite p-groups, Characters, Group pseudo-algebra.

Mots-clés. p-groupes finis, Caractères, Pseudo-algèbre de groupe.

2020 Mathematics Subject Classification. 20C15, 20D15.

Manuscript received 11 May 2024, revised 16 July 2024, accepted 26 July 2024.

1. Introduction

All groups considered in this article are finite. As usual, G will always be a finite group, and $k(G)$ denotes the number of conjugacy classes of G. We write $\mathrm{Irr}(G)$ to denote the set of complex irreducible characters of G and $\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$. Let $\chi \in \mathrm{Irr}(G)$. The codegree of χ is defined as

$$\mathrm{cod}\,\chi = \frac{|G : \ker\chi|}{\chi(1)},$$

which was introduced by Qian, Wang and Wei in [9]. The concept has been studied extensively and proved to have interesting connections with some algebraic structure of finite groups (see, for example, [4, 6–8, 10]).

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ISSN (electronic): 1778-3569 https://comptes-rendus.academie-sciences.fr/mathematique/

In [7], A. Moretó first introduced the concept of the group pseudo-algebra, which is defined as the multi-set

$$C(G) = \{(d, m_G(d)) \mid d \in \mathrm{Cod}(G)\},$$

where $\mathrm{Cod}(G) = \{\mathrm{cod}\,\chi \mid \chi \in \mathrm{Irr}(G)\}$ and $m_G(d)$ is the number of irreducible characters having codegree d. He showed that if two finite abelian groups have the same group pseudo-algebra, then they are isomorphic. Additionally, a natural question arises: must groups have the same order if they have the same group pseudo-algebra? A particular case of the question asks whether $G \cong A$ provided that $C(G) = C(A)$, where G is a finite group and A is an abelian p-group for some prime p. He gave an affirmative answer when either A is cyclic or the exponent of A does not exceed p^2 (see [7, Theorem 3.4]). Thus, the next natural case to look is when the exponent of A is p^3. We will prove that if G is a nonabelian group and A is an abelian p-group of exponent p^3 so that $C(G) = C(A)$, then $p = 2$. This result guides us in constructing examples, suggesting that the question does not always yield a positive answer.

Theorem 1. *Let p be a prime. There exists an abelian p-group A and a group G with $C(G) = C(A)$ so that G is not isomorphic to A. Hence, groups may have different orders even though they have the same group pseudo-algebra.*

From the above theorem, it is clear that the question does not always have a positive answer when the abelian p-group A has three generators. Thus, it might be a good idea to focus on the case when A has two generators, in particular, when $A \cong C_{p^n} \times C_p$, where p is a prime. We first consider metacyclic groups G satisfying $C(G) = C(A)$ and it turns out that G must be isomorphic to A. Applying this result, we can prove that for a group G and $A \cong C_{p^n} \times C_p$, if $C(G) = C(A)$, then either $G \cong A$ or $|G : G'| = p^2$ and $p > 2$.

Theorem 2. *Let G be a group and $A \cong C_{p^n} \times C_p$, where p is a prime and $n \geqslant 3$ is an integer. Suppose that $C(G) = C(A)$. Then G is a p-group and one of the following holds:*

(1) *$G \cong A$.*
(2) *$|G : G'| = p^2$, $p > 2$ and $Z(G)$ is noncyclic. In addition, there is a unique maximal subgroup X of G' which is normal in G so that the factor group G/X is nonabelian of order p^3 and of exponent p.*

Applying the above result, we show that $G \cong A$ if $A \cong C_{p^3} \times C_p$ and $C(G) = C(A)$. We also demonstrate that under the same hypothesis as stated in the above theorem, if, in addition, G has either a metacyclic maximal subgroup or a two-generator derived subgroup G', then $G \cong A$.

Theorem 3. *Let G be a group and $A \cong C_{p^n} \times C_p$, where p is a prime. Suppose that $C(G) = C(A)$. Then $G \cong A$ if one of the following holds:*

(1) *G has a metacyclic maximal subgroup,*
(2) *The derived subgroup G' is generated by two elements,*
(3) *The derived subgroup G' is abelian.*

2. Main Results

In this section, we start by stating a fact that will be used frequently. Theorem A in [9] yields that if G is a group such that $\mathrm{Cod}(G)$ is a set of powers of a prime p, then G is a p-group. Now we prove the following basic lemmas.

Lemma 4. *Let G be a nonabelian group and A be an abelian p-group of order p^a for some prime p. Suppose that $C(G) = C(A)$. Then $|\mathrm{cd}(G)| \geqslant 3$.*

Proof. Since A is abelian, we have that $A \cong \mathrm{Irr}(A)$ and so $\mathrm{Cod}(A)$ coincides with the set of element orders of A. Hence $\mathrm{Cod}(G) = \mathrm{Cod}(A)$ is a set of powers of p. It follows that G is a p-group. Assume that $|\mathrm{cd}(G)| < 3$. Then $|\mathrm{cd}(G)| = 2$ as G is nonabelian. So we may assume $\mathrm{cd}(G) = \{1, p^e\}$ for some positive integer e. Let $|A| = p^a$, $|G| = p^n$ and $|G : G'| = p^r$. By $C(G) = C(A)$, we have that $k(G) = k(A) = p^a$. Then $p^r < p^a < p^n$. Notice that $|G| = |G : G'| + (k(G) - |G : G'|)p^{2e}$. Thus $p^{n-r} - 1 = (p^{a-r} - 1)p^{2e}$, contrary to the fact that p^{2e} does not divide $p^{n-r} - 1$. Hence $|\mathrm{cd}(G)| \geqslant 3$, as wanted. □

Now we delve deeper into the degree set of G. In particular, we consider the case when $\mathrm{cd}(G) = \{1, p, p^2\}$.

Lemma 5. *Let G be a nonabelian group and A be an abelian p-group of order p^a for some prime p. Suppose that $C(G) = C(A)$ and $\mathrm{cd}(G) = \{1, p, p^2\}$. Then $p = 2$ and $|G| = 2^{a+2}$. In particular, if we write $|G : G'| = p^r$ and $k_1 = |\{\chi \in \mathrm{Irr}(G) \mid \chi(1) = p\}|$ and $k_2 = |\{\chi \in \mathrm{Irr}(G) \mid \chi(1) = p^2\}|$, then $k_1 = p^a - p^r - p^{r-2}$ and $k_2 = p^{r-2}$.*

Proof. Following the same reasoning process as in Lemma 4, G is a p-group. Since $C(G) = C(A)$, we have that $k(G) = k(A) = p^a$ and so $|G| > p^a$ as G is nonabelian. On the other hand, we have that

$$\begin{aligned} |G| &= |G : G'| + k_1 p^2 + k_2 p^4 \\ &\leqslant p^r + p^2 + (p^a - p^r - 1)p^4 \\ &= p^{a+4} - p^{r+4} + p^r - p^4 + p^2 \\ &< p^{a+4}. \end{aligned}$$

Hence, $|G| = p^{a+1}, p^{a+2}$, or p^{a+3}. Notice that $|G : G'| = k(G) - k_1 - k_2$ and so $|G| = k(G) + k_1(p^2 - 1) + k_2(p^4 - 1)$. Hence $(p^2 - 1) \mid (|G| - k(G))$, which indicates that $|G| = p^{a+2}$. By [5, Theorem 3], such groups do not exist if p is odd. Hence, $p = 2$. Now it is easy to see that $k_1 = p^a - p^r - p^{r-2}$ and $k_2 = p^{r-2}$. □

With the above lemmas, we are prepared to provide an example for Theorem 1, and it is advisable to look at 2-groups. Let $A \cong C_{2^3} \times C_2 \times C_2$ be an abelian 2-group. Then $C(A) = \{(1,1), (2,7), (2^2, 8), (2^3, 16)\}$. Since $\chi(1) < \mathrm{cod}\,\chi$ for all non-principal characters χ of G, it follows that $\mathrm{cd}(G)$ is a subset of $\{1, 2, 2^2\}$. By Lemma 4 and Lemma 5, if there is a group G so that $C(G) = C(A)$, then $|G| = 2^7$ and $\mathrm{cd}(G) = \{1, 2, 4\}$. We notice that such a group does exist. For example, using GAP, G can be one of SmallGroup(128, 755), SmallGroup(128, 756), SmallGroup(128, 773). To enhance the readability, we present the information of the irreducible representations of SmallGroup(128, 773) (see https://people.maths.bris.ac.uk/~matyd/GroupNames/128/C4sC4s7D4.html)

dim	1	1	1	1	1	2	2	2	2	2	2	4
type	+	+	+	+	+	+	+	+	+			+
image	C_1	C_2	C_2	C_2	C_2	D_4	D_4	D_4	D_8	SD_{16}	$C_4 \circ D_4$	$C_8 \rtimes C_2^2$
kernel	$C_4 \rtimes C_4 \rtimes_7 D_4$	$C_2^2{\cdot}_7 C_4^2$	$C_2^3{\cdot}_{65} C_2^3$	$C_2 \times D_4 \rtimes C_4$	$C_2 \times C_4 \rtimes_1 D_4$	$C_4 \rtimes C_4$	$C_2^2 \times C_4$	$C_2 \times D_4$	$C_2 \times C_4$	$C_2 \times C_4$	$C_2 \times C_4$	C_2^2
# reps	1	1	1	4	1	2	2	4	4	4	6	2

Since there is a counterexample when A has three generators, we move on to the case when $A = C_{p^n} \times C_p$, where p is a prime. We first give the following lemma.

Lemma 6. *Let G be a group and $A = C_{p^n} \times C_p$, where p is a prime. Suppose that $C(G) = C(A)$. Then G is a p-group and $G/G' \cong C_{p^m} \times C_p$ for some integer $m \leqslant n$.*

Proof. It is clear that [9, Theorem A] implies that G is a p-group. By [7, Lemma 3.3], $G/\Phi(G) \cong A/\Phi(A) \cong C_p \times C_p$. Notice that $G' \leqslant \Phi(G)$ and $\Phi(G)/G' \cong \Phi(G/G')$. Then G/G' has two generators. It follows that $G/G' \cong C_{p^m} \times C_{p^l}$ for some positive integers m and l. If both m and l are greater than 1, then it is easy to see that G has more irreducible characters of codegree p^2 than the abelian group A. This is impossible. Hence, without loss of generality, we can let $l = 1$. On the other hand, it is clear that $m \leqslant n$. The proof is complete now. □

We now consider a class of two-generator groups: metacyclic groups. We obtain the following result.

Theorem 7. *Let G be a metacyclic group and $A = C_{p^n} \times C_p$, where p is a prime. Suppose that $C(G) = C(A)$. Then G is abelian and $G \cong A$.*

Proof. We write $G = HK$, where H and K are cyclic subgroups of G and H is normal in G. If either one of these has index p, then by ([1, Theorem 1.2]), $k(G)$ can be computed and it is not a power of p. Therefore, both H and K have index greater than or equal to p^2. Then the group G maps onto a group, say M, of order p^4, which is a product of two cyclic groups of order p^2. Clearly, the irreducible characters of M can be viewed as irreducible characters of G. If M is abelian, then $M \cong C_{p^2} \times C_{p^2}$. Notice that $C(G) = C(A) = \{(1,1),(p,p^2-1),(p^2,p^2(p-1)),\dots,(p^n,p^n(p-1))\}$. Hence, M has more irreducible characters of codegree p^2 than G. Thus M is nonabelian. It follows that $Z(M)$ is noncyclic of order p^2 and so $\mathrm{cd}(M) = \{1,p\}$ and $|\ker\chi| = p$ for all nonlinear characters $\chi \in \mathrm{Irr}(M)$. In other words, all nonlinear characters of M have codegree p^2. Notice that $M' \leqslant Z(M)$ and M' is cyclic. So M' has order p. Since M/M' must be isomorphic to a subgroup of A, we have that $M/M' \cong C_{p^2} \times C_p$. Hence, there are $p^2(p-1)$ linear characters having codegree p^2. Together with those nonlinear characters, there are more than $p^2(p-1)$ irreducible characters of codegree p^2. □

Next, we give a proof of Theorem 2.

Proof of Theorem 2. If G is abelian, then (1) follows. Assume now that G is nonabelian. By Lemma 6, we can write $G/G' \cong C_{p^{a-1}} \times C_p$, where $2 \leqslant a \leqslant n$. A proof similar to Theorem 7 shows that G is a p-group, $k(G) = k(A) = p^{n+1}$ and $|G| \geqslant p^{n+3}$. Since G is a nonabelian p-group with two generators, by [3, Lemma 2.2] we can let X be the unique maximal subgroup of G' which is normal in G. Consider the factor group $\overline{G} = G/X$. Then $|\overline{G}| = p^{a+1}$, $\overline{G}/\overline{G}' \cong G/G'$, and $\overline{G}' = G'/X$ has order p. Notice that for any irreducible character $\chi \in \mathrm{Irr}(\overline{G})$ with $\chi(1) > 1$, $\mathrm{cod}\,\chi = \frac{p^{a+1}}{\chi(1)|\ker\chi|} \geqslant p^a$. Then $\chi(1) \leqslant p$ and $\ker\chi = 1$. This implies that $\mathrm{cd}(\overline{G}) = \{1,p\}$ and all nonlinear irreducible characters are faithful. Hence $\overline{G}'$ is the unique minimal normal subgroup of $\overline{G}$.

Write $\overline{G}/\overline{G}' = \overline{C}/\overline{G}' \times \overline{D}/\overline{G}'$, where $\overline{C}/\overline{G}' \cong C_{p^{a-1}}$ and $\overline{D}/\overline{G}' \cong C_p$. Next, we will discuss in two cases.

Case 1: $|G:G'| > p^2$. We claim that that G is metacyclic. By [3, Theorem 2.3], we only need to show that $\overline{G}$ is metacyclic. If $\Phi(\overline{C}) = 1$, then $\overline{C}$ is elementary abelian and so $\overline{C}/\overline{G}' \cong C_p$ and $a = 2$, contrary to $|G:G'| > p^2$. As $\Phi(\overline{C})$ char $\overline{C} \trianglelefteq \overline{G}$, it follows that $\Phi(\overline{C}) \trianglelefteq \overline{G}$. Hence by the uniqueness of $\overline{G}'$, we have that $\overline{G}' \leqslant \Phi(\overline{C})$. Notice that $(\overline{C}/\overline{G}')/(\Phi(\overline{C})/\overline{G}') \cong \overline{C}/\Phi(\overline{C})$ is cyclic. Then $\overline{C}$ is cyclic. Since $\overline{G}/\overline{C}$ is cyclic, $\overline{G}$ is metacyclic. Hence, the above claim holds. It follows from Theorem 7 that G is abelian, which is a contradiction. Hence, this case cannot happen.

Case 2: $|G:G'| = p^2$. If $p = 2$, then $|G:G'| = 4$ and such groups have been classified. By the equation $|G| = |G:G'| + (k(G) - |G:G'|)\cdot 2^2$, it is easy to see that $k(G)$ is not a power of 2, contrary to the hypothesis $C(G) = C(A)$. Hence $p > 2$. Since $\overline{G}/\overline{G}' \cong G/G' \cong C_p \times C_p$, it follows that $\overline{G}$ has order p^3. Now we only need to show that $\overline{G}$ has exponent p. If $\overline{G}$ has exponent p^2, then the group $\overline{C}$ defined above is cyclic of order p^2 and hence $\overline{G}$ is metacyclic and so G must be metacyclic. It follows from Theorem 7 that G is abelian, which is a contradiction. Hence $\overline{G}$ is of exponent p and so (2) follows. □

By Lemma 4 and Lemma 5, we have that if $A \cong C_{p^3} \times C_p$ and G is a group satisfying $C(G) = C(A)$, then $G \cong A$. Theorem 3 immediately follows from Theorem 2 and a result of Blackburn. He proved that if a p-group G and its derived subgroup G' are generated by two elements, then G' is abelian (see [2, Theorem 4]).

Proof of Theorem 3. If G is abelian, there is nothing to prove. If G is not isomorphic to A, then by Theorem 2, we have that $|G : G'| = p^2$ and $p > 2$. If G has a metacyclic maximal subgroup M, then $G' \leqslant \Phi(G)$ is a subgroup of M and so G' is metacyclic, which indicates that G' is generated by two elements. Hence by Blackburn's result, G' is abelian. Now we have that G' is abelian in all three cases. Notice that $\chi(1) \mid |G : G'|$ for all irreducible characters χ of G. Then $\mathrm{cd}(G)$ is a subset of $\{1, p, p^2\}$. It follows from Lemma 4 and 5 that $p = 2$. This is a contradiction. Therefore, $G \cong A$, as desired. □

Acknowledgements

The work in this paper was completed by the second author (Ph.D. student) under the supervision of the first author at Kent State University. The contents of this paper appear as part of the second author's Ph.D. dissertation. We would like to thank Professor Steve Gagola for several helpful discussions while writing this paper. In particular, the proof of Theorem 7 presented here is largely due to an idea of his which is much simpler than our original proof. We would like to thank the referee for a careful reading of the paper.

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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