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Approximation and extension of Hermitian metrics on holomorphic vector bundles over Stein manifolds

Approximation et extension des métriques hermitiennes sur les fibrés vectoriels holomorphes sur les variétés de Stein

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Abstract. We show that a singular Hermitian metric on a holomorphic vector bundle over a Stein manifold which is negative in the sense of Griffiths (resp. Nakano) can be approximated by a sequence of smooth Hermitian metrics with the same curvature negativity. We also show that a smooth Hermitian metric on a holomorphic vector bundle over a Stein manifold restricted to a submanifold which is negative in the sense of Griffiths (resp. Nakano) can be extended to the whole bundle with the same curvature negativity.

Résumé. Nous montrons qu'une métrique hermitienne singulière sur un fibré vectoriel holomorphe sur une variété de Stein qui est négative au sens de Griffiths (resp. Nakano) peut ê tre approximé par une séquence de métriques hermitiennes lisses avec la même négativité de courbure. Nous montrons également qu'une métrique hermitienne lisse sur un fibré vectoriel holomorphe sur une variété de Stein restreinte à une sous-variété ce qui est négatif au sens de Griffiths (resp. Nakano) peut être étendu à l'ensemble du faisceau avec la même négativité de courbure.

Keywords. Approximation, extension, singular hermitian metric, Griffiths negative, Nakano negative.

Mots-clés. Approximation, extension, métrique hermitienne singulière, négativité de Griffiths, négativité de Nakano.

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1. Introduction

It is known that any plurisubharmonic function on a Stein manifold can be globally approximated point-wise by a decreasing sequence of smooth plurisubharmonic functions [4]. For Stein manifolds, another important result states that any plurisubharmonic function on the submanifold of a Stein manifold can always be extended to a plurisubharmonic function on the ambient space [7]. Note that a plurisubharmonic function on a complex manifold can be viewed as a positively curved singular Hermitian metric on the trivial line bundle over the manifold. The aim of the present work is to prove similar results for positively curved (in certain sense, to be clarified later) singular Hermitian metrics on holomorphic vector bundles over Stein manifolds.

To state the main results, we first recall some notions about curvature positivity of singular Hermitian metrics on holomorphic vector bundles.

Let $\pi : E \to X$ be a holomorphic vector bundle over a complex manifold X. Then a singular Hermitian metric is a measurable section h of $E^* \otimes \overline{E^*}$ that gives a Hermitian inner product on E_x, the fiber of E over x, for almost all $x \in X$. Given such a singular Hermitian metric h on E, we can define a dual singular Hermitian metric h^* in the dual bundle E^* of E in a natural way.

The following definition is given in [2].

Definition 1. *Let $\pi : E \to X$ be a holomorphic vector bundle over a complex manifold X. A singular Hermitian metric h on E is negatively curved in the sense of Griffiths if $\|u\|_h^2$ is plurisubharmonic for any local holomorphic section u of E, and we say that h is positively curved in the sense of Griffiths if the dual metric h^* on the dual bundle E^* of E is negatively curved in the sense of Griffiths.*

For smooth Hermitian metrics on holomorphic vector bundles, Berndtsson discovered an equivalent characterization (see [1]), as follows. Let h be a smooth Hermitian metric on E. For any local coordinate system $\{z = (z_1, \dots, z_n), U\}$ on X and any n-tuple local holomorphic sections $v = (v_1, \dots, v_n)$ of E over U, we set

$$T_v^h := \sum_{j,k=1}^{n} (v_j, v_k)_h \widehat{dz_j \wedge d\bar{z}_k},$$

where $\widehat{dz_j \wedge d\bar{z}_k}$ is the wedge product of all dz_s and $d\bar{z}_s$ except dz_j and $d\bar{z}_k$, multiplied by a constant of absolute value 1, such that

$$i\,dz_j \wedge d\bar{z}_k \wedge \widehat{dz_j \wedge d\bar{z}_k} = i\,dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge i\,dz_n \wedge d\bar{z}_n =: dV_z.$$

Then h is negatively curved in the sense of Nakano if and only if if $i\partial\bar{\partial} T_v^h \geq 0$ for all choices of v.

If h is assumed to be singular, T_v^h can also be defined as a current on U of bi-degree $(n-1, n-1)$, and hence $i\partial\bar{\partial} T_v^h$ is an (n, n)-current on U. Following Berndtsson's observation, Raufi induces the following definition [6] of Nakano negativity and dual Nakano positivity for singular Hermtian metric on holomorphic vector bundles.

Definition 2. *Let $\pi : E \to X$ be a holomorphic vector bundle over a complex manifold X of dimension n. A singular Hermitian metric h on E is negatively curved in the sense of Nakano, if*

$$i\partial\bar{\partial} T_v^h \geq 0$$

holds for any n-tuple local holomorphic sections $v = (v_1, \dots, v_n)$, and we say that h is dual Nakano positive if h^* is negatively curved in the sense of Nakano.

In Section 2, we will explain that the curvature positivity defined in Definitions 1 and 2 reduces to the standard definitions in the case of smooth Hermitian metrics.

The first result of the note is the following

Theorem 3. *Let (E,h) be a singular hermitian holomorphic vector bundle over a Stein manifold X. Assume h is negatively curved in the sense of Griffiths (resp. Nakano). Then there exists a sequence of smooth hermitian metrics $\{h_k\}_{k=1}^{\infty}$ with negative curvature in the sense of Griffiths (resp. Nakano), such that for any compact subset $K \subset X$, there is $k(K)$, for $k \geq k(K)$, h_k decrease to h pointwise on K.*

A local version, namely, the case that X is a polydisc and E is trivial, of the Theorem 3 was proved in [2] and [6].

The second result is about extension of Hermitian metrics.

Theorem 4. *Let E be a holomorphic vector bundle over a Stein manifold X, and Y be a closed submanifold of X. Assume that h is a smooth hermitian metric on $E|_Y$ with negative curvature in the sense of Griffiths (resp. Nakano), then h can be extended to a smooth hermitian metric $\widetilde{h}$ on E with negative curvature in the sense of Griffiths (resp. Nakano).*

We do not know if the result in Theorem 4 still holds if h is assumed to be singular.

2. On different formations for curvature positivity for smooth Hermitian metrics on holomorphic vector bundles

In this section, we explain that the curvature positivity defined in Definitions 1 and 2 reduce to the standard definitions in the case of smooth Hermitian metrics. The result in this section is already known, but we also give the related details since the lack of exact references.

2.1. *On Griffiths negativity*

We first consider the case of line bundle. Let L be a holomorphic line bundle over a complex manifold X and h be a smooth metric on L. If e is a holomorphic local frame of L on some open set $U \subset X$ and $|e|_h^2 = e^{\phi}$ for some smooth function ϕ on U, then the curvature of (L,h) is negative if and only if ϕ is a plurisubhamronic function on U (here negative means semi-nagative). Let $s = fe$ be a holomorphic section of L on U, then $|s|_h^2 = |f|^2 e^{\phi}$ is of course plurisubhamronic if ϕ is plurisubhamronic. So we get the conclusion that the norm square of any local holomorphic section of L with respect to h is plurisubhamronic if the curvature of (L,h) is negative.

On the other hand, assume that the converse is true, that is $|s|_h^2 = |f|^2 e^{\phi}$ is plurisubhamronic for any local holomorphic section s of L, we can see that the curvature of (L,h) is negative. In fact, we can just consider nonvanishing section s and write $f = e^{g/2}$ for some local holomorphic function g, then $e^{\mathrm{Re}g+\phi}$ is plurisubhamronic, and it follows that $\mathrm{Re}g + \phi$ satisfies the maximum value principle for all local holomorphic function g, which implies that ϕ itself is plurisubhamronic.

In conclusion, (L,h) has negative curvature if and only if $|s|_h^2$ is plurisubhamronic for any local holomorphic section s of L.

We now move to the discussion of Hermitian holomorphic vector bundles. Let E be a holomorphic vector bundle over X and h be a smooth Hermitian metric on E. We assume that the curvature of (E,h) is negative in the sense of Griffiths. Then it is well known that any holomorphic subbundle of E with the induced metric has negative curvature in the sense of Griffiths (see [3, Proposition 6.10]). Now let s be a nonvanishing local holomorphic section of E, then s generates a local subbundle, say L, of rank 1 of E. It follows that $(L, h|_L)$ has negative curvature, and by the discussion above we know that $|s|_h^2$ is a local plurisubhamronic function. Clearly the same statement still true if s has zeros since then $|s|_h^2$ satisfies the mean value inequality near the zeros.

We now consider the converse. The aim is to prove that if $|s|_h^2$ is plurisubhamronic for any local holomorphic section s of E, then the curvature of (E,h) is negative in the sense of Griffiths.

Fix any point $p \in X$ and a local holomorphic coordinate system $(z_1, \dots, z_n)$ on X around p, it is known (see [3, Proposition 12.10]) that there exists a local holomorphic frame $e_1, \dots, e_r$ of E near p such that

$$\langle e_\lambda, e_\mu \rangle_h = \delta_{\lambda\mu} - \sum_{j,k=1}^{n} c_{jk\lambda\mu} z_j \bar{z}_k + O(|z|^3),\ 1 \le \lambda, \mu, \le r.$$

For any constants c_λ, considering the local holomorphic section $s = \sum_\lambda c_\lambda e_\lambda$ of E near p, then we have

$$|s|_h^2 = \sum_{\lambda=1}^{r} |c_\lambda|^2 - \sum_{j,k=1}^{n} \sum_{\lambda,\mu=1}^{r} c_{jk\lambda\mu} c_\lambda \bar{c}_\mu z_j \bar{z}_k + O(|z|^3).$$

So the plurisubharmonicity of $|s|_h^2$ implies that the matrix

$$-\left(\frac{\partial |s|_h^2}{\partial z_j \partial \bar{z}_k} \right)_{n\times n} (0) = \left(\sum_{\lambda,\mu=1}^{r} c_{jk\lambda\mu} c_\lambda \bar{c}_\mu \right)_{n\times n}$$

is negative, which is exactly equivalent to the Griffiths negativity of the curvature of (E, h) at p.

In conclusion, we get the following

Proposition 5. *Let E be a holomorphic vector bundle over X and h be a smooth Hermitian metric on E, The curvature of (E, h) is negative in the sense of Griffiths if and only if $|s|_h^2$ is plurisubhamronic for any local holomorphic section s of E.*

2.2. *On Nakano negativity*

Let X be a complex manifold of dimension n, E be a holomorphic vector bundle over X of rank r, and h be a smooth Hermitian metric on E. For any local coordinate system $\{z = (z_1, \dots, z_n), U\}$ on X, the curvature of the Chern connection is a $(1,1)$-form of operators

$$\Theta = \sum_{j,k=1}^{n} \Theta_{jk} \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k.$$

E is said to be semi-negative in the sense of Nakano if for any n-tuple local holomorphic sections $v = (v_1, \dots, v_n)$ of E over U,

$$\sum_{j,k=1}^{n} (\Theta_{j,k} v_j, v_k) \le 0.$$

If we set

$$T_v^h := \sum_{j,k=1}^{n} (v_j, v_k)_h \widehat{\mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k},$$

where $\widehat{\mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k}$ is the wedge product of all $\mathrm{d}z_s$ and $\mathrm{d}\bar{z}_s$ except $\mathrm{d}z_j$ and $\mathrm{d}\bar{z}_k$, multiplied by a constant of absolute value 1, such that

$$\mathrm{i}\,\mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k \wedge \widehat{\mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k} = \mathrm{i}\,\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 \wedge \cdots \wedge \mathrm{i}\,\mathrm{d}z_n \wedge \mathrm{d}\bar{z}_n =: \mathrm{d}V_z.$$

A direct calculation (see [1, Section 2]) shows that

$$\mathrm{i}\partial\bar{\partial} T_v^h = -\sum_{j,k=1}^{n} (\Theta_{jk} v_j, v_k)\, \mathrm{d}V_z.$$

Therefore, h is negatively curved in the sense of Nakano if and only if if $\mathrm{i}\partial\bar{\partial} T_v^h \ge 0$ for all choices of v.

3. Approximation of singular Hermitian metrics

We put the following obvious results as a lemma.

Lemma 6. *Let E be a holomorphic vector bundle over a complex manifold X, h_1 and h_2 be two singular hermitian metrics on E. Then*

(1) *if h_1 and h_2 are both with negative curvature in the sense of Griffiths, so is h_1+h_2;*
(2) *if h_1 and h_2 are both with negative curvature in the sense of Nakano, so is h_1+h_2.*

From the Oka–Grauert principle for holomorphic vector bundles, we have the following

Lemma 7. *Let E be a holomorphic vector bundle over a Stein manifold X. There is a holomorphic vector bundle F on X, such that $E\oplus F$ is a trivial holomorphic vector bundle.*

Proof. The statement is a corollary of the Oka–Grauert principle, which says that any topological vector bundle over a Stein manifold admits one and only one holomorphic structure, up to isomorphism (see [5, Theorem 5.3.1]). It is also known that E can be realized as a holomorphic subbundle of a trivial vector bundle say $X\times\mathbb{C}^N$ (see [5, Corollary 7.3.2]). Hence there exists a topological vector bundle F over X such that $E\oplus F$ is a topologically trivial vector bundle over X. By the Oka–Grauert principle, there is a unique holomorphic structure on F. Since $E\oplus F$ is trivial as a topological vector bundle, applying again the Oka–Grauert principle to $E\oplus F$, we know that $E\oplus F$ is also trivial as a holomorphic vector bundle. □

Lemma 8. *For any vector bundle E on a Stein manifold X, there is a smooth Hermitian metric h on E such that $\mathrm{i}\Theta_h$ is both Nakano positive and dual Nakano positive. Furthermore, there is a smooth Hermitian metric h' on E such that $\mathrm{i}\Theta_{h'}$ is Nakano negative.*

Proof. The proof of the first statement is as follows. We fix a smooth hermitian metric h_0 on E at first, and denote by h_0^* the metric on E^* dual to h_0. We also fix an exhaustive smooth strictly plurisubharmonic function ϕ on X. Then we can choose a smooth increasing convex function u, such that the curvature of $h=\mathrm{e}^{-u(\phi)}\,h_0$ is Nakano positive and $h^*:=\mathrm{e}^{u(\phi)}\,h_0^*$ is Nakano negative, whic means that $\mathrm{i}\Theta_h$ is both Nakano positive and dual Nakano positive.

We can get the second statement by applying the first stament to E^*, which is also a vector bundle on the Stein manifold X. □

We need the following proposition.

Proposition 9 (see [3, Proposition 6.10, p. 340]). *Let $0\to S\to E\to Q\to 0$ be an exact sequence of hermitian vector bundles, and h is a smooth hermitian metric on E. Give S and Q the induced metric by E. Then*

(1) *If $\mathrm{i}\Theta_E$ is Griffith (semi-)positive, so is $\mathrm{i}\Theta_Q$.*
(2) *If $\mathrm{i}\Theta_E$ is Griffith (semi-)negative, so is $\mathrm{i}\Theta_S$.*
(3) *If $\mathrm{i}\Theta_E$ is Nakano (semi-)negative, so is $\mathrm{i}\Theta_S$.*

B. Berndtsson and M. Păun [2] proved the following theorem in the case that h is negatively curved in the sense of Griffiths by convolution in an approximate way. H. Raufi [6] observed that the same technique yields a similar result when h is negatively curved in the sense of Nakano.

Theorem 10 (see [2] and [6]). *Let h be a singular hermitian metric on a trivial vector bundle E over a polydisc, and assume that h is negatively curved in the sense of Griffiths (resp. Nakano). There exists a sequence of smooth hermitian metrics $\{h_\nu\}_{\nu=1}^{\infty}$ with negative curvature in the sense of Griffiths (resp. Nakano), decreasing to h pointwise on any smaller polydisc.*

We now give the proof of Theorem 3.

Theorem 11 (= Theorem 3). *Let (E,h) be a singular hermitian holomorphic vector bundle over a Stein manifold X. Assume h is negatively curved in the sense of Griffiths (resp. Nakano). Then there exists a sequence of smooth hermitian metrics $\{h_k\}_{k=1}^{\infty}$ with negative curvature in the sense of Griffiths (resp. Nakano), such that for any compact subset $K \subset X$, there is $k(K)$, for $k \geq k(K)$, h_k decrease to h pointwise on K.*

Proof. We give the proof for the case of Griffiths negativity. The proof for the case of Nakano negativity is similar.

From Lemma 7, there is a holomorphic vector bundle F on X, such that $E \oplus F$ is a trivial holomorphic vector bundle. As the restriction of trivial hermitian metric of trivial vector bundle on F which denoted by g is Griffiths negative, so (h,g) is negatively curved in the sense of Griffiths. If we get an approximation of (h,g) by hermitian metrics with Griffiths negative curvatures, by Proposition 9, the restriction of the hermitian metrics to E which are Griffiths negative is an approximation of h. Without of loss generality, we may assume that E is the trivial vector bundle $X \times \mathbb{C}^r$, and h is a hermitian matrix function on X.

By the imbedding theorem, there is a proper holomorphic imbedding $\iota : X \to \mathbb{C}^N$. So we may view X as a closed complex submanifold of $\mathbb{C}^N$. There exist finitely many holomorphic functions $f_1, \ldots, f_m$ on $\mathbb{C}^N$, such that $\{z \in \mathbb{C}^N : f_1(z) = \cdots = f_m(z) = 0\} = X$. Set $|f|^2 = \sum_{j=1}^m |f_j|^2$. By Corollary 1 of [8], there is a Stein open neighborhood $U \subset \mathbb{C}^N$ of X, and a holomorphic retraction $\pi : U \to X$. Hence, $\pi^*(h) = h \circ \pi$ is a singular hermitian metric on $U \times \mathbb{C}^r$, which is an extension of h and negatively curved in the sense of Griffiths.

We will construct h_k by induction. For $z \in \mathbb{C}^N$, set $|z| = \sqrt{\sum_{j=1}^N |z_j|^2}$, and $X_k = \{z \in X : |z| \leq k\}$. Denote $d_k = \operatorname{dist}(X_k, \partial U)$. Let ρ be a smooth radical function on $\mathbb{C}^N$, $\rho \geq 0$, $\operatorname{supp} \rho \subset B_1$ and $\int_{\mathbb{C}^n} \rho \, dV = 1$. Set $\rho_\epsilon(z) := \frac{\rho(z/\epsilon)}{\epsilon^{2N}}$ for $\epsilon > 0$. Hence by Theorem 10, $(h \circ \pi) * \rho_{\epsilon_k}$ is Griffiths negative on

$$V_k := \{z \in \mathbb{C}^N : \operatorname{dist}(z, X_k) < d_k - \epsilon_k\}$$

for $\epsilon_k < d_k$. We may choose $\{\epsilon_k\}$ inductively, such that ϵ_k is decreasing to 0, as $k \to \infty$. Let χ_k be a smooth function, such that $\chi_k \equiv 1$ in a neighborhood U_k of X_k, and $\operatorname{supp} \chi_k \subset V_k$. Let $\delta_k = \frac{1}{4} \operatorname{dist}(X_k, \partial U_k) > 0$, and

$$u_k(z) = \max_{\delta_k} \left\{ |z|^2 - k^2 - \delta_k, 0 \right\} + |f(z)|^2,$$

where $\max_\delta\{x,y\}$ means the convolution of the maximum function (see [3]). Notice that

$$u_k|_{X_k} = 0 \text{ and } u_k(z) = |z|^2 - k^2 - \delta_k + |f(z)|^2 \geq c_k, \text{ for } z \in U_k^c \tag{3.1}$$

for some constant $c_k > 0$.

Claim. *There is a sequence of smooth increasing convex functions $\{v_k\}$, with $v_k(0) = 0$, such that*

$$\widetilde{h}_k := e^{v_k(u_k)} \left(\chi_k (h \circ \pi) * \rho_{\epsilon_k} + \epsilon_k I_r \right)$$

is a Griffiths negative hermitian metric for the trivial vector bundle $\mathbb{C}^N \times \mathbb{C}^r$, where I_r denotes the $r \times r$ unit matrix.

If the claim is proved, let $h_k = \widetilde{h}_k|_X$, h_k is Griffiths negative, and

$$h_k|_{X_k} = (h \circ \pi) * \rho_{\epsilon_k}|_{X_k} + \epsilon_k I_r \text{ for } k \geq 1.$$

As ϵ_k decreases to 0, then $\{h_k\}$ satisfies all the conditions.

We will prove the claim.

Note that,

$$i\Theta_{h_k} = i\Theta_{\chi_k (h\circ\pi) * \rho_{\epsilon_k}|_{X_k} + \epsilon_k I_r} - i\partial\bar{\partial} v_k(u_k) \otimes I_r.$$

From Lemma 6, $\chi_k(h\circ\pi)*\rho_{\epsilon_k}|_{X_k}+\epsilon_k I_r$ is Griffiths negative on U_k. As $\chi_k=0$ on V_k^c, we only need to consider $V_k\backslash U_k$. Since $\chi_k(h\circ\pi)*\rho_{\epsilon_k}|_{X_k}+\epsilon_k I_r$ is a smooth hermitian metric, $\overline{V_k}\backslash U_k$ is compact, and u_k is strictly plurisubharmonic on $\overline{V_k}\backslash U_k$, we can find a smooth increasing convex function v_k, such that $v_k(0)=0$ and $\widetilde{h}_k$ is Griffiths negative. □

Remark 12. By the dual proposition, we can also get similar results for the approximation of singular metric which is positively curved in the sense of Griffiths or dual Nakano positive on Stein manifold with increasing sequence of metrics with Griffiths positive or dual Nakano positive curvatures.

4. Extensions of Hermitian metrics

In this section we give the proof of Theorem 4.

Theorem 13 (= Theorem 4). *Let E be a holomorphic vector bundle over a Stein manifold X, and Y be a closed submanifold of X. Assume that h is a smooth hermitian metric on $E|_Y$ with negative curvature in the sense of Griffiths (resp. Nakano), then h can be extended to a smooth hermitian metric $\widetilde{h}$ on E with negative curvature in the sense of Griffiths (resp. Nakano).*

Proof. We give the proof for the case of Griffiths negativity. The proof for the case of Nakano negativity is similar.

From Lemma 7, there is a holomorphic vector bundle F on X, such that $E\oplus F$ is the trivial holomorphic vector bundle $X\times\mathbb{C}^r$. The restriction of trivial hermitian metric of trivial vector bundle on F which denoted by g is Griffiths negative.

As Y is a closed submanifold of the Stein manifold X, there exists an open neighborhood U of Y in X, with a holomorphic retraction $\pi: U\to Y$. So we get a smooth hermitian metric $(h\circ\pi, g)$ on $(E\oplus F)|_U$ with negative Griffiths curvature.

Let χ be a smooth function satisfying $\chi\equiv 1$ in an open neighborhood V of Y and supp $\chi\subset U$. Set

$$h'=\chi(h\circ\pi,g)+(1-\chi)I_r,$$

then h' is a smooth Hermitian metric on $E\oplus F$. As $h'|_V=(h\circ\pi,g)$, h' is Griffiths negative on V. So we will focus on $V^c:=X\setminus V$.

There exist finitely many holomorphic functions $f_1,\dots,f_m$ on X, such that

$$Y=\{z\in X: f_1(z)=\cdots=f_m(z)=0\}.$$

Let $|f|^2=\sum_{j=1}^m|f_j|^2$, and $\phi\geq 0$ be a smooth strictly plurisubharmonic exhaustion function on X. There is a continuous function v on $[0,\infty)$, such that

$$\mathrm{i}\Theta_{h'}\leq v(\phi)\,\mathrm{i}\partial\bar{\partial}\phi I_r \tag{4.1}$$

and

$$\frac{1}{|f|^2}\leq v(\phi)\ \text{ on }\ V^c. \tag{4.2}$$

Let u be a smooth increasing convex function on $(-1,\infty)$, such that

$$u(t)\geq 0\ \text{ and }\ u'(t)\geq v^2(t)\ \ \forall\, t\in[0,\infty). \tag{4.3}$$

Let

$$\widetilde{h}=\mathrm{e}^{|f|^2\mathrm{e}^{u(\phi)}}h',\ \text{ and }\ \psi=\log|f|^2+u(\phi).$$

Then $\widetilde{h}$ is a smooth hermitian metric on $E\oplus F$, and

$$\widetilde{h}|_Y=h'|_Y=(h,g|_Y).$$

Since ψ is plurisubharmonic, we have that $\mathrm{e}^{\psi}=|f|^2\mathrm{e}^{u(\phi)}$ is also plurisubharmonic. As

$$\mathrm{i}\Theta_{\tilde{h}}=\mathrm{i}\Theta_{h'}-\mathrm{i}\partial\bar{\partial}(|f|^2\mathrm{e}^{u(\phi)})I_r, \tag{4.4}$$

and h' is Griffiths negative on V, Therefore, $\widetilde{h}$ is also Griffiths negative on V. On V^c, notice

$$\begin{aligned} \mathrm{i}\partial\bar{\partial}(|f|^2 \mathrm{e}^{u(\phi)}) &= \mathrm{i}\partial\bar{\partial}\,\mathrm{e}^{\psi} \\ &= \mathrm{e}^{\psi}(\mathrm{i}\partial\bar{\partial}\psi + \mathrm{i}\partial\psi\wedge\bar{\partial}\psi) \\ &\geq |f|^2 \mathrm{e}^{u(\phi)}\,\mathrm{i}\partial\bar{\partial}(\log|f|^2 + u(\phi)) \\ &\geq |f|^2 \mathrm{e}^{u(\phi)}\, u'(\phi)\mathrm{i}\partial\bar{\partial}\phi \\ &\geq \nu(\phi)\mathrm{i}\partial\bar{\partial}\phi, \end{aligned} \tag{4.5}$$

the last inequality holds form inequality (4.2) and (4.3).

From inequality (4.1), (4.4) and (4.5), we have that $\widetilde{h}$ is Griffiths semi-negative on V^c. Therefore, $\widetilde{h}$ is is Griffiths semi-negative on X. Let

$$\iota : E \longrightarrow E\oplus F,\ \ \iota(\alpha) = (\alpha, 0).$$

$\iota^*(\widetilde{h})$ is a smooth Griffiths semi-negative hermitian metric on E by Proposition 9, with $\iota^*(\widetilde{h})|_Y = h$. □

Remark 14. For the extension theorems, if the metric being extended is Griffiths nagative or Nakano negative, we can get a Griffiths negative or Nakano negative extension metric. The proof is almost same as above, if one notice that there is a sequence of holomorphic functions $\{f_j\}$ on X, such that $|f|^2 := \sum_{j=1}^{\infty}|f_j|^2$ converges uniformly on compact subsets of X, $Y = \{|f|^2 = 0\}$, and $i\partial\bar{\partial}|f|^2 + \omega_Y > 0$ on every point $p \in Y \subset X$ for any given Kähler form ω_Y on Y.

Theorem 15. *Let E be a holomorphic vector bundle over a Stein manifold X, and Y be a closed submanifold of X. Assume h is a smooth hermitian metric on $E|_Y$ which is Nakano (semi-)positive. Then h can be extended to a smooth hermitian metric $\widetilde{h}$ on E which is Nakano (semi-)positive.*

Proof. As Y be a closed submanifold of the Stein manifold X, by [8], there is a Stein neighborhood $U \subset X$ of Y, and a holomorphic retraction $\rho : U \to Y$. By the Oka–Grauert principle, $\rho^*(E|_Y)$ is isomorphic to $E|_U$, and $\rho^*(E|_Y)|_Y = E|_Y$. Therefore, we may view $\rho^*(h)$ as a smooth hermitian metric on $E|_U$, and $\rho^*(h)|_Y = h$. Choose a smooth hermitian metric h_1 on E, and a smooth function χ on X such that $\chi = 1$ in an open neighborhood of Y, and $\mathrm{supp}\chi \subset U$. Then we get a smooth hermitian metric

$$h' = \chi\rho^*(h) + (1-\chi)h_1$$

on E.

There is a sequence of holomorphic functions $\{f_j\}$ on X, such that $|f|^2 := \sum_{j=1}^{\infty}|f_j|^2$ converges uniformly on compact subsets of X, $Y = \{|f|^2 = 0\}$, and $\mathrm{i}\partial\bar{\partial}|f|^2 + \omega_Y > 0$ on every point $p \in Y \subset X$ for any Kähler form ω_Y on Y.

By a similar procedure of the proof of Theorem 4, we can find a smooth plurisubharmonic function φ on X, such that $\mathrm{e}^{-|f|^2\mathrm{e}^{\varphi}}h'$ is Nakano (semi)-positive on X. It is obvious $\mathrm{e}^{-|f|^2\mathrm{e}^{\varphi}}h'$ is an extension of h. □

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Declaration of interests

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