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Dynamics near the origin of the long range scattering for the one-dimensional Schrödinger equation

Dynamique près de l'origine du scattering longue portée pour l'équation de Schrödinger unidimensionnelle

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Abstract. We consider the cubic Schrödinger equation on the line, for which the scattering theory requires modifications due to long range effects. We revisit the construction of the modified wave operator, and recall the construction of its inverse, in order to describe the asymptotic behavior of these operators near the origin. At leading order, these operators, whose definition includes a nonlinear modification in the phase compared to the linear dynamics, correspond to the identity. We compute explicitly the first corrector in the asymptotic expansion, and justify this expansion by error estimates.

Résumé. Nous considèrons l'équation de Schrödinger cubique sur la droite, pour laquelle la théorie du scattering demande des modifications dues aux effets à longue portée. Nous reprenons la construction de l'opérateur d'onde modifié, et rappelons la construction de son inverse, afin de décrire le comportement de ces opérateurs près de l'origine. Au premier ordre, ces opérateurs, dont la définition contient une modification non linéaire de la phase par rapport à la dynamique linéaire, coïncident avec l'identité. Nous calculons explicitement le premier correcteur du développement asymptotique, et justifions ce développement par des estimations d'erreur.

Keywords. Nonlinear Schrödinger equation, long range scattering, asymptotic expansion, error estimate.

Mots-clés. Équation de Schrödinger non linéaire, scattering longue portée, développement asymptotique, estimation d'erreur.

2020 Mathematics Subject Classification. 35Q55, 35P25, 35B40.

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1. Introduction

We consider the cubic Schrödinger equation on the line,

$$i\partial_t u + \frac{1}{2}\partial_x^2 u = \lambda |u|^2 u, \quad x \in \mathbb{R}, \tag{1.1}$$

with $\lambda \in \mathbb{R}$. It is well known that in terms of scattering, this equation corresponds to a borderline case, where long range effects appear (see [1]). The goal of this paper is to analyze the modified scattering map near the origin, as well as the modified wave operator and its inverse. We first recall a few aspects of scattering theory for the nonlinear Schrödinger equation in the short range case, then turn to the specificities of this long range setting.

1.1. *Short range scattering*

Denote by $U(t)$ the Schrödinger group,

$$U(t) := e^{i\frac{t}{2}\partial_x^2}.$$

In view of the explicit formula for $U(t)$ as a convolution operator, we have

$$U(t)f(x) \underset{t\to\pm\infty}{\sim} e^{i\frac{x^2}{2t}} \frac{1}{(it)^{1/2}} \widehat{f}\left(\frac{x}{t}\right), \tag{1.2}$$

where we normalize the Fourier transform as follows,

$$\mathscr{F} f(\xi) = \widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)\, e^{-ix\xi}\, dx.$$

See Lemma 5 for a more precise statement regarding (1.2). Consider, for the sake of comparison with (1.1), the case of a quintic, defocusing nonlinearity

$$i\partial_t u + \frac{1}{2}\partial_x^2 u = |u|^4 u, \quad x \in \mathbb{R}. \tag{1.3}$$

The discussion for, e.g., the two-dimensional or the three-dimensional cubic Schrödinger equation would be similar. Given

$$u_- \in \Sigma := \left\{f \in H^1(\mathbb{R});\ \|f\|_\Sigma := \|f\|_{L^2} + \|\partial_x f\|_{L^2} + \|xf\|_{L^2} < \infty\right\}, \tag{1.4}$$

there exists a unique $u_0 \in \Sigma = H^1 \cap \mathscr{F}(H^1)$ such that the (unique, global) solution u to (1.3) with $u_{|t=0} = u_0$ satisfies

$$\|U(-t)u(t) - u_-\|_\Sigma \underset{t\to-\infty}{\longrightarrow} 0.$$

We recall that $U(t)$ is unitary on H^1, but not on $\mathscr{F}(H^1)$, this is why the quantity measured above is not $u(t) - U(t)u_-$. The map $u_- \mapsto u_0$ is classically referred to as *wave operator* (see e.g. [7]).

Conversely, given $u_0 \in \Sigma$, there exists $u_+ \in \Sigma$ such that

$$\|U(-t)u(t) - u_+\|_\Sigma \underset{t\to+\infty}{\longrightarrow} 0.$$

The map $u_- \mapsto u_+$ is called the *scattering operator*. It can be defined for other (defocusing) nonlinearities, but strictly supercubic, in view of [1], where it is proved that, typically for (1.1), it is not possible to compare the nonlinear dynamics with the linear one for large time (see also [7]). We will see in the next subsection that in the cubic case (1.1), nontrivial long range effects must be taken into account, and that these effects are explicit.

In general, rather little is known regarding properties of the scattering map S. For instance, it is proven in [3] that for smooth, power-like nonlinearities such that short range scattering is known, the wave and scattering operators are analytic. The formula for the associated Taylor series is given, and the formula differs whether the expansion is considered at the origin or at a nontrivial state. Such asymptotic expansions (not necessarily in the analytic case) have proven useful in the context of inverse problems, see e.g. [21, 22] and references therein.

1.2. *Long range case: modified scattering*

The picture is different in the case of (1.1): the nonlinear dynamics cannot be compared to the linear one, unless one considers the trivial case $u \equiv 0$ ([1]), so a modified scattering theory has been developed in order to describe the large time behavior of u, sharing some similarities with the linear case, presented in e.g. [6].

Given asymptotic states u_- and u_+, we introduce the long range phase corrections (different whether $t \to -\infty$ of $t \to +\infty$),

$$S_\pm(t,x) := \mp\lambda \left|\widehat{u}_\pm\left(\frac{x}{t}\right)\right|^2 \log|t|. \tag{1.5}$$

Loosely speaking, the existence of modified wave operator reads as follows: given u_- (sufficiently small), there exists a solution u to (1.1) such that

$$u(t,x) \underset{t\to-\infty}{\sim} e^{iS_-(t,x)}U(t)u_-(x) \underset{t\to-\infty}{\sim} e^{iS_-(t,x)+i\frac{x^2}{2t}} \frac{1}{(it)^{1/2}}\widehat{u}_-\left(\frac{x}{t}\right),$$

where the second approximation stems from (1.2). Note that the phase modification S_- is by no means negligible as $t \to -\infty$: it accounts for long range effects, as established initially in [28]. We denote by $u_{|t=0} = W_-^{\rm mod}(u_-)$ the modified wave operator.

The modified asymptotic completeness is similar: given u_0 (sufficiently small), there exists u_+ such that the solution u to (1.1) with $u_{|t=0} = u_0$ satisfies

$$u(t) \underset{t\to+\infty}{\sim} e^{iS_+(t)}U(t)u_+.$$

Such a result was proven initially in [15]. The modified scattering map is given by $S^{\rm mod}(u_-) = u_+$. At this stage, we have not addressed the function spaces in which the above asymptotics have been proven.

In [28], the existence of modified wave operators was established for $u_- \in \mathscr{F}(H^2)$, with $\|\widehat{u}_-\|_{L^\infty}$ sufficiently small, and the solution u to (1.1) has an L^2 regularity, $u_0 \in L^2(\mathbb{R})$. We emphasize, as the notation will be used many times, that the space $\mathscr{F}(H^s)$ is characterized by

$$\mathscr{F}(H^s) = \left\{f \in \mathscr{S}'(R),\ \|f\|^2_{\mathscr{F}(H^s)} := \int_{\mathbb{R}} \langle x\rangle^{2s}|f(x)|^2\,dx < \infty\right\}, \quad \langle x\rangle = \sqrt{1+x^2}.$$

The result of [28] also addresses the case where $u_- \in \mathscr{H}$ (defined below, see (1.6)) where, provided again that $\|\widehat{u}_-\|_{L^\infty}$ sufficiently small, the solution u to (1.1) has an H^1 regularity, and convergence holds in this space.

In [15], the asymptotic completeness was proven for $u_0 \in H^\gamma \cap \mathscr{F}(H^\gamma)$ with $\gamma > 1/2$ and $\|u_0\|_{H^\gamma\cap\mathscr{F}(H^\gamma)}$ sufficiently small. The obtained asymptotic state u_+ is such that $\widehat{u}_+ \in L^2 \cap L^\infty$.

Denote

$$\begin{aligned}\mathscr{H} &:= \{f \in H^1(\mathbb{R});\ \langle x\rangle\partial_x f, \langle x\rangle^3 f \in L^2(\mathbb{R})\}\\ &= \{f \in \mathscr{S}'(\mathbb{R}); \|f\|_{\mathscr{H}} := \|\langle x\rangle\partial_x f\|_{L^2} + \|\langle x\rangle^3 f\|_{L^2} < \infty\}.\end{aligned} \tag{1.6}$$

In [2], the main result of [28] was adapted for $u_- \in \mathscr{H}$ with $\|\widehat{u}_-\|_{L^\infty}$ sufficiently small, and the regularity of the solution u to (1.1) was proven to be at least Σ, hence $u_0 \in \Sigma$, making it possible to connect this result with the asymptotic completeness from [15], thus defining a map $u_- \mapsto u_+$ from (a subset of) $\mathscr{H}$ to $L^2 \cap L^\infty$. When invoking the modified scattering operator, we refer to this notion.

This gap in regularity between u_- and u_+ was considerably diminished in [17], where, with the same notations as above, the authors consider the setting (along with smallness conditions)

$$u_- \in \mathscr{F}(H^\alpha), \quad u_0 \in \mathscr{F}(H^\beta), \quad u_+ \in \mathscr{F}(H^\delta),$$

with the constraints $1/2 < \delta < \beta < \alpha < 1$, allowing these three indices to be arbitrarily close one from another. This is achieved by adapting the notion of (modified) asymptotic completeness in order to avoid loss of differentiability issues, in space dimensions two and three, where the borderline nonlinearity in terms of scattering is $|u|^{2/d}u$.

We also emphasize that there are many references addressing the theory of long range scattering for nonlinear Schrödinger equations (e.g. [4, 18, 19, 24–27]), as well as for Schrödinger-like equations, like Hartree equation (e.g. [8, 12, 13, 29]), derivative nonlinear Schrödinger equation (e.g. [14, 16]), Maxwell–Schrödinger system (e.g. [9, 10]), wave-Schrödinger system (e.g. [11]), not to mention other dispersive equations.

We note that the very definition of the modified wave and scattering operators encodes the fact that the nonlinearity is cubic, and recovering the nonlinearity from the scattering map does not make sense, contrary to the case of [21, 22]. In the case where λ is allowed to depend on x in a somehow perturbative way, Chen and Murphy [5] showed that the inverse of the modified wave operator uniquely determines λ. One of the tools of the proof there is to study the behavior of this operator near the origin, thanks to a rather implicit expansion ([5, Proposition 4.1]). We present a more explicit formula in the next subsection, when λ is constant.

To fix the ideas, we summarize the above discussion by introducing the following definition, where regularity aspects are left out for simplicity.

Definition 1. *Given u_-, the modified wave operator acting on u_- is given by $u_{|t=0} = W_-^{\mathrm{mod}}(u_-)$, where u solves* (1.1)*, and satisfies*

$$u(t,x) = \mathrm{e}^{\mathrm{i}S_-(t,x)}\,U(t)u_-(x) + o(1) \text{ in } L^2(\mathbb{R}) \text{ as } t \longrightarrow -\infty,$$

where S_- is defined in (1.5).

The modified scattering operator is defined by $S^{\mathrm{mod}}(u_-) = u_+$ if the above solution u satisfies in addition

$$u(t,x) = \mathrm{e}^{\mathrm{i}S_+(t,x)}\,U(t)u_+(x) + o(1) \text{ in } L^2(\mathbb{R}) \text{ as } t \longrightarrow +\infty,$$

where S_+ is defined in (1.5).

1.3. *Main result*

In the same spirit as what has been achieved for the wave and scattering operators in the short range case, we consider the asymptotic behavior of the modified wave and scattering operators, with two restrictions compared to [3]. First, we shall confine ourselves to the asymptotic expansion near the origin. Second, we compute only the first two terms of this asymptotic expansion. We will see that this already requires some amount of work, but that the same method should provide some, if not all, higher order terms in the expansion at the origin. On the other hand, the description of these operators near a nontrivial state certainly requires a different approach.

We emphasize that the question addressed here is different from the (higher order) asymptotic expansion of the large time behavior of $u(t)$, as studied in [23]. Our main results are gathered in the following statement:

Theorem 2. *Let $v_- \in \mathcal{H}$ and $v_0 \in \Sigma$.*

- *For any $0<\eta<2$, we have, in Σ and as $\varepsilon \to 0$,*

$$W_-^{\mathrm{mod}}(\varepsilon v_-) = \varepsilon v_- + \varepsilon^3 w_2 + \mathcal{O}\left(\varepsilon^{5-\eta}\right),$$

where $w_2 \in \Sigma$ is defined by

$$w_2 = -\mathrm{i}\lambda \int_{-\infty}^{-1} \left(U(-\tau)\left(|U(\tau)v_-|^2 U(\tau)v_-\right) + \frac{1}{|\tau|}\mathscr{F}^{-1}\left(|\widehat{v}_-|^2\widehat{v}_-\right)\right)\mathrm{d}\tau \\ - \mathrm{i}\lambda \int_{-1}^{0} U(-\tau)\left(|U(\tau)v_-|^2 U(\tau)v_-\right)\mathrm{d}\tau.$$

- *For any* $0<\eta<2$, *we have, in* L^2 *and as* $\varepsilon\to 0$,

$$\left(W_+^{\rm mod}\right)^{-1}(\varepsilon v_0)=\varepsilon v_0+\varepsilon^3\mu_2+\mathcal{O}\left(\varepsilon^{5-\eta}\right),$$

where $\mu_2=\mu_2(v_0)\in L^2$ *is defined by*

$$\mu_2=-i\lambda\int_0^1\left(U(-s)\left(|U(s)v_0|^2U(s)v_0\right)\right)\mathrm{d}s \\ +\lambda\int_1^\infty\left(M(-\tau)\mathcal{F}^{-1}\left(\left|\widehat{\overline{M(\tau)v_0}}\right|^2\widehat{\overline{M(\tau)v_0}}\right)-\mathcal{F}^{-1}\left(|\widehat{v}_0|^2\widehat{v}_0\right)\right)\frac{\mathrm{d}\tau}{\tau},$$

and $M(t)$ *stands for the multiplication by* $e^{i\frac{x^2}{2t}}$.

- *For any* $0<\eta<2$, *we have, in* L^2 *and as* $\varepsilon\to 0$,

$$S^{\rm mod}(\varepsilon v_-)=\varepsilon v_-+\varepsilon^3\nu_2+\mathcal{O}\left(\varepsilon^{5-\eta}\right),$$

where $\nu_2\in L^2$ *is defined by* $\nu_2=w_2+\mu_2(v_-)$, *where* w_2 *is given by the first point, and* μ_2 *by the second.*

Remark 3. The functions μ_2 and ν_2 are more regular than merely L^2, as we will see in Section 6 that they belong to $H^\gamma\cap\mathcal{F}(H^\gamma)$ for any $0<\gamma<1$.

It may be surprising that the leading order behavior of the modified wave and scattering operators at the origin is the identity: we recall that from their definition, these operators are already nontrivial, and account for long range effects. We also emphasize that the first corrector has a more involved expression than in the short range case, a case where we would simply have (typically for the two-dimensional and three-dimensional cubic Schrödinger equations, see e.g. [3])

$$w_2=-i\lambda\int_{-\infty}^0\left(U(-s)\left(|U(s)v_0|^2U(s)v_0\right)\right)\mathrm{d}s,$$
$$\mu_2=-i\lambda\int_0^\infty\left(U(-s)\left(|U(s)v_0|^2U(s)v_0\right)\right)\mathrm{d}s.$$

Note that in the (one-dimensional) long range case, these integrals diverge.

As evoked above, in principle, the method of proof that we present allows to compute (some) higher order terms in the asymptotic expansions near the origin.

1.4. *Outline*

In Section 2, we recall several properties which are classical in the context of scattering theory for nonlinear Schrödinger equations, and which are of constant use in this paper. In Section 3, we revisit the construction of the modified wave operator $W_-^{\rm mod}$, in such a way that the leading order behavior of this operator at the origin, presented in Section 4, is rather straightforward. In Section 4, we also consider the first corrector in the asymptotic expansion of $W_-^{\rm mod}$ at the origin, an aspect which requires some extra work. In Section 5, we recall the modified asymptotic completeness result established in [15], and infer the leading of behavior of $S^{\rm mod}$ at the origin. The first corrector is derived in Section 6, where the last two error estimates announced in Theorem 2 are proved.

1.5. *Notations*

We recall the classical factorization of the Schrödinger group,

$$U(t)=e^{i\frac{t}{2}\partial_x^2}=M(t)D(t)\mathcal{F}M(t),$$

where the multiplication $M(t)$, the dilation $D(t)$ and the Fourier transform $\mathcal{F}$ are defined by

$$\begin{aligned} M(t) &= e^{i\frac{x^2}{2t}}, \quad D(t)f(x) = \frac{1}{(it)^{1/2}} f\left(\frac{x}{t}\right), \\ \mathcal{F}f(\xi) &= \widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)\, e^{-ix\xi}\, dx. \end{aligned} \tag{1.7}$$

Note that each of these three operators is unitary on $L^2(\mathbb{R})$, and that (1.2) reads $U(t) \approx M(t)D(t)\mathcal{F}$, see also Lemma 5. We recall that the space Σ is defined by

$$\Sigma = H^1 \cap \mathcal{F}(H^1) = \left\{f \in H^1(\mathbb{R});\ \|f\|_\Sigma := \|f\|_{L^2} + \|\partial_x f\|_{L^2} + \|xf\|_{L^2} < \infty\right\}.$$

We note that Σ is a Banach algebra, invariant under the Fourier transform, and such that $\Sigma \hookrightarrow L^1 \cap L^\infty$. In addition, Σ is invariant under the action of $U(t)$, for any $t \in \mathbb{R}$.

For functions $f^\varepsilon, g^\varepsilon \geqslant 0$ depending on time t and ε, the notation

$$f^\varepsilon \lesssim g^\varepsilon$$

means that there exists C independent of t and ε such that

$$f^\varepsilon \leqslant C g^\varepsilon.$$

2. Technical preliminaries

In this section, we gather classical estimates which can be found in several references cited in the introduction.

Lemma 4. *The operator*

$$J(t) = x + it\partial_x$$

satisfies the following properties:

- *$J(t) = U(t)xU(-t)$, and therefore J commutes with the linear part of* (1.1),

$$\left[J(t), i\partial_t + \frac{1}{2}\partial_x^2\right] = 0. \tag{2.1}$$

- *It can be factorized as*

$$J(t) = it\, e^{i\frac{x^2}{2t}}\, \partial_x\left(e^{-i\frac{x^2}{2t}}\ \cdot\right).$$

As a consequence, J yields weighted Gagliardo–Nirenberg inequalities. For $2 \leqslant r \leqslant \infty$, there exists $C(r)$ depending only on r such that

$$\|f\|_{L^r} \leqslant \frac{C(r)}{|t|^{\delta(r)}} \|f\|_{L^2}^{1-\delta(r)} \|J(t)f\|_{L^2}^{\delta(r)}, \quad \delta(r) := \frac{1}{2} - \frac{1}{r}. \tag{2.2}$$

Also, if $F(z) = G(|z|^2)z$ is C^1, then $J(t)$ acts like a derivative on $F(w)$:

$$J(t)\,(F(w)) = \partial_z F(w) J(t) w - \partial_{\bar{z}} F(w) \overline{J(t) w}. \tag{2.3}$$

Lemma 5. *Denote by*

$$\mathcal{R}(t) = M(t)D(t)\mathcal{F}\,(M(t) - 1)\,\mathcal{F}^{-1},$$

where the above terms are defined in (1.7). *The following estimates hold for $|t| \geqslant 1$:*

(1) *For all $s > 1/2$ and $0 \leqslant \theta \leqslant 1$,*

$$\|\mathcal{R}(t)f\|_{L^\infty_x} \lesssim \frac{1}{|t|^{1/2+\theta}} \|f\|_{H^{s+2\theta}}, \quad \forall f \in \mathcal{S}(\mathbb{R}).$$

(2) *For all $0 \leqslant \theta \leqslant 1$,*

$$\|\mathcal{R}(t)f\|_{L^2_x} \lesssim \frac{1}{|t|^{\theta}} \|f\|_{H^{2\theta}}, \quad \forall f \in \mathcal{S}(\mathbb{R}).$$

(3) *For all* $0 \leqslant \theta \leqslant 1$,

$$\|J(t)\mathscr{R}(t)f\|_{L^2_x} \lesssim \frac{1}{|t|^\theta}\|f\|_{H^{1+2\theta}}, \quad \forall f \in \mathscr{S}(\mathbb{R}).$$

(4) *We have*

$$\|\partial_x\mathscr{R}(t)f\|_{L^2_x} \lesssim \frac{1}{|t|}\|f\|_{H^1} + \frac{1}{|t|}\|xf\|_{H^2}, \quad \forall f \in \mathscr{S}(\mathbb{R}).$$

Remark 6. We emphasize that $\mathscr{R}(t)$ does not map $L^\infty(\mathbb{R})$ into itself, since $\mathscr{F}(M(t)-1)\mathscr{F}^{-1} = U\left(\frac{1}{t}\right)-1$ and $U\left(\frac{1}{t}\right)\left(e^{-it\frac{x^2}{2}}\right) = \delta_0$.

Even though such estimates can be found in the existing literature, we give a proof, as the ideas will be resumed when $M(t)-1$ is present.

Proof. In view of the definition of $\mathscr{R}$, we readily have

$$\|\mathscr{R}(t)f\|_{L^\infty_x} = \frac{1}{|t|^{1/2}}\|\mathscr{F}(M(t)-1)\mathscr{F}^{-1}f\|_{L^\infty}.$$

Using Hausdorff–Young inequality and the general estimate

$$|M(t)-1| = 2\left|\sin\left(\frac{x^2}{4t}\right)\right| \lesssim \left|\frac{x^2}{t}\right|^\theta, \quad \forall\, 0 \leqslant \theta \leqslant 1, \tag{2.4}$$

we find

$$\|\mathscr{R}(t)f\|_{L^\infty_x} \lesssim \frac{1}{|t|^{1/2}}\left\|\left|\frac{x^2}{t}\right|^\theta \mathscr{F}^{-1}f\right\|_{L^1}.$$

Using the easy property $\mathscr{F}(H^s) \hookrightarrow L^1$ for $s > 1/2$ (the same is true on $\mathbb{R}^d$ provided that $s > d/2$, from Cauchy–Schwarz inequality),

$$\|\mathscr{R}(t)f\|_{L^\infty_x} \lesssim \frac{1}{|t|^{1/2+\theta}}\left\|\langle x\rangle^s |x|^{2\theta}\mathscr{F}^{-1}f\right\|_{L^2} \lesssim \frac{1}{|t|^{1/2+\theta}}\|f\|_{H^{s+2\theta}},$$

hence the first inequality. For the second one, since $D(t)$ is unitary on L^2, Plancherel formula and (2.4) yield

$$\|\mathscr{R}(t)f\|_{L^2_x} = \|(M(t)-1)\mathscr{F}^{-1}f\|_{L^2} \lesssim \frac{1}{|t|^\theta}\left\||x|^{2\theta}\mathscr{F}^{-1}f\right\|_{L^2} \lesssim \frac{1}{|t|^\theta}\|f\|_{H^{2\theta}}.$$

For the third inequality, we use the formula $J(t) = U(t)xU(-t)$, along with the factorization

$$U(-t) = iM(-t)\mathscr{F}^{-1}D\left(\frac{1}{t}\right)M(-t), \tag{2.5}$$

to obtain

$$\|J(t)\mathscr{R}(t)f\|_{L^2_x} = \|x(M(t)-1)\mathscr{F}^{-1}f\|_{L^2} \lesssim \frac{1}{|t|^\theta}\left\||x|^{1+2\theta}\mathscr{F}^{-1}f\right\|_{L^2} \lesssim \frac{1}{|t|^\theta}\|f\|_{H^{1+2\theta}}.$$

To estimate $\partial_x\mathscr{R}(t)f$, two terms appear, whether the derivative hits the first factor $M(t)$ or not, and we have

$$\|\partial_x\mathscr{R}(t)f\|_{L^2_x} \leqslant \left\|\frac{x}{t}D(t)\mathscr{F}(M(t)-1)\mathscr{F}^{-1}f\right\|_{L^2} + \frac{1}{|t|}\left\|\partial_x\mathscr{F}(M(t)-1)\mathscr{F}^{-1}f\right\|_{L^2}.$$

The first term on the right hand side is equal to

$$\begin{aligned}\left\|x\mathscr{F}(M(t)-1)\mathscr{F}^{-1}f\right\|_{L^2} &= \left\|\partial_x(M(t)-1)\mathscr{F}^{-1}f\right\|_{L^2}\\ &\leqslant \left\|\frac{x}{t}\mathscr{F}^{-1}f\right\|_{L^2} + \left\|(M(t)-1)\partial_x\mathscr{F}^{-1}f\right\|_{L^2}\\ &\lesssim \frac{1}{|t|}\|f\|_{H^1} + \frac{1}{|t|^\theta}\|xf\|_{H^{2\theta}},\end{aligned}$$

for any $\theta \in [0,1]$. Note that

$$\frac{1}{|t|}\left\|\partial_x \mathscr{F}(M(t)-1)\mathscr{F}^{-1}f\right\|_{L^2} = \frac{1}{|t|}\left\|x(M(t)-1)\mathscr{F}^{-1}f\right\|_{L^2} \lesssim \frac{1}{|t|}\left\|x\mathscr{F}^{-1}f\right\|_{L^2},$$

hence the result, by choosing $\theta = 1$. □

3. Modified wave operator

A key aspect in the proofs of the existence of modified wave operators for (1.1), in particular to get minimal regularity assumptions on the asymptotic state u_-, is to consider a suitable approximate solution near $t=-\infty$. Introduce two such approximate solutions, u_1 and u_2, defined for $t \leqslant -10$, given by

$$u_1(t,x) := \frac{1}{(\mathrm{i}t)^{1/2}} \mathrm{e}^{\mathrm{i}\frac{x^2}{2t}}\, \widehat{u}_-\left(\frac{x}{t}\right) \mathrm{e}^{\mathrm{i}S_-(t,x)} = M(t)D(t)\widehat{w}(t,x),$$

where

$$\widehat{w}(t) := \widehat{u}_-\, \mathrm{e}^{\mathrm{i}\lambda|\widehat{u}_-|^2\log|t|},$$

where we resume the same notations as in [17], and

$$u_2(t) := U(t)\mathscr{F}^{-1}\widehat{w}.$$

As $t \to -\infty$, u_1 and u_2 are close, in view of Lemma 5, since $u_2(t)-u_1(t) = \mathscr{R}(t)\widehat{w}$. Our goal is to keep track of the smallness of the asymptotic state as precisely as possible, in the sense that we consider $u_-^\varepsilon = \varepsilon v_-$ for some fixed v_-, assume that $\varepsilon > 0$ is small, and we want to get larger powers of ε in the error terms whenever we can. It turns out that apparently, u_2 is a better candidate than u_1 in this direction. More explicitly, the Duhamel formula for $u-u_1$ (see [17]) contains a linear term $(R\widehat{w})$, which we want to remove. We first gather some estimates on u_1.

Lemma 7. *Let $u_- \in \mathscr{H}$: $\|u_1(t)\|_{L^2} = \|u_2(t)\|_{L^2} = \|u_-\|_{L^2}$ for all $t \leqslant -10$. There exists $C>0$ independent of u_- such that for all $t \leqslant -10$,*

$$\|u_1(t)\|_{L^\infty} \leqslant \frac{\|\widehat{u}_-\|_{L^\infty}}{\sqrt{|t|}},$$
$$\|\partial_x u_1(t)\|_{L^2} \leqslant C\left(\|u_-\|_{\mathscr{H}} + \|u_-\|^3_{\mathscr{H}}\right),$$
$$\|J(t)u_1(t)\|_{L^2} \leqslant C\|u_-\|_{\mathscr{H}}\left(1+\|u_-\|^2_{\mathscr{H}}\log|t|\right).$$

Proof. The conservation of the L^2-norm is obvious, as well as the estimate

$$\|u_1(t)\|_{L^\infty} \leqslant \frac{\|\widehat{u}_-\|_{L^\infty}}{\sqrt{|t|}}.$$

The expression of u_1 yields directly

$$\|\partial_x u_1(t)\|_{L^2} \leqslant \|\partial_x \widehat{u}_-\|_{L^2} + |\lambda|\frac{\log|t|}{|t|}\|\widehat{u}_-\|^2_{L^\infty}\|\partial_x\widehat{u}_-\|_{L^2}.$$

In view of the factorization for J stated in Lemma 4,

$$\begin{aligned} J(t)u_1(t) &= \mathrm{i}tM(t)\partial_x\left(D(t)\widehat{u}_-\,\mathrm{e}^{\mathrm{i}S_-(t)}\right)\\ &= \mathrm{e}^{\mathrm{i}\frac{x^2}{2t}+\mathrm{i}S_-(t)}\left(\mathrm{i}D(t)(\partial_x\widehat{u}_-) - \lambda D(t)\widehat{u}_- \times \log|t| \times \partial_x|\widehat{u}_-|^2\left(\frac{x}{t}\right)\right), \end{aligned}$$

thus

$$\|J(t)u_1(t)\|_{L^2} \lesssim \|\widehat{u}_-\|_{H^1} + \log|t|\|\widehat{u}_-\|^2_{L^\infty}\|\widehat{u}_-\|_{H^1}.$$

The lemma follows. □

Lemma 8. *Let $u_- \in \mathcal{H}$. There exists $C>0$ independent of u_- such that for all $t \leqslant -10$,*

$$\|\mathcal{R}(t)\widehat{w}\|_{L^\infty} \leqslant \frac{C}{\sqrt{|t|}}\left(\|\widehat{u}_-\|_{H^1} + \|\widehat{u}_-\|_{H^1}^3\right),$$

$$\|\mathcal{R}(t)\widehat{w}\|_{L^\infty} \leqslant \frac{C}{|t|^{3/2}}\left(\|u_-\|_{\mathcal{H}} + (\log|t|)^3\|u_-\|_{\mathcal{H}}^7\right).$$

$$\|U(-t)\mathcal{R}(t)\widehat{w}\|_\Sigma \leqslant \frac{C}{|t|}\left(\|u_-\|_{\mathcal{H}} + (\log|t|)^3\|u_-\|_{\mathcal{H}}^7\right).$$

Proof. If we use the first point of Lemma 5 with $\theta = 0$ and $s > 1/2$, the phase factor causes the appearance of a $\log|t|$ term, since the H^1-norm of $\widehat{w}$ is involved. So we rather take $\theta > 0$ and $s > 1/2$ such that $s + 2\theta = 1$, and so, for $t \leqslant -10$,

$$\|\mathcal{R}(t)\widehat{w}\|_{L^\infty} \lesssim \frac{\|\widehat{w}\|_{H^1}}{|t|^{1/2+\theta}} \lesssim \frac{1}{|t|^{1/2+\theta}}\left(\|\widehat{u}_-\|_{H^1} + (\log|t|)\|\widehat{u}_-\|_{L^\infty}^2\|\widehat{u}_-\|_{H^1}\right)$$
$$\lesssim \frac{1}{\sqrt{|t|}}\left(\|\widehat{u}_-\|_{H^1} + \|\widehat{u}_-\|_{H^1}^3\right),$$

where the logarithmic factor was left out since $\theta > 0$. It is actually at the level of this estimate that the H^1-norm of $\widehat{u}_-$ appears, instead of merely its L^∞-norm.

To obtain the stronger time decay, we choose $s = \theta = 1$ in the first point of Lemma 5, hence, since $H^3(\mathbb{R})$ is an algebra,

$$\|\mathcal{R}(t)\widehat{w}\|_{L^\infty} \lesssim \frac{\|\widehat{w}\|_{H^3}}{|t|^{3/2}} \lesssim \frac{1}{|t|^{3/2}}\left(\|\widehat{u}_-\|_{H^3} + (\log|t|)^3\|\widehat{u}_-\|_{H^3}^7\right).$$

In view of Lemma 4,

$$\|U(-t)\mathcal{R}\widehat{w}\|_\Sigma = \|\mathcal{R}\widehat{w}\|_{L^2} + \|\partial_x\mathcal{R}\widehat{w}\|_{L^2} + \|J(t)\mathcal{R}\widehat{w}\|_{L^2},$$

and Lemma 5 yields

$$\|U(-t)\mathcal{R}(t)\widehat{w}\|_\Sigma \lesssim \frac{1}{|t|}\|\widehat{w}\|_{H^3} + \frac{1}{|t|}\|x\widehat{w}\|_{H^2}.$$

Using the fact that $H^s(\mathbb{R})$ is an algebra for $s > 1/2$, we infer

$$\|U(-t)\mathcal{R}(t)\widehat{w}\|_\Sigma \lesssim \frac{1}{|t|}\left(\|\widehat{u}_-\|_{H^3} + (\log|t|)^3\|\widehat{u}_-\|_{H^3}^7\right) + \frac{1}{|t|}\left(\|x\widehat{u}_-\|_{H^2} + (\log|t|)^2\|x\widehat{u}_-\|_{H^2}^5\right)$$
$$\lesssim \frac{1}{|t|}\left(\|u_-\|_{\mathcal{H}} + \sum_{j=0,1}(\log|t|)^{2+j}\|u_-\|_{\mathcal{H}}^{5+2j}\right),$$

hence the lemma. □

Proposition 9. *There exist $\delta_0 > 0$, C, and a polynomial P, such that the following holds. For any $u_- \in L^2$ with $u_- \in \mathcal{H}$, where $\mathcal{H}$ is defined in* (1.6), *and such that $\|\widehat{u}_-\|_{H^1} \leqslant \delta_0$, there exists a unique $u_0 \in \Sigma$ such that the solution $u \in C(\mathbb{R},\Sigma)$ to* (1.1) *with $u_{|t=0} = u_0$ satisfies, for $t \leqslant -10$,*

$$\|U(-t)\left(u(t) - u_2(t)\right)\|_\Sigma \leqslant C\|u_-\|_{\mathcal{H}}^3 P\left(\|u_-\|_{\mathcal{H}}\right)\frac{(\log|t|)^4}{|t|}.$$

Remark 10. The choice of the approximate solution u_2 is tailored to ensure that, with the proof presented below, the error term is superlinear in some norm of u_-, uniformly in the limit $t \to -\infty$. On the other hand, the smallness assumption, which is a consequence of the first estimate in Lemma 8, is stronger than in the cited references, since Sobolev embedding yields $H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$.

Proof. Like in [28] or [17], the proof relies on a fixed point argument near $t = -\infty$. Resuming the computations from [17] (and keeping the same notations),

$$i\partial_t\widehat{w} = \frac{\lambda}{t}|\widehat{w}|^2\widehat{w},$$

and so we check that u_2 solves the equation

$$\mathrm{i}\partial_t \mathscr{F} U(-t) u_2 = \lambda \mathscr{F} U(-t)\left(\frac{1}{t} U(t)\mathscr{F}^{-1}\left(|\widehat{w}|^2\widehat{w}\right)\right).$$

Therefore, we want to solve

$$\mathrm{i}\partial_t \mathscr{F} U(-t)(u-u_2) = \lambda \mathscr{F} U(-t)\left(|u|^2 u - \frac{1}{t} U(t)\mathscr{F}^{-1}\left(|\widehat{w}|^2\widehat{w}\right)\right).$$

We insert the term $|u_2|^2 u_2$ into the above equation, and use the identity

$$\begin{aligned}\frac{1}{t} U(t)\mathscr{F}^{-1}\left(|\widehat{w}|^2\widehat{w}\right) &= \frac{1}{t} MD\mathscr{F}M\mathscr{F}^{-1}\left(|\widehat{w}|^2\widehat{w}\right)\\ &= \frac{1}{t}\mathscr{R}(t)\left(|\widehat{w}|^2\widehat{w}\right) + \frac{1}{t} M(t)D(t)\left(|\widehat{w}|^2\widehat{w}\right),\end{aligned}$$

with

$$\frac{1}{t} M(t)D(t)\left(|\widehat{w}|^2\widehat{w}\right) = M(t)|M(t)D(t)\widehat{w}|^2 D(t)\widehat{w} = |u_1|^2 u_1,$$

so Duhamel's formula reads, along with the requirement $U(-t)(u(t)-u_2(t)) \to 0$ as $t\to-\infty$,

$$\begin{aligned}u(t) = u_2(t) - \mathrm{i}\lambda\int_{-\infty}^t U(t-\tau)\left(|u|^2u - |u_2|^2 u_2\right)(\tau)\,\mathrm{d}\tau + \mathrm{i}\lambda\int_{-\infty}^t U(t-\tau)\left(|u_2|^2 u_2 - |u_1|^2 u_1\right)(\tau)\,\mathrm{d}\tau\\ - \mathrm{i}\lambda\int_{-\infty}^t U(t-\tau)\left(\frac{1}{\tau}\mathscr{R}(\tau)\left(|\widehat{w}|^2\widehat{w}\right)\right)(\tau)\,\mathrm{d}\tau. \quad (3.1)\end{aligned}$$

The last two lines correspond to source terms, involving only the various approximate solutions, and can be estimated thanks to Lemma 5, as $u_2 = u_1 + \mathscr{R}\widehat{w}$.

Denote by $\Phi(u) = u_2 + \Phi_1(u) + \Phi_2 + \Phi_3$ the right hand side of (3.1), where Φ_j corresponds to the j-th line (note that Φ_2 and Φ_3 do not depend on u). We have

$$\begin{aligned}\left|\partial_x\left(|u_2|^2 u_2 - |u_1|^2 u_1\right)\right| &= \left|\partial_x\left(|\mathscr{R}\widehat{w} + u_1|^2(\mathscr{R}\widehat{w} + u_1) - |u_1|^2 u_1\right)\right|\\ &\lesssim \left(|\mathscr{R}\widehat{w}|^2 + |u_1|^2\right)|\partial_x\mathscr{R}\widehat{w}| + (|\mathscr{R}\widehat{w}| + |u_1|)\,|\mathscr{R}\widehat{w}||\partial_x u_1|.\end{aligned}$$

Using the formula $J(t) = U(t)xU(-t)$ from Lemma 4, as well as (2.3) that state that $J(t)$ can be thought of as a derivative here, we find

$$\begin{aligned}\|U(-t)\Phi_2(t)\|_\Sigma \lesssim \int_{-\infty}^t \left(\|\mathscr{R}(\tau)\widehat{w}\|_{L^\infty}^2 + \|u_1(\tau)\|_{L^\infty}^2\right)\|U(-\tau)\mathscr{R}\widehat{w}\|_\Sigma\,\mathrm{d}\tau\\ + \int_{-\infty}^t (\|\mathscr{R}(\tau)\widehat{w}\|_{L^\infty} + \|u_1(\tau)\|_{L^\infty})\,\|\mathscr{R}(\tau)\widehat{w}\|_{L^\infty}\|U(-\tau)u_1(\tau)\|_\Sigma\,\mathrm{d}\tau.\end{aligned}$$

We invoke Lemmas 7 and 8 to estimate the above terms. More precisely, then factor $\|\mathscr{R}(\tau)\widehat{w}\|_{L^\infty}$ is controlled by the second estimate of Lemma 8, and we get, for $\|\widehat{u}_-\|_{H^1}\leqslant 1$ and $t\leqslant -10$:

$$\|U(-t)\Phi_2(t)\|_\Sigma \lesssim \|\widehat{u}_-\|_{H^1}\|u_-\|_{\mathscr{H}}^2 P_2(\|u_-\|_{\mathscr{H}})\int_{-\infty}^t \frac{(\log|\tau|)^4}{\tau^2}\mathrm{d}\tau,$$

for some polynomial P_2. The term Φ_3 is estimated thanks to Lemma 5, and the choice of parameter is motivated by the previous estimate, keeping in mind that a factor $\frac{1}{\tau}$ is already present in the definition of Φ_3. For the L^2-estimate, we thus choose $\theta = 1$ in Lemma 5, so that

$$\begin{aligned}\|\Phi_3(t)\|_{L^2} &\lesssim \int_{-\infty}^t \left\||\widehat{w}|^2\widehat{w}\right\|_{H^2}\frac{\mathrm{d}\tau}{\tau^2} \lesssim \int_{-\infty}^t \|\widehat{w}\|_{L^\infty}^2\|\widehat{w}\|_{H^2}\frac{\mathrm{d}\tau}{\tau^2}\\ &\lesssim \|\widehat{u}_-\|_{L^\infty}^2\|\widehat{u}_-\|_{H^2}\int_{-\infty}^t\left(1 + (\log|\tau|)^2\|\widehat{u}_-\|_{H^2}^4\right)\frac{\mathrm{d}\tau}{\tau^2}.\end{aligned}$$

The norm $\|xU(-t)\Phi_3(t)\|_{L^2} = \|J(t)\Phi_3(t)\|_{L^2}$ is estimate similarly, by setting again $\theta = 1$ (in the third point of Lemma 5, and we get, since the H^2-norm is replaced by an H^3-norm in the above computation,

$$\|J(t)\Phi_3(t)\|_{L^2} \lesssim \|\widehat{u}_-\|_{L^\infty}^2 \|\widehat{u}_-\|_{H^3} \int_{-\infty}^t \left(1 + (\log|\tau|)^3 \|\widehat{u}_-\|_{H^3}^6\right) \frac{\mathrm{d}\tau}{\tau^2}.$$

The choice of suitable parameters for the $\dot{H}^1$-norm was already made in the statement of Lemma 5, and we have

$$\|\partial_x \Phi_3(t)\|_{L^2} \lesssim \int_{-\infty}^t \left(\||\widehat{w}|^2 \widehat{w}\|_{H^1} + \|x|\widehat{w}|^2 \widehat{w}\|_{H^2}\right) \frac{\mathrm{d}\tau}{\tau^2}.$$

The term involving the H^1-norm has been estimated above, so we focus on the other norm. Direct computations yield

$$\left|\partial_x^2 \left(x|\widehat{w}|^2 \widehat{w}\right)\right| \lesssim |x||\widehat{w}|^2 |\partial_x^2 \widehat{w}| + |\widehat{w}|^2 |\partial_x \widehat{w}| + |x||\widehat{w}||\partial_x \widehat{w}|^2,$$

and we get

$$\begin{aligned}\|\partial_x \Phi_3(t)\|_{L^2} \lesssim \|\widehat{u}_-\|_{L^\infty}^2 \|\widehat{u}_-\|_{H^1} \int_{-\infty}^t \left(1 + (\log|\tau|)\|\widehat{u}_-\|_{H^1}^2\right) \frac{\mathrm{d}\tau}{\tau^2} \\ + \|\widehat{u}_-\|_{L^\infty} \|u_-\|_{\mathscr{H}}^2 \int_{-\infty}^t \left(1 + (\log|\tau|)^2 \|u_-\|_{\mathscr{H}}^4\right) \frac{\mathrm{d}\tau}{\tau^2}.\end{aligned}$$

The source term is therefore controlled, for $t \leqslant -10$, by:

$$\|U(-t)(\Phi_2(t) + \Phi_3(t))\|_\Sigma \lesssim \|\widehat{u}_-\|_{H^1} \|u_-\|_{\mathscr{H}}^2 P(\|u_-\|_{\mathscr{H}}) \int_{-\infty}^t \frac{\left(\log|\tau|\right)^4}{\tau^2} \mathrm{d}\tau, \tag{3.2}$$

for some polynomial P whose precise expression is irrelevant.

The end of the proof consists of a fixed point argument. Mimicking [28], for $\alpha \in]1/2, 1[$ arbitrary (but fixed), and $T \gg 1$, introduce the space

$$\begin{aligned}X^\alpha(T) = \Big\{u \in L^\infty(]-\infty, -T], L^2) \text{ such that } t \longmapsto U(-t)u(t) \in L^\infty(]-\infty, -T], \Sigma), \\ \sup_{t \leqslant -T} |t|^\alpha \|U(-t)(u(t) - u_2(t))\|_\Sigma \leqslant \|\widehat{u}_-\|_{H^1}\Big\}.\end{aligned}$$

and define on $X^\alpha(T)$ the metric

$$d(u, v) = \sup_{t \leqslant -T} |t|^\alpha \|u(t) - v(t)\|_{L^2}.$$

Note that here, we choose to measure distance by considering the L^2 norm only: $X^\alpha(T)$, equipped with this distance, is a complete metric space.

For $u \in X^\alpha(T)$, $\Phi_1(u)$ is estimated by

$$\begin{aligned}\|U(-t)\Phi_1(u(t))\|_{L^2} &\lesssim \int_{-\infty}^t \left(\|u(\tau)\|_{L^\infty}^2 + \|u_2(\tau)\|_{L^\infty}^2\right) \|u(\tau) - u_2(\tau)\|_{L^2} \mathrm{d}\tau \\ &\lesssim \|\widehat{u}_-\|_{H^1}^2 \int_{-\infty}^t \frac{\mathrm{d}\tau}{\tau^{1+\alpha}} \lesssim \frac{\|\widehat{u}_-\|_{H^1}^2}{|t|^\alpha}.\end{aligned}$$

For $A \in \{\partial_x, J(t)\}$, we have

$$\left|A\left(|u|^2 u - |u_2|^2 u_2\right)\right| \lesssim |u|^2 |A(u - u_2)| + (|u| + |u_2|)\, |A u_2| |u - u_2|. \tag{3.3}$$

Therefore,

$$\|U(-t)\Phi_1(u(t))\|_\Sigma \lesssim \int_{-\infty}^t \|u(\tau)\|_{L^\infty}^2 \|U(-\tau)(u(\tau)-u_2(\tau))\|_\Sigma \,\mathrm{d}\tau \\ + \int_{-\infty}^t (\|u(\tau)\|_{L^\infty}+\|u_2(\tau)\|_{L^\infty})\|U(-\tau)u_2(\tau)\|_\Sigma \|u(\tau)-u_2(\tau)\|_{L^\infty}\mathrm{d}\tau \\ \lesssim \|\widehat{u}_-\|_{H^1}^2 P_3(\|u_-\|_{\mathscr{H}}) \int_{-\infty}^t \frac{\log|\tau|}{\tau^{1+\alpha}}\,\mathrm{d}\tau,$$

for some polynomial P_3 whose precise expression is irrelevant, where we have used Lemmas 7 and 8 to estimate $\|U(-\tau)u_2(\tau)\|_\Sigma$. The L^∞-norm of $u(\tau)-u_2(\tau)$ is controlled by $|\tau|^{-1/2-\alpha}$ in view of Lemma 4 and the definition of $X^\alpha(T)$. Therefore, by choosing T sufficiently large, we check that Φ maps $X^\alpha(T)$ to itself.

To conclude by a fixed point argument, we show that Φ is a contraction provided that $\|\widehat{u}_-\|_{H^1}$ is sufficiently small. Indeed, for $u, v \in X^\alpha(T)$,

$$\Phi(u)(t)-\Phi(v)(t) = -\mathrm{i}\lambda \int_{-\infty}^t U(t-\tau)\left(|u|^2u-|v|^2v\right)(\tau)\,\mathrm{d}\tau,$$

and so, for $t \leqslant -10$,

$$\|\Phi(u)(t)-\Phi(v)(t)\|_{L^2} \lesssim \int_{-\infty}^t \left(\|u(\tau)\|_{L^\infty}^2+\|v(\tau)\|_{L^\infty}^2\right)\|u(\tau)-v(\tau)\|_{L^2}\,\mathrm{d}\tau \\ \lesssim \|\widehat{u}_-\|_{H^1}^2 d(u,v)\int_{-\infty}^t \frac{\mathrm{d}\tau}{\tau^{1+\alpha}},$$

where we have used

$$\|u(\tau)\|_{L^\infty}+\|v(\tau)\|_{L^\infty} \leqslant 2\|u_2(\tau)\|_{L^\infty}+C\|u(\tau)-u_2(\tau)\|_{H^1}+C\|v(\tau)-u_2(\tau)\|_{H^1} \\ \lesssim \frac{\|\widehat{u}_-\|_{H^1}}{\sqrt{|\tau|}},$$

since $\alpha > 1/2$. Therefore,

$$d(\Phi(u),\Phi(v)) \lesssim \|\widehat{u}_-\|_{H^1}^2 d(u,v),$$

hence the result provided that $\|\widehat{u}_-\|_{H^1}$ is sufficiently small. □

4. Behavior of the modified wave operator near the origin

4.1. *Leading order asymptotic behavior*

Let $u_-^\varepsilon = \varepsilon v_-$ with $v_- \in \mathscr{H}$ independent of ε. Denote by u^ε the solution provided by Proposition 9. Then like u_-^ε, u^ε is of order ε, and the remainder in Proposition 9 is $\mathcal{O}(\varepsilon^3)$. The long range phase correction S_-^ε is $\mathcal{O}(\varepsilon^2\log|t|)$, for $t \leqslant -10$: its contribution is negligible for times t such that $\varepsilon^2|\log|t|| \ll 1$, so in particular we can match with a linear solution at $t_\gamma^\varepsilon = -1/\varepsilon^\gamma$,

$$\mathrm{i}\partial_t \mathbf{u}_1^\varepsilon + \frac{1}{2}\partial_x^2 \mathbf{u}_1^\varepsilon = 0,$$

with

$$\mathbf{u}^\varepsilon_{1|t=-1/\varepsilon^\gamma} = \frac{1}{(\mathrm{i}t)^{1/2}}\mathrm{e}^{\mathrm{i}\frac{x^2}{2t}}\mathscr{F}(u_-^\varepsilon)\left(\frac{x}{t}\right)\Big|_{t=-1/\varepsilon^\gamma}.$$

The right hand side is $M(t)D(t)\mathscr{F}u_-^\varepsilon$ evaluated at $t=-1/\varepsilon^\gamma$. In view of Lemma 5, up to a small error term,

$$\mathbf{u}_1^\varepsilon(t) = U(t)u_-^\varepsilon, \text{ or, equivalently, } \mathbf{u}_1^\varepsilon = \varepsilon v_1, \text{ with } v_1(t) = U(t)v_-.$$

We choose this definition for the first order approximation $\mathbf{u}_1^\varepsilon$. Let $t \geqslant -1/\varepsilon^\gamma$. Duhamel's formula implies

$$U(-t)\left(u^\varepsilon(t)-v_1^\varepsilon(t)\right) = U(-s)\left(u^\varepsilon(s)-u_2^\varepsilon(s)+u_2^\varepsilon(s)-\mathbf{u}_1^\varepsilon(s)\right)\Big|_{s=-1/\varepsilon^\gamma} - \mathrm{i}\lambda\int_{-1/\varepsilon^\gamma}^t U(-\tau)\left(|u^\varepsilon|^2u^\varepsilon\right)(\tau)\,\mathrm{d}\tau,$$

that is

$$U(-t)u^\varepsilon(t)-u_-^\varepsilon = U(-s)\left(u^\varepsilon(s)-u_2^\varepsilon(s)\right)\Big|_{s=-1/\varepsilon^\gamma} + U(-s)u_2^\varepsilon(s)\Big|_{s=-1/\varepsilon^\gamma} - u_-^\varepsilon - \mathrm{i}\lambda\int_{-1/\varepsilon^\gamma}^t U(-\tau)\left(|u^\varepsilon|^2u^\varepsilon\right)(\tau)\mathrm{d}\tau,$$

The first term on the right hand side is estimated thanks to Proposition 9:

$$\left\|U(-s)\left(u^\varepsilon(s)-u_2^\varepsilon(s)\right)\Big|_{s=-1/\varepsilon^\gamma}\right\|_\Sigma \lesssim \varepsilon^{3+\gamma}|\log\varepsilon|.$$

On the other hand,

$$\left\|U(-s)u_2^\varepsilon(s)\Big|_{s=-1/\varepsilon^\gamma} - u_-^\varepsilon\right\|_\Sigma = \|\mathscr{F}^{-1}\widehat{w}_{|s=-1/\varepsilon^\gamma} - u_-^\varepsilon\|_\Sigma = \|\widehat{w}_{|s=-1/\varepsilon^\gamma} - \mathscr{F}(u_-^\varepsilon)\|_\Sigma.$$

Recalling that

$$\widehat{w} = \mathscr{F}(u_-^\varepsilon)\exp\left(\mathrm{i}|\mathscr{F}(u_-^\varepsilon)|^2\log|t|\right),$$

we have directly

$$\begin{aligned}\|\widehat{w}-\mathscr{F}(u_-^\varepsilon)\|_\Sigma &\lesssim \|\langle x\rangle\left(\widehat{w}-\mathscr{F}(u_-^\varepsilon)\right)\|_{L^2} + \left\|\partial_x\widehat{u_-^\varepsilon}\left(\exp\left(\mathrm{i}|\mathscr{F}(u_-^\varepsilon)|^2\log|t|\right)-1\right)\right\|_{L^2}\\ &\quad + |\log\varepsilon|\left\|\mathscr{F}(u_-^\varepsilon)\partial_x|\mathscr{F}(u_-^\varepsilon)|^2\right\|_{L^2}\\ &\lesssim \varepsilon^3|\log\varepsilon|.\end{aligned}$$

The integral term can be estimated in Σ by

$$\int_{-1/\varepsilon^\gamma}^t \|u^\varepsilon(\tau)\|_{L^\infty}^2\|U(-\tau)u^\varepsilon(\tau)\|_\Sigma\,\mathrm{d}\tau.$$

Since $\|U(T)u^\varepsilon(-T)\|_\Sigma \lesssim \varepsilon$ from Lemmas 7 and 8, and Proposition 9 (where T is independent of ε), we have the following uniform estimate from [15, Theorem 1.1] (see also [20]), provided that $\varepsilon > 0$ is sufficiently small,

$$\|u^\varepsilon(\tau)\|_{L^\infty} \lesssim \frac{\varepsilon}{\langle\tau\rangle^{1/2}}, \quad \forall\tau\in\mathbb{R}. \tag{4.1}$$

Writing

$$\|U(-\tau)u^\varepsilon(\tau)\|_\Sigma \leqslant \|U(-\tau)\left(u^\varepsilon(\tau)-\mathbf{u}_1^\varepsilon(\tau)\right)\|_\Sigma + \|U(-\tau)\mathbf{u}_1^\varepsilon(\tau)\|_\Sigma,$$

and using the obvious fact that $\|U(-\tau)\mathbf{u}_1^\varepsilon(\tau)\|_\Sigma = \varepsilon\|v_-\|_\Sigma$, we have, for $t \geqslant -1/\varepsilon^\gamma$,

$$\left\|U(-t)u^\varepsilon(t)-u_-^\varepsilon\right\|_\Sigma \lesssim \varepsilon^3|\log\varepsilon| + \varepsilon^2\int_{-1/\varepsilon^\gamma}^t \left\|U(-\tau)u^\varepsilon(\tau)-u_-^\varepsilon\right\|_\Sigma\frac{\mathrm{d}\tau}{\langle\tau\rangle} + \varepsilon^3\int_{-1/\varepsilon^\gamma}^t\frac{\mathrm{d}\tau}{\langle\tau\rangle}.$$

Therefore, Gronwall lemma yields

$$\sup_{-1/\varepsilon^\gamma\leqslant t\leqslant T}\left\|U(-t)u^\varepsilon(t)-u_-^\varepsilon\right\|_\Sigma \lesssim \varepsilon^3|\log\varepsilon|\,\mathrm{e}^{C\varepsilon^2|\log\varepsilon|+C\varepsilon^2\log\langle T\rangle},$$

for some constant C independent of ε and T. In particular, for any $\delta\in\,]0,1]$, the right hand side is $\mathscr{O}(\varepsilon^{3-\delta})$ on $[-1/\varepsilon^\gamma, 1/\varepsilon^\beta]$ for any $\beta > 0$. This estimate implies the property

$$W_-^{\mathrm{mod}}(\varepsilon v_-) = \varepsilon v_- + \mathscr{O}(\varepsilon^{3-\eta}),$$

for any $\eta > 0$. In the next subsection, we improve this estimate by describing the first corrector term.

4.2. *Next term in the asymptotic expansion*

Following e.g. [3], a refined asymptotic expansion for u^ε, at least on bounded time intervals, is given by the solution $\mathbf{u}_2^\varepsilon$ to

$$i\partial_t \mathbf{u}_2^\varepsilon + \frac{1}{2}\partial_x^2 \mathbf{u}_2^\varepsilon = \lambda |\mathbf{u}_1^\varepsilon|^2 \mathbf{u}_1^\varepsilon,$$

where the Cauchy data must be chosen carefully. Formally, as $\mathbf{u}_1^\varepsilon$ is of order ε, $\mathbf{u}_2^\varepsilon$ is of order ε^3, provided that its Cauchy datum is $\mathcal{O}(\varepsilon^3)$. Set

$$\mathbf{u}_{\rm app}^\varepsilon = \mathbf{u}_1^\varepsilon + \mathbf{u}_2^\varepsilon.$$

We now have

$$\begin{aligned} &U(-t)\left(u^\varepsilon(t) - \mathbf{u}_{\rm app}^\varepsilon(t)\right) \\ &\quad = U(-s)\left(u^\varepsilon(s) - u_2^\varepsilon(s) + u_2^\varepsilon(s) - \mathbf{u}_{\rm app}^\varepsilon(s)\right)\Big|_{s=-1/\varepsilon^\gamma} - i\lambda \int_{-1/\varepsilon^\gamma}^t U(-\tau)\left(|u^\varepsilon|^2 u^\varepsilon - |\mathbf{u}_1^\varepsilon|^2 \mathbf{u}_1^\varepsilon\right)(\tau)\,d\tau. \end{aligned}$$

We have seen before that

$$\left\| U(-s)\left(u^\varepsilon(s) - u_2^\varepsilon(s)\right)\Big|_{s=-1/\varepsilon^\gamma}\right\|_\Sigma \lesssim \varepsilon^{3+\gamma}|\log\varepsilon| = o(\varepsilon^3).$$

So if we want to catch the ε^3 term, this is good. Making $U(-s)(u_2^\varepsilon(s) - \mathbf{u}_{\rm app}^\varepsilon(s))$ small ($o(\varepsilon^3)$ in Σ) at $s = -1/\varepsilon^\gamma$ is what should tell us how to choose the data for $\mathbf{u}_2^\varepsilon$. Indeed, the integral term is estimated in L^2 (Σ will require more care) by

$$\int_{-1/\varepsilon^\gamma}^t \left(\|u^\varepsilon(\tau)\|_{L^\infty}^2 + \|\mathbf{u}_1^\varepsilon(\tau)\|_{L^\infty}^2\right)\|u^\varepsilon(\tau) - v_1^\varepsilon(\tau)\|_{L^2}\,d\tau \lesssim \varepsilon^2 \int_{-1/\varepsilon^\gamma}^t \|u^\varepsilon(\tau) - \mathbf{u}_1^\varepsilon(\tau)\|_{L^2}\frac{d\tau}{\langle\tau\rangle},$$

and we now write

$$\|u^\varepsilon(\tau) - \mathbf{u}_1^\varepsilon(\tau)\|_{L^2} \leqslant \|u^\varepsilon(\tau) - \mathbf{u}_{\rm app}^\varepsilon(\tau)\|_{L^2} + \|\mathbf{u}_2^\varepsilon(\tau)\|_{L^2},$$

so the last term yields a smaller contribution (in terms of ε at least) than in the previous subsection.

Thus, everything seems to boil down to choosing $\mathbf{u}_2^\varepsilon$ at time $-1/\varepsilon^\gamma$ in an efficient way, so that

$$\left\| U(-s)(u_2^\varepsilon(s) - \mathbf{u}_{\rm app}^\varepsilon(s))\Big|_{s=-1/\varepsilon^\gamma}\right\|_\Sigma = o(\varepsilon^3).$$

By construction,

$$U(-s)(u_2^\varepsilon(s))\Big|_{s=-1/\varepsilon^\gamma} = \mathcal{F}^{-1}\left(\mathcal{F}(u_-^\varepsilon)\exp\left(-i\lambda\gamma|\mathcal{F}(u_-^\varepsilon)|^2\log\varepsilon\right)\right).$$

Taking into account the second term in the asymptotic expansion of the exponential in $\widehat{w}$, we define $\mathbf{u}_2^\varepsilon$ by

$$U(-t)\mathbf{u}_2^\varepsilon(t) = -i\lambda\gamma\varepsilon^3(\log\varepsilon)\mathcal{F}^{-1}\left(|\mathcal{F}(v_-)|^2\mathcal{F}(v_-)\right) - i\lambda\varepsilon^3\int_{-1/\varepsilon^\gamma}^t U(-\tau)\left(|U(\tau)v_-|^2 U(\tau)v_-\right)d\tau. \quad (4.2)$$

Lemma 11. *Let $v_- \in \mathcal{H}$. For every $t \in \mathbb{R}$, $U(-t)\mathbf{u}_2^\varepsilon(t)/\varepsilon^3$ converges in $L^2 \cap L^\infty(\mathbb{R})$, and the limit is independent of γ. This limit is given by*

$$\begin{aligned} U(-t)v_2(t) = -i\lambda\int_{-\infty}^{-1}&\left(U(-\tau)\left(|U(\tau)v_-|^2 U(\tau)v_-\right) + \frac{1}{|\tau|}\mathcal{F}^{-1}\left(|\mathcal{F}(v_-)|^2\mathcal{F}(v_-)\right)\right)d\tau \\ &- i\lambda\int_{-1}^t U(-\tau)\left(|U(\tau)v_-|^2 U(\tau)v_-\right)d\tau. \end{aligned}$$

The limit also holds in Σ, and for all $t \in \mathbb{R}$,

$$\left\| U(-t)\left(\frac{\mathbf{u}_2^\varepsilon(t)}{\varepsilon^3} - v_2(t)\right)\right\|_\Sigma \lesssim \varepsilon^\gamma.$$

Remark 12. We emphasize that even though the limit v_2 is independent of γ, the error estimate improves for large values of γ. In view of the definition of $\mathbf{u}_2^\varepsilon$, the matching condition at $t=-1/\varepsilon^\gamma$ between u_2^ε and $\mathbf{u}_{\rm app}^\varepsilon$ is not better than $\mathcal{O}\left(\varepsilon^5(\log\varepsilon)^2\right)$, so it is no use (in this paper) to consider $\gamma>2$.

Proof. The proof consists in writing a precise asymptotic expansion of the argument of the integral involved in (4.2). Writing, for any $\theta\in\,]0,1]$,

$$M(t)=\mathrm{e}^{\mathrm{i}\frac{x^2}{2t}}=1+\mathcal{O}\left(\left|\frac{x^2}{t}\right|^\theta\right),$$

we have, in the same vein as in Lemma 5,

$$U(\tau)v_-(x)=M(\tau)D(\tau)\mathcal{F}v_-+R_1(\tau,x),$$

where, as $\tau\to-\infty$

$$\|R_1(\tau)\|_{L^2}=\mathcal{O}\left(\frac{1}{|\tau|^\theta}\right),\quad \|R_1(\tau)\|_{L^\infty}=\mathcal{O}\left(\frac{1}{|\tau|^{\theta+1/2}}\right).$$

We infer, as $\tau\to-\infty$,

$$|U(\tau)v_-|^2U(\tau)v_-=\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\frac{\mathrm{e}^{\mathrm{i}\frac{x^2}{2\tau}}}{|\tau|^{3/2}}\left(|\mathcal{F}(v_-)|^2\mathcal{F}(v_-)\right)\left(\frac{x}{\tau}\right)+R_2(\tau,x),$$

with

$$\|R_2(\tau)\|_{L^2}=\mathcal{O}\left(\frac{1}{|\tau|^{1+\theta}}\right),\quad \|R_2(\tau)\|_{L^\infty}=\mathcal{O}\left(\frac{1}{|\tau|^{3/2+\theta}}\right).$$

In view of (2.5), we next compute

$$\begin{aligned}U(-\tau)\left(|U(\tau)v_-|^2U(\tau)v_-\right)&=\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\frac{x^2}{2\tau}}\mathcal{F}^{-1}D\left(\frac{1}{\tau}\right)\left(\frac{\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}}{|\tau|^{3/2}}\left(|\mathcal{F}(v_-)|^2\mathcal{F}(v_-)\right)\left(\frac{x}{\tau}\right)\right)\\&\quad+\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\frac{x^2}{2\tau}}\mathcal{F}^{-1}D\left(\frac{1}{\tau}\right)\left(\mathrm{e}^{-\mathrm{i}\frac{x^2}{2\tau}}R_2(\tau,x)\right)\\&=-\frac{\mathrm{e}^{-\mathrm{i}\frac{x^2}{2\tau}}}{|\tau|}\mathcal{F}^{-1}\left(|\mathcal{F}(v_-)|^2\mathcal{F}(v_-)\right)(x)+R_3(\tau,x)\\&=-\frac{1}{|\tau|}\mathcal{F}^{-1}\left(|\mathcal{F}(v_-)|^2\mathcal{F}(v_-)\right)(x)+R_4(\tau,x),\end{aligned}$$

where

$$\|R_4(\tau)\|_{L^2\cap L^\infty}=\mathcal{O}\left(\frac{1}{|\tau|^{1+\theta}}\right).$$

Set $\theta=1$, and write

$$\begin{aligned}\frac{1}{\lambda\varepsilon^3}U(-t)\mathbf{u}_2^\varepsilon(t)&=-\mathrm{i}\gamma(\log\varepsilon)\mathcal{F}^{-1}\left(|\mathcal{F}(v_-)|^2\mathcal{F}(v_-)\right)-\mathrm{i}\int_{-1/\varepsilon^\gamma}^{-1}U(-\tau)\left(|U(\tau)v_-|^2U(\tau)v_-\right)\mathrm{d}\tau\\&\quad-\mathrm{i}\int_{-1}^{t}U(-\tau)\left(|U(\tau)v_-|^2U(\tau)v_-\right)\mathrm{d}\tau\\&=-\mathrm{i}\gamma(\log\varepsilon)\mathcal{F}^{-1}\left(|\mathcal{F}(v_-)|^2\mathcal{F}(v_-)\right)-\mathrm{i}\int_{-1/\varepsilon^\gamma}^{-1}\left(-\frac{1}{|\tau|}\mathcal{F}^{-1}\left(|\mathcal{F}(v_-)|^2\mathcal{F}(v_-)\right)+R_4(\tau)\right)\mathrm{d}\tau\\&\quad-\mathrm{i}\int_{-1}^{t}U(-\tau)\left(|U(\tau)v_-|^2U(\tau)v_-\right)\mathrm{d}\tau\\&=-\mathrm{i}\int_{-\infty}^{-1}R_4(\tau)\mathrm{d}\tau+\underbrace{\mathcal{O}\left(\int_{-\infty}^{-1/\varepsilon^\gamma}\frac{\mathrm{d}\tau}{\tau^2}\right)}_{=\mathcal{O}(\varepsilon^\gamma)}-\mathrm{i}\int_{-1}^{t}U(-\tau)\left(|U(\tau)v_-|^2U(\tau)v_-\right)\mathrm{d}\tau.\end{aligned}$$

Therefore,

$$U(-t)v_2(t)=-\mathrm{i}\lambda\int_{-\infty}^{-1}R_4(\tau)\mathrm{d}\tau-\mathrm{i}\lambda\int_{-1}^{t}U(-\tau)\left(|U(\tau)v_-|^2U(\tau)v_-\right)\mathrm{d}\tau,$$

and, by construction,

$$R_4(\tau) = U(-\tau)\left(|U(\tau)v_-|^2 U(\tau)v_-\right) + \frac{1}{|\tau|}\mathscr{F}^{-1}\left(|\mathscr{F}(v_-)|^2\mathscr{F}(v_-)\right),$$

hence the formula of the lemma. In particular, by the same arguments as above, the second integral diverges logarithmically as $t \to +\infty$. We leave out the convergence in Σ (since $v_- \in \mathscr{H}$, so we can pay some momentum estimate when controlling $M-1$), which relies on similar ideas, like the proof of Lemma 5. Note that the present argument □

We can now improve the error estimate proven in the previous subsection with only the leading order approximation $\mathbf{u}_1^\varepsilon$. By construction,

$$\left\| U(-s)(u_2^\varepsilon(s) - \mathbf{u}_{\rm app}^\varepsilon(s))\Big|_{s=-1/\varepsilon^\gamma}\right\|_\Sigma = \mathscr{O}\left(\varepsilon^5(\log\varepsilon)^2\right),$$

and we have, if $0<\gamma<2$,

$$\left\| U(-t)\left(u^\varepsilon(t) - \mathbf{u}_{\rm app}^\varepsilon(t)\right)\right\|_\Sigma \lesssim \varepsilon^{3+\gamma}|\log\varepsilon| + \int_{-1/\varepsilon^\gamma}^t \left\| U(-\tau)\left(|u^\varepsilon|^2u^\varepsilon - |\mathbf{u}_1^\varepsilon|^2\mathbf{u}_1^\varepsilon\right)(\tau)\right\|_\Sigma d\tau.$$

In view of (3.3),

$$\begin{aligned}\|U(-\tau)\left(|u^\varepsilon|^2u^\varepsilon - |\mathbf{u}_1^\varepsilon|^2\mathbf{u}_1^\varepsilon\right)(\tau)\|_\Sigma &\lesssim \|u^\varepsilon(\tau)\|_{L^\infty}^2 \|U(-\tau)\left(u^\varepsilon - \mathbf{u}_1^\varepsilon\right)(\tau)\|_\Sigma \\ &\quad + \left(\|u^\varepsilon(\tau)\|_{L^\infty} + \|\mathbf{u}_1^\varepsilon(\tau)\|_{L^\infty}\right)\|U(-\tau)\mathbf{u}_1^\varepsilon(\tau)\|_\Sigma \|u^\varepsilon(\tau) - \mathbf{u}_1^\varepsilon(\tau)\|_{L^\infty}.\end{aligned}$$

Recalling (4.1), we readily have

$$\|u^\varepsilon(\tau)\|_{L^\infty} \lesssim \frac{\varepsilon}{\langle\tau\rangle^{1/2}}.$$

On the other hand,

$$\|U(-\tau)\mathbf{u}_1^\varepsilon(\tau)\|_\Sigma = \varepsilon\|v_-\|_\Sigma,$$

and Sobolev embedding yields, along with (2.2),

$$\|\mathbf{u}_1^\varepsilon(\tau)\|_{L^\infty} \lesssim \frac{\varepsilon}{\langle\tau\rangle^{1/2}},$$

as well as

$$\|u^\varepsilon(\tau) - \mathbf{u}_1^\varepsilon(\tau)\|_{L^\infty} \lesssim \frac{1}{\langle\tau\rangle^{1/2}}\|U(-\tau)\left(u^\varepsilon - v_1^\varepsilon\right)(\tau)\|_\Sigma.$$

Therefore,

$$\begin{aligned}\|U(-\tau)\left(|u^\varepsilon|^2u^\varepsilon - |\mathbf{u}_1^\varepsilon|^2\mathbf{u}_1^\varepsilon\right)(\tau)\|_\Sigma &\lesssim \frac{\varepsilon^2}{\langle\tau\rangle}\|U(-\tau)\left(u^\varepsilon - \mathbf{u}_1^\varepsilon\right)(\tau)\|_\Sigma \\ &\lesssim \frac{\varepsilon^2}{\langle\tau\rangle}\left(\left\|U(-\tau)(u^\varepsilon - \mathbf{u}_{\rm app}^\varepsilon)(\tau)\right\|_\Sigma + \|U(-\tau)\mathbf{u}_2^\varepsilon(\tau)\|_\Sigma\right).\end{aligned}$$

We see that there exists C such that for all $T \geqslant 1$,

$$\|U(-\tau)\mathbf{u}_2^\varepsilon(\tau)\|_\Sigma \lesssim \varepsilon^3\log\langle T\rangle, \quad \forall\tau\in\left[-\frac{1}{\varepsilon^\gamma}, T\right],$$

where the logarithmic correction is necessarily present for large positive τ, as pointed out above. Gronwall lemma now yields, for all $0<\gamma<2$,

$$\sup_{-1/\varepsilon^\gamma\leqslant t\leqslant T}\left\| U(-t)\left(u^\varepsilon(t) - \mathbf{u}_{\rm app}^\varepsilon(t)\right)\right\|_\Sigma \lesssim \varepsilon^{3+\gamma}|\log\varepsilon|\, e^{C\varepsilon^2|\log\varepsilon| + C\varepsilon^2\log\langle T\rangle},$$

for some constant C independent of ε and T. In particular, the right hand side is $o(\varepsilon^3)$ on $[-1/\varepsilon^\gamma, 1/\varepsilon^\beta]$ for any $\beta>0$.

The first point of Theorem 2 follows, by considering the above error estimate at time $t=0$ and setting $w_2 = v_{2|t=0}$.

5. Asymptotic completeness

5.1. *Main steps of the construction of u_+*

We recall the main steps from the proof of [15, Theorem 1.2], a result which we state in the particular case that we shall consider (the initial data may have a different regularity there, and an extra short range nonlinearity can be incorporated). In passing, we keep track of the size of the remainders more precisely.

Theorem 13 (From [15]). *Let $u_0 \in \Sigma$, with $\|u_0\|_\Sigma = \varepsilon' \leqslant \delta$, where $\delta > 0$ is sufficiently small. There exist unique functions $W \in L^\infty \cap L^2$ and $\Phi \in L^\infty$ such that, for $t \geqslant 1$ and $C\delta < \alpha < 1/4$,*

$$\left\|\mathscr{F}\left(U(-t)u(t)\right)\exp\left(\mathrm{i}\lambda\int_1^t |\widehat{u}(\tau)|^2\frac{\mathrm{d}\tau}{\tau}\right) - W\right\|_{L^2\cap L^\infty} \leqslant C(\varepsilon')^3 t^{-\alpha + C(\varepsilon')^2}, \tag{5.1}$$

and

$$\left\|\lambda\int_1^t |\widehat{u}(\tau)|^2\frac{\mathrm{d}\tau}{\tau} - \lambda|W|^2\log t - \Phi\right\|_{L^\infty} \leqslant C(\varepsilon')^3 t^{-\alpha + C(\varepsilon')^2}. \tag{5.2}$$

In particular,

$$u(t,x) = \frac{1}{(\mathrm{i}t)^{1/2}} W\left(\frac{x}{t}\right)\exp\left(\mathrm{i}\frac{x^2}{2t} - \mathrm{i}\lambda\left|W\left(\frac{x}{t}\right)\right|^2\log t - \mathrm{i}\Phi\left(\frac{x}{t}\right)\right) + \rho(t,x),$$

with

$$\|\rho(t)\|_{L^2} \leqslant C\varepsilon' t^{-\alpha + C(\varepsilon')^2}, \quad \|\rho(t)\|_{L^\infty} \leqslant C\varepsilon' t^{-1/2-\alpha + C(\varepsilon')^2}.$$

In [15], the time decay in (5.2) is raised to the power $2/3$, because the setting is more general and includes the case of dimension three. We explain below why this power can be discarded in the above statement. The asymptotic state u_+ is naturally given by

$$u_+ = \mathscr{F}^{-1}\left(W\mathrm{e}^{-\mathrm{i}\Phi}\right).$$

Sketch of the proof. As announced above, we recall the main steps from [15], and slightly improve some estimates in terms of the powers of ε'.

As evoked before in the case of (4.1), (the proof of) [15, Theorem 1.1] provides global estimates for u (proven in [15, Lemmas 3.2 and 3.3], see also [20]):

$$\|u(t)\|_{L^\infty} \lesssim \frac{\varepsilon'}{\langle t\rangle^{1/2}}, \quad \|U(-t)u(t)\|_\Sigma \lesssim \varepsilon' \langle t\rangle^{C(\varepsilon')^2}. \tag{5.3}$$

Denote

$$v(t) = U(-t)u(t), \quad \widehat{w} = \widehat{v}\underbrace{\exp\left(\mathrm{i}\lambda\int_1^t |\widehat{v}(\tau)|^2\frac{\mathrm{d}\tau}{\tau}\right)}_{=:B(t)}.$$

Then (1.1) is recasted as

$$\mathrm{i}\partial_t\widehat{w} = \frac{\lambda}{t}B(t)\left(I_1 + I_2\right), \tag{5.4}$$

where

$$I_1(t) = \mathscr{F}\left(M(-t)-1\right)\mathscr{F}^{-1}\left(|\widehat{M(t)v}|^2\widehat{M(t)v}\right), \quad I_2(t) = \left|\widehat{M(t)v}\right|^2\widehat{M(t)v} - |\widehat{v}|^2\widehat{v}.$$

These source terms are controlled as follows, for $t \geqslant 1$:

$$\|I_1(t)\|_{L^2\cap L^\infty} + \|I_2(t)\|_{L^2\cap L^\infty} \lesssim \frac{\|v\|_\Sigma^3}{t^\alpha},$$

provided that $\alpha < 1/2$. In view of (5.3) and the definition of v, this entails

$$\|I_1(t)\|_{L^2\cap L^\infty} + \|I_2(t)\|_{L^2\cap L^\infty} \lesssim (\varepsilon')^3 t^{-\alpha + 3C(\varepsilon')^2}.$$

Renaming $3C$ to C, and integrating (5.4), we readily find that there exists $W \in L^2 \cap L^\infty$ such that

$$\|\widehat{w}(t) - W\|_{L^2\cap L^\infty} \lesssim (\varepsilon')^3 t^{-\alpha + C(\varepsilon')^2}, \tag{5.5}$$

thus proving (5.1), since

$$|\widehat{v}(t,\xi)| = |\mathscr{F}(U(-t)u(t))(\xi)| = \left|\mathrm{e}^{-\mathrm{i}t\frac{\xi^2}{2}}\,\widehat{u}(t,\xi)\right| = |\widehat{u}(t,\xi)|.$$

The phase corrector Φ is obtained by introducing the function

$$\Psi(t) = \lambda \int_1^t \left(|\widehat{w}(\tau)|^2 - |\widehat{w}(t)|^2\right)\frac{\mathrm{d}\tau}{\tau}. \tag{5.6}$$

We have, for $t > s > 2$,

$$\Psi(t) - \Psi(s) = \lambda \int_s^t \left(|\widehat{w}(\tau)|^2 - |\widehat{w}(t)|^2\right)\frac{\mathrm{d}\tau}{\tau} + \lambda\left(|\widehat{w}(t)|^2 - |\widehat{w}(s)|^2\right)\log s.$$

In view of the above estimates, for $t_2 > t_1 > 2$,

$$\begin{aligned}\left||\widehat{w}(t_2)|^2 - |\widehat{w}(t_1)|^2\right| &\lesssim (|\widehat{w}(t_2)| + |\widehat{w}(t_1)|)\,|\widehat{w}(t_2) - \widehat{w}(t_1)| \\ &\lesssim (|\widehat{w}(t_2)| + |\widehat{w}(t_1)|)\,(\varepsilon')^3 t_1^{-\alpha + C(\varepsilon')^2}.\end{aligned}$$

On the other hand, we have

$$|\widehat{w}(t)| \lesssim \varepsilon',$$

from (5.5). Therefore,

$$\left||\widehat{w}(t_2)|^2 - |\widehat{w}(t_1)|^2\right| \lesssim (\varepsilon')^4 t_1^{-\alpha + C(\varepsilon')^2}, \quad t_2 > t_1 > 2, \tag{5.7}$$

and so

$$|\Psi(t) - \Psi(s)| \lesssim (\varepsilon')^4 s^{-\alpha + C(\varepsilon')^2} \log s.$$

In particular, there exists $\Phi \in L^\infty$ such that, for $t > 2$,

$$|\Phi - \Psi(t)| \lesssim (\varepsilon')^4 t^{-\alpha + C(\varepsilon')^2} \log t. \tag{5.8}$$

The estimate (5.2) then follows from the identity

$$\lambda \int_1^t |\widehat{w}(\tau)|^2 \frac{\mathrm{d}\tau}{\tau} = \lambda |W|^2 \log t + \Phi + \Psi(t) - \Phi + \lambda\left(|\widehat{w}(t)|^2 - |W|^2\right)\log t,$$

using (5.5) and (5.8), possibly modifying the constant C, and using the fact that the condition on α in Theorem 13 is open.

In view of (5.1) and (5.2), we find

$$\mathscr{F}(U(-t)u(t)) = W\,\mathrm{e}^{-\mathrm{i}\lambda|W|^2\log t - \mathrm{i}\Phi} + r(t),$$

where

$$\|r(t)\|_{L^2\cap L^\infty} \lesssim (\varepsilon')^3 t^{-\alpha + C(\varepsilon')^2}.$$

Writing

$$1 = U(t)U(-t) = M(t)D(t)\mathscr{F}M(t)U(-t) = M(t)D(t)\mathscr{F}U(-t) + \mathscr{R}(t)\mathscr{F}U(-t),$$

we infer

$$u(t,x) = M(t)D(t)\mathscr{F}U(-t)u(t) + \rho_1(t,x),$$

with, in view of Lemma 5,

$$\|\rho_1(t)\|_{L^2} \lesssim \frac{\varepsilon'}{\sqrt{t}}\|\mathscr{F}U(-t)u(t)\|_{L^2} \lesssim \frac{\varepsilon'}{\sqrt{t}},$$

and, since $\alpha < 1/4$, using the first point of Lemma 5 with $\theta = \alpha$ and $s = 1 - 2\alpha$,

$$\|\rho_1(t)\|_{L^\infty} \lesssim \frac{\varepsilon'}{t^{1/2+\alpha}}\|\mathscr{F}U(-t)u(t)\|_{H^1} \lesssim \varepsilon' t^{-1/2-\alpha+C(\varepsilon')^2}.$$

We conclude thanks to the estimates

$$\|M(t)D(t)r(t)\|_{L^2} = \|r(t)\|_{L^2} \lesssim (\varepsilon')^3 t^{-\alpha+C(\varepsilon')^2},$$
$$\|M(t)D(t)r(t)\|_{L^\infty} = \frac{1}{\sqrt{t}}\|r(t)\|_{L^\infty} \lesssim (\varepsilon')^3 t^{-1/2-\alpha+C(\varepsilon')^2},$$

since $u(t) = M(t)D(t)\big(W\,\mathrm{e}^{-\mathrm{i}\lambda|W|^2\log t - \mathrm{i}\Phi}\big) + M(t)D(t)r(t) + \rho_1(t)$. □

5.2. *Modified scattering operator: leading order behavior near the origin*

In view of Theorem 13, the asymptotic state that we consider is $u_+^\varepsilon = \mathscr{F}^{-1}\big(W^\varepsilon\,\mathrm{e}^{-\mathrm{i}\Phi^\varepsilon}\big)$, with $u_0^\varepsilon = u^\varepsilon_{|t=0}$. The main remark at this stage is that the reduction presented in the proof of Theorem 13 boils down the analysis of the asymptotic behavior of u_+^ε as $\varepsilon \to 0$ to a regular asymptotic expansion. In order to treat the last two cases of Theorem 2, we suppose that, for $0 < \eta < 2$,

$$u_0^\varepsilon = \varepsilon v_0 + \varepsilon^3 w_2 + \mathscr{O}\left(\varepsilon^{5-\eta}\right),$$

with $v_0, w_2 \in \Sigma$ and the remainder term is (measured) in Σ. For the last point of Theorem 2, we assume $w_2 = \rho^\varepsilon = 0$, while for the second point, $v_0 = v_-$ and $w_2 = v_{2|t=0}$ as in the first point.

We keep the same notations as the proof of Theorem 13, with now ε instead of ε'. The definition of $\widehat{w}$, in particular the term B, requires as a first step the asymptotic description of u^ε at time $t = 1$ instead of only $t = 0$.

In view of [15, Theorem 1.1], as in (5.3), we have

$$\|u^\varepsilon(t)\|_{L^\infty} \lesssim \frac{\varepsilon}{\langle t\rangle^{1/2}}, \quad \|U(-t)u^\varepsilon(t)\|_\Sigma \lesssim \varepsilon\,\langle t\rangle^{C\varepsilon^2}.$$

In view of the proof of (5.1) and (5.2), we directly know that

$$\|\partial_t \widehat{w}\|_{L^1([1,\infty[,L^2\cap L^\infty)} \lesssim \varepsilon^3,$$

and so

$$\widehat{w}(t) = \widehat{w}(1) + \mathscr{O}(\varepsilon^3) \text{ in } L^\infty([1,\infty[, L^2 \cap L^\infty).$$

By definition, $\widehat{w}(1) = \widehat{v}(1) = \mathscr{F}\left(U(-1)u^\varepsilon(1)\right) = \varepsilon\widehat{v}_0 + \mathscr{O}(\varepsilon^3)$ in Σ.

Regarding Φ, the definition (5.6), and (5.7), yield

$$\|\Phi\|_{L^\infty} \leqslant \|\Phi - \Psi(t)\|_{L^\infty} + \|\Psi(t)\|_{L^\infty} \lesssim \varepsilon^4,$$

where the right hand side is estimated uniformly in $t \geqslant 1$. Therefore,

$$u_+^\varepsilon = \mathscr{F}^{-1}\left(W\,\mathrm{e}^{-\mathrm{i}\Phi}\right) = \varepsilon v_0 + \mathscr{O}(\varepsilon^3) \text{ in } L^2.$$

Therefore, at leading order, we have

$$S^{\mathrm{mod}}(\varepsilon v_-) = \varepsilon v_- + \mathscr{O}(\varepsilon^3), \quad \left(W_+^{\mathrm{mod}}\right)^{-1}(\varepsilon v_0) = \varepsilon v_0 + \mathscr{O}(\varepsilon^3) \quad \text{in } L^2.$$

6. Higher order asymptotic expansion of the final state

The higher order asymptotic expansion for u_+^ε, involving an ε^3 term, requires more work, even at the formal level. We first show how to derive this term, then, in a final subsection, prove the error estimates announced in Theorem 2.

6.1. *Derivation of the first corrector*

We first examine the value of u^ε at time $t=1$, in view of future connections with Theorem 13. The idea is the same as in Section 4.2: for finite time, since we consider small data, the asymptotic expansion is given by Picard's scheme, and we have directly from Section 4,

$$u^\varepsilon_{|t=1} = \varepsilon U(1)v_0 + \varepsilon^3 \widetilde{w}_2 + \varepsilon^{5-\eta}\widehat{\rho}^\varepsilon,$$

with

$$\widetilde{w}_2 = U(1)w_2 - i\lambda\int_0^1 U(1-s)\left(|U(s)v_0|^2 U(s)v_0\right)ds,$$

and $\|\rho^\varepsilon\|_\Sigma \lesssim 1$ as $\varepsilon\to 0$.

To derive the first corrector, we plug the asymptotic expansion

$$\widehat{w}(t) = \varepsilon\widehat{v}_0 + \varepsilon^3\widehat{v}_2(t) + \varepsilon^{5-\eta}\widehat{r^\varepsilon}(t)$$

into (5.4), with $r^\varepsilon = \mathcal{O}(1)$ in some topology we shall precise. We first proceed formally, and then check that the above ansatz indeed provides a corrector of u^ε_+ in L^2. We compute successively

$$|\widehat{v}|^2 = |\widehat{w}|^2 = \varepsilon^2|\widehat{v}_0|^2 + \mathcal{O}(\varepsilon^4),$$

hence, for bounded t,

$$B(t) = e^{i\lambda\int_1^t |\widehat{v}(\tau)|^2\frac{d\tau}{\tau}} = 1 + i\lambda\varepsilon^2|\widehat{v}_0|^2\log t + \mathcal{O}(\varepsilon^4),$$

and

$$\widehat{v} = \widehat{w}\overline{B} = \varepsilon\widehat{v}_0 + \varepsilon^3\left(\widehat{v}_2 - i\lambda|\widehat{v}_0|^2\widehat{v}_0\log t\right) + \mathcal{O}(\varepsilon^{5-\eta}).$$

We then expand the factor I_1^ε and I_2^ε from (5.4):

$$I_1^\varepsilon(t) = \varepsilon^3\mathcal{F}(M(-t)-1)\mathcal{F}^{-1}\left(\left|\widehat{M(t)v_0}\right|^2\widehat{M(t)v_0}\right) + \mathcal{O}(\varepsilon^5),$$

$$I_2^\varepsilon(t) = \varepsilon^3\left(\left|\widehat{M(t)v_0}\right|^2\widehat{M(t)v_0} - |\widehat{v}_0|^2\widehat{v}_0\right) + \mathcal{O}(\varepsilon^5).$$

Therefore, the natural candidate for v_2 satisfies:

$$\partial_t\widehat{v}_2 = \frac{\lambda}{t}(J_1+J_2),$$

where

$$J_1 = \mathcal{F}(M(-t)-1)\mathcal{F}^{-1}\left(\left|\widehat{M(t)v_0}\right|^2\widehat{M(t)v_0}\right),$$

$$J_2 = \left|\widehat{M(t)v_0}\right|^2\widehat{M(t)v_0} - |\widehat{v}_0|^2\widehat{v}_0.$$

The Cauchy datum for v_2 is given at $t=1$: on the one hand, the above asymptotic expansion implies

$$\widehat{v}_{|t=1} = \varepsilon\widehat{v}_0 + \varepsilon^3\widehat{v}_{2|t=1} + \mathcal{O}(\varepsilon^{5-\eta}),$$

while on the other hand, the analysis of Section 4 yields

$$U(-t)u^\varepsilon(t)_{|t=1} = \varepsilon v_0 + \varepsilon^3 U(-1)\widetilde{w}_2 + \mathcal{O}(\varepsilon^{5-\eta}),$$

so we infer

$$v_{2|t=1} = U(-1)\widetilde{w}_2 = w_2 - i\lambda\int_0^1 U(-s)\left(|U(s)v_0|^2U(s)v_0\right)ds.$$

As $v_0\in\Sigma$, the proof of Theorem 13 readily yields $\frac{1}{t}(J_1+J_2)\in L^1([1,\infty[;L^2\cap L^\infty)$. Since Σ is a Banach algebra, invariant under the Fourier transform, and such that $\Sigma\hookrightarrow L^1\cap L^\infty$, we have $v_{2|t=1}\in\Sigma$, and so $v_2\in L^\infty([1,\infty[;L^2\cap L^\infty)$ and

$$\widehat{v}_2(t)\underset{t\to\infty}{\longrightarrow}\widehat{v}_{2|t=1} + \lambda\int_1^\infty\left(J_1(\tau)+J_2(\tau)\right)\frac{d\tau}{\tau}\quad\text{in } L^2\cap L^\infty.$$

However, in the asymptotic expansion of B, I_1 and I_2, the more precise information that we need is that the remainder term is $\mathcal{O}(\varepsilon^5)$ together with some algebraic decay in time, so we obtain an asymptotic expansion for $\widehat{w}$ in L^2. This requires more effort, and the proof is presented in the next subsection.

6.2. *Justification of the second order asymptotic expansion*

As a first step, we analyze the regularity of the corrector v_2. To do so, we introduce an extra function space: for $0<\gamma<1$, set

$$\Sigma^\gamma = H^\gamma \cap \mathscr{F}(H^\gamma) = \{f\in H^\gamma(\mathbb{R}),\ \|f\|_{\Sigma^\gamma} := \|f\|_{H^\gamma} + \|\langle x\rangle^\gamma f\|_{L^2}<\infty\}.$$

We note that like Σ, Σ^γ is invariant under the Fourier transform, as well as the action of $U(t)$, and, when $\gamma>1/2$, it is an algebra embedded into $L^1\cap L^\infty$.

Lemma 14. *Let $v_0, w_2\in\Sigma$. Recall that the first corrector is defined by*

$$\widehat{v}_2(t) = \widehat{w}_2 - i\lambda\int_0^1 \mathscr{F}U(-s)\left(|U(s)v_0|^2U(s)v_0\right)ds + \lambda\int_1^t (J_1(\tau)+J_2(\tau))\frac{d\tau}{\tau},$$

where J_1 and J_2 are defined by

$$J_1(t) = \mathscr{F}\left(M(-t)-1\right)\mathscr{F}^{-1}\left(\left|\widehat{M(t)v_0}\right|^2\widehat{M(t)v_0}\right),$$

$$J_2(t) = \left|\widehat{M(t)v_0}\right|^2\widehat{M(t)v_0} - |\widehat{v}_0|^2\widehat{v}_0.$$

Then $J_1, J_2\in L^\infty([1,\infty[,\Sigma)$, with

$$\sup_{t\geqslant 1}\|J_1(t)\|_\Sigma + \sup_{t\geqslant 1}\|J_2(t)\|_\Sigma \lesssim \|v_0\|_\Sigma^3,$$

and, for any $1/2<\gamma<1$,

$$\|J_1(t)\|_{\Sigma^\gamma} + \|J_2(t)\|_{\Sigma^\gamma} \lesssim \frac{\|v_0\|_\Sigma^3}{t^{(1-\gamma)/2}},$$

and therefore $v_2\in L^\infty_{\rm loc}([1,\infty[,\Sigma)\cap L^\infty([1,\infty[,\Sigma^\gamma)$. Finally, $\widehat{v}_2(t)\to\widehat{v}_2^\infty$ in Σ^γ as $t\to\infty$, where

$$\widehat{v}_2^\infty = \widehat{w}_2 - i\lambda\int_0^1 \mathscr{F}U(-s)\left(|U(s)v_0|^2U(s)v_0\right)ds + \lambda\int_1^\infty \left(J_1(\tau)+J_2(\tau)\right)\frac{d\tau}{\tau}.$$

Proof of Lemma 14. Since $w_2, v_0\in\Sigma$, the first two terms defining $\widehat{v}_2$ belong to Σ (as it is an algebra). For any (fixed) t, $J_1, J_2\in\Sigma$, and $v_2(t)\in\Sigma$: the uniform bound in time is straightforward, but to get some time decay, we pay a little regularity. Let $0<\gamma<1$: like in the proof of Lemma 5, write, for $t\geqslant 1$,

$$\begin{aligned}\|J_1(t)\|_{\Sigma^\gamma} &= \left\|\left(M(-t)-1\right)\mathscr{F}^{-1}\left(\left|\widehat{M(t)v_0}\right|^2\widehat{M(t)v_0}\right)\right\|_{\Sigma^\gamma}\\ &\lesssim \frac{1}{t^{(1-\gamma)/2}}\left\||x|^{1-\gamma}\mathscr{F}^{-1}\left(\left|\widehat{M(t)v_0}\right|^2\widehat{M(t)v_0}\right)\right\|_{\Sigma^\gamma}\\ &\lesssim \frac{1}{t^{(1-\gamma)/2}}\left\|\mathscr{F}^{-1}\left(\left|\widehat{M(t)v_0}\right|^2\widehat{M(t)v_0}\right)\right\|_{\Sigma} \lesssim \frac{\|v_0\|_\Sigma^3}{t^{(1-\gamma)/2}},\end{aligned}$$

since Σ is an algebra. The assumption $\gamma>1/2$ simplifies the computations in the case of J_2, as Σ^γ is an algebra, and the first inequality below is straightforward:

$$\begin{aligned}\|J_2(t)\|_{\Sigma^\gamma} &\lesssim \left(\|M(t)v_0\|_{\Sigma^\gamma}^2 + \|v_0\|_{\Sigma^\gamma}^2\right)\|(M(t)-1)v_0\|_{\Sigma^\gamma}\\ &\lesssim \|v_0\|_\Sigma^2\frac{1}{t^{(1-\gamma)/2}}\||x|^{1-\gamma}v_0\|_{\Sigma^\gamma} \lesssim \frac{\|v_0\|_\Sigma^3}{t^{(1-\gamma)/2}}.\end{aligned}$$

The lemma follows, since the extra decay in time, $t^{(\gamma-1)/2}$, ensures the convergence of the integral in the last term defining $\widehat{v}_2$. □

We recall two uniform estimates which will be of constant use in the course of the proof:

$$\|\widehat{v}(t)\|_{L^\infty}+\left\|\widehat{M(t)v}\right\|_{L^\infty}\lesssim\varepsilon. \tag{6.1}$$

The first quantity is estimated in the proof of Theorem 13, thanks to (5.5). The second one is controlled thanks to (5.3), since, using (2.5),

$$\left\|\widehat{M(t)v}\right\|_{L^\infty}=\|\mathscr{F}M(t)U(-t)u(t)\|_{L^\infty}=\left\|D\left(\frac{1}{t}\right)M(-t)u(t)\right\|_{L^\infty}=\sqrt{t}\,\|u(t)\|_{L^\infty}.$$

The justification of the formal asymptotic expansion derived in the previous subsection relies on a Gronwall type argument. The error that we want to control is $\widehat{w}-\varepsilon\widehat{v}_0-\varepsilon^3\widehat{v}_2$, which solves, by construction (keeping in mind that v_0 does not depend on time),

$$\partial_t\left(\widehat{w}-\varepsilon\widehat{v}_0-\varepsilon^3\widehat{v}_2\right)=\frac{\lambda}{t}B(I_1^\varepsilon+I_2^\varepsilon)-\frac{\lambda}{t}\varepsilon^3(J_1+J_2).$$

We rewrite the right hand side as

$$\frac{\lambda}{t}B\left(I_1^\varepsilon+I_2^\varepsilon-\varepsilon^3J_1-\varepsilon^3J_2\right)+\frac{\lambda}{t}(B-1)\varepsilon^3(J_1+J_2),$$

and the last term will be considered as a source term. Indeed, from the definition

$$B(t)=e^{i\lambda\int_1^t|\widehat{v}(\tau)|^2\frac{d\tau}{\tau}}=e^{i\lambda\int_1^t|\widehat{w}(\tau)|^2\frac{d\tau}{\tau}},$$

so from (6.1),

$$\|B(t)-1\|_{L^\infty}\lesssim\varepsilon^2\log t, \tag{6.2}$$

and Lemma 14 yields

$$\left\|\varepsilon^3(B-1)(J_1+J_2)\right\|_{L^2}\lesssim\varepsilon^5\frac{\log t}{t^{(1-\gamma)/2}}, \tag{6.3}$$

for any $1/2<\gamma<1$.

We now focus on the term $B(I_1^\varepsilon+I_2^\varepsilon-\varepsilon^3J_1-\varepsilon^3J_2)$, to estimate it in L^2, and examine successively $B(I_1^\varepsilon-\varepsilon^3J_1)$ and $B(I_2^\varepsilon-\varepsilon^3J_2)$. First, by definition,

$$I_1^\varepsilon-\varepsilon^3J_1=\mathscr{F}\left(M(-t)-1\right)\mathscr{F}^{-1}\left(\left|\widehat{M(t)v}\right|^2\widehat{M(t)v}-\varepsilon^3\left|\widehat{M(t)v_0}\right|^2\widehat{M(t)v_0}\right),$$

so the term $v-\varepsilon v_0$ is naturally factored out, and we must relate it to the left hand side, involving $w-\varepsilon v_0-\varepsilon^3v_2$. Since $B\overline{B}=1$, we can write

$$\widehat{\rho}:=\widehat{w}-\varepsilon\widehat{v}_0-\varepsilon^3\widehat{v}_2=B\widehat{v}-\varepsilon\widehat{v}_0-\varepsilon^3\widehat{v}_2=B\left(\widehat{v}-\varepsilon\overline{B}\widehat{v}_0-\varepsilon^3\overline{B}\widehat{v}_2\right),$$

so we have

$$\widehat{v}-\varepsilon\widehat{v}_0=\overline{B}\left(\widehat{w}-\varepsilon\widehat{v}_0-\varepsilon^3\widehat{v}_2\right)+\varepsilon(\overline{B}-1)\widehat{v}_0+\varepsilon^3\overline{B}\widehat{v}_2. \tag{6.4}$$

The first term on the right hand side will be treated differently from the last two terms, which will be considered as source terms. Using the formula

$$|z_2|^2z_2-|z_1|^2z_1=|z_2|^2(z_2-z_1)+z_1\operatorname{Re}(z_2-z_1)(\overline{z}_2+\overline{z}_1),$$

with $z_2=\mathscr{F}(Mv)$ and $z_1=\varepsilon\mathscr{F}(Mv_0)$, as well as (6.4), yielding

$$\mathscr{F}\left(M(v-\varepsilon v_0)\right)=\mathscr{F}M\mathscr{F}^{-1}\left(\overline{B}\left(\widehat{w}-\varepsilon\widehat{v}_0-\varepsilon^3\widehat{v}_2\right)+\varepsilon(\overline{B}-1)\widehat{v}_0+\varepsilon^3\overline{B}\widehat{v}_2\right), \tag{6.5}$$

we write

$$\|I_1^\varepsilon-\varepsilon^3J_1\|_{L^2}\leqslant\|G_1\|_{L^2}+\|S_1\|_{L^2}+\|S_2\|_{L^2},$$

where, simplifying the algebraic structure for the sake of presentation,

$$G_1=\mathscr{F}\left(M^{-1}-1\right)\mathscr{F}^{-1}\left(\left(|\mathscr{F}(Mv)|^2+\varepsilon^2|\mathscr{F}(Mv_0)|^2\right)\mathscr{F}M\mathscr{F}^{-1}\left(\overline{B}\widehat{\rho}\right)\right),$$

$$S_1=\varepsilon\mathscr{F}\left(M^{-1}-1\right)\mathscr{F}^{-1}\left(\left(|\mathscr{F}(Mv)|^2+\varepsilon^2|\mathscr{F}(Mv_0)|^2\right)\mathscr{F}M\mathscr{F}^{-1}\left((\overline{B}-1)\widehat{v}_0\right)\right),$$

$$S_2=\varepsilon^3\mathscr{F}\left(M^{-1}-1\right)\mathscr{F}^{-1}\left(\left(|\mathscr{F}(Mv)|^2+\varepsilon^2|\mathscr{F}(Mv_0)|^2\right)\mathscr{F}M\mathscr{F}^{-1}\left(\overline{B}\widehat{v}_2\right)\right).$$

We have left out the time variable in the expression of M in order to lighten the notation, and do so below for the same reason. The Gronwall term G_1 is estimated by merely using the boundedness on L^2 of $\mathscr{F}\left(M^{-1}-1\right)\mathscr{F}^{-1}$, and the unitarity of $\mathscr{F}M\mathscr{F}^{-1}$, and we write

$$\|G_1\|_{L^2} \lesssim \left(\|\mathscr{F}(Mv)\|_{L^\infty}^2 + \varepsilon^2\|\mathscr{F}(Mv_0)\|_{L^\infty}^2\right)\|\widehat{\rho}\|_{L^2} \lesssim \varepsilon^2\|\widehat{\rho}\|_{L^2},$$

by using (6.1). For the source term S_1 and S_2, we invoke the same arguments as in the proof of Lemma 5, and the fact that $\mathscr{F}M\mathscr{F}^{-1}$ is unitary on H^s for any $s \geqslant 0$:

$$\begin{aligned}
\|S_1\|_{L^2} &\lesssim \frac{\varepsilon}{\sqrt{t}}\left\|\left(|\mathscr{F}(Mv)|^2 + \varepsilon^2|\mathscr{F}(Mv_0)|^2\right)\mathscr{F}M\mathscr{F}^{-1}\left((\overline{B}-1)\widehat{v}_0\right)\right\|_{H^1}\\
&\lesssim \frac{\varepsilon}{\sqrt{t}}\left(\|v\|_{\mathscr{F}(H^1)}^2 + \varepsilon^2\|v_0\|_{\mathscr{F}(H^1)}^2\right)\|(\overline{B}-1)\widehat{v}_0\|_{H^1}\\
&\lesssim \frac{\varepsilon^3}{\sqrt{t}} t^{2C\varepsilon^2}\|(\overline{B}-1)\widehat{v}_0\|_{H^1},
\end{aligned}$$

where we have used (5.3) for the last inequality. The last term is controlled by

$$\|(\overline{B}-1)\widehat{v}_0\|_{H^1} \leqslant \|\overline{B}-1\|_{L^\infty}\|v_0\|_\Sigma + \|\widehat{v}_0\partial_x\overline{B}\|_{L^2} \lesssim \varepsilon^2\log t + \|\widehat{v}_0\partial_x\overline{B}\|_{L^2},$$

in view of (6.2). Writing

$$\widehat{v}_0\partial_x\overline{B} = -2i\lambda\widehat{v}_0\int_1^t \operatorname{Re}\left(\overline{\widehat{v}}\partial_x\widehat{v}\right)\frac{d\tau}{\tau},$$

(5.3) now yields

$$\|\widehat{v}_0\partial_x\overline{B}\|_{L^2} \lesssim \|v_0\|_{L^2}\int_1^t \|\widehat{v}(\tau)\|_{L^\infty}\|\partial_x\widehat{v}(\tau)\|_{L^2}\frac{d\tau}{\tau} \lesssim \varepsilon^2\int_1^t \frac{d\tau}{\tau^{1-C\varepsilon^2}} \lesssim \varepsilon^2 t^{C\varepsilon^2}.$$

The estimate for S_2 is rather similar:

$$\begin{aligned}
\|S_2\|_{L^2} &\lesssim \frac{\varepsilon^3}{t^{1/2}}\left\|\left(|\mathscr{F}(Mv)|^2 + \varepsilon^2|\mathscr{F}(Mv_0)|^2\right)\mathscr{F}M\mathscr{F}^{-1}\left(\overline{B}\widehat{v}_2\right)\right\|_{H^1}\\
&\lesssim \frac{\varepsilon^3}{t^{1/2}}\left(\|\mathscr{F}(Mv)\|_{H^1}^2 + \varepsilon^2\|\mathscr{F}(Mv_0)\|_{H^1}^2\right)\|\overline{B}\widehat{v}_2\|_{H^1} \lesssim \frac{\varepsilon^5}{t^{1/2-2C\varepsilon^2}}\|\overline{B}\widehat{v}_2\|_{H^1}.
\end{aligned}$$

The last term is controlled via Leibniz formula, invoking Lemma 14,

$$\|\overline{B}\widehat{v}_2\|_{H^1} \leqslant \|\widehat{v}_2\|_{H^1} + \|\widehat{v}_2\partial_x\overline{B}\|_{L^2} \lesssim \log t + \varepsilon^2 t^{C\varepsilon^2},$$

and the source terms are estimated by

$$\|S_1\|_{L^2} + \|S_2\|_{L^2} \lesssim \varepsilon^5\frac{\log t}{t^{1/2-3C\varepsilon^2}}.$$

We now turn to the L^2 estimate of $I_2^\varepsilon - \varepsilon^3 J_2$, and proceed along the same spirit.

$$I_2^\varepsilon - \varepsilon^3 J_2 = \left|\widehat{M(t)v}\right|^2\widehat{M(t)v} - |\widehat{v}|^2\widehat{v} - \varepsilon^3\left|\widehat{M(t)v_0}\right|^2\widehat{M(t)v_0} + \varepsilon^3|\widehat{v}_0|^2\widehat{v}_0.$$

We distinguish $\left|\widehat{M(t)v}\right|^2\widehat{M(t)v} - \varepsilon^3\left|\widehat{M(t)v_0}\right|^2\widehat{M(t)v_0}$ and $|\widehat{v}|^2\widehat{v} - \varepsilon^3|\widehat{v}_0|^2\widehat{v}_0$. Discarding the precise algebraic structure like before,

$$\left|\widehat{M(t)v}\right|^2\widehat{M(t)v} - \varepsilon^3\left|\widehat{M(t)v_0}\right|^2\widehat{M(t)v_0} \approx \left(\left|\widehat{M(t)v}\right|^2 + \varepsilon^2\left|\widehat{M(t)v_0}\right|^2\right)\mathscr{F}M(v-\varepsilon v_0).$$

The last factor is again rewritten thanks to (6.5), and we estimate $I_2^\varepsilon - \varepsilon^3 J_2$ as

$$\|I_2^\varepsilon - \varepsilon^3 J_2\|_{L^2} \leqslant \|G_2\|_{L^2} + \|S_3\|_{L^2} + \|S_4\|_{L^2},$$

where

$$G_2=\left(\left|\widehat{M(t)v}\right|^2+\varepsilon^2\left|\widehat{M(t)v_0}\right|^2\right)\mathscr{F}M\mathscr{F}^{-1}\left(\overline{B}\left(\widehat{w}-\varepsilon\widehat{v}_0-\varepsilon^3\widehat{\nu}_2\right)\right)-\left(|\widehat{v}|^2+\varepsilon^2|\widehat{v_0}|^2\right)\overline{B}\left(\widehat{w}-\varepsilon\widehat{v}_0-\varepsilon^3\widehat{\nu}_2\right),$$

$$S_3=\varepsilon\left(\left|\widehat{M(t)v}\right|^2+\varepsilon^2\left|\widehat{M(t)v_0}\right|^2\right)\mathscr{F}M\mathscr{F}^{-1}\left(\left(\overline{B}-1\right)\widehat{v}_0\right)-\varepsilon\left(|\widehat{v}|^2+\varepsilon^2|\widehat{v_0}|^2\right)\left(\overline{B}-1\right)\widehat{v}_0,$$

$$S_4=\varepsilon^3\left(\left|\widehat{M(t)v}\right|^2+\varepsilon^2\left|\widehat{M(t)v_0}\right|^2\right)\mathscr{F}M\mathscr{F}^{-1}\widehat{\nu}_2-\varepsilon^3\left(|\widehat{v}|^2+\varepsilon^2|\widehat{v_0}|^2\right)\widehat{\nu}_2.$$

For the Gronwall term G_2, we proceed like for G_1, and write

$$\|G_2\|_{L^2}\lesssim\varepsilon^2\|\widehat{w}-\varepsilon\widehat{v}_0-\varepsilon^3\widehat{\nu}_2\|_{L^2}=\varepsilon^2\|\widehat{\rho}\|_{L^2}.$$

The source terms S_3 and S_4 are readily of size $\mathscr{O}(\varepsilon^5)$; we recover some decay in time by making the quantity $M-1$ appear systematically. In the case of S_3, we write

$$S_3=\varepsilon\left(\left|\widehat{M(t)v}\right|^2+\varepsilon^2\left|\widehat{M(t)v_0}\right|^2\right)\mathscr{F}M\mathscr{F}^{-1}\left(\left(\overline{B}-1\right)\widehat{v}_0\right)$$
$$-\varepsilon\left(|\widehat{v}|^2+\varepsilon^2|\widehat{v_0}|^2\right)\left(\overline{B}-1\right)\widehat{v}_0\pm\varepsilon\left(\left|\widehat{M(t)v}\right|^2+\varepsilon^2\left|\widehat{M(t)v_0}\right|^2\right)\left(\overline{B}-1\right)\widehat{v}_0,$$

and estimate as follows:

$$\|S_3\|_{L^2}\lesssim\varepsilon^3\|(M-1)\mathscr{F}^{-1}\left(\left(\overline{B}-1\right)\widehat{v}_0\right)\|_{L^2}+\|\left(\overline{B}-1\right)\widehat{v}_0\|_{L^\infty}\|(M-1)\widehat{v}\|_{L^2}\left(\|\widehat{Mv}\|_{L^\infty}+\|\widehat{v}\|_{L^\infty}\right)$$
$$\lesssim\frac{\varepsilon^3}{\sqrt{t}}\|x\mathscr{F}^{-1}\left(\left(\overline{B}-1\right)\widehat{v}_0\right)\|_{L^2}+\varepsilon^5\frac{\log t}{\sqrt{t}}\times\|x\widehat{v}\|_{L^2},$$

where we have used (6.1) and (6.2). We have already estimated the H^1-norm of $\left(\overline{B}-1\right)\widehat{v}_0$,

$$\|\left(\overline{B}-1\right)\widehat{v}_0\|_{H^1}\lesssim\varepsilon^2\left(\log t+t^{C\varepsilon^2}\right),$$

and therefore

$$\|S_3\|_{L^2}\lesssim\varepsilon^5\frac{\log t}{t^{1/2-C\varepsilon^2}}.$$

The term S_4 is controlled similarly, by using the same ideas as above, and we come up with:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\widehat{\rho}\|_{L^2}\lesssim\frac{\varepsilon^2}{t}\|\widehat{\rho}\|_{L^2}+\varepsilon^5\frac{\log t}{t^{3/2-3C\varepsilon^2}}.$$

Gronwall lemma then implies, provided that $\varepsilon>0$ is sufficiently small:

Proposition 15. *Suppose that u^ε solves (1.1), with $u^\varepsilon_{|t=0}=u_0^\varepsilon\in\Sigma$ such that*

$$u_0^\varepsilon=\varepsilon v_0+\varepsilon^3w_2+\mathscr{O}\left(\varepsilon^{5-\eta}\right),$$

for $v_0,w_2\in\Sigma$, and some $0<\eta<2$. Then we have

$$\sup_{t\geqslant1}\|\widehat{w}(t)-\varepsilon\widehat{v}_0-\varepsilon^3\widehat{\nu}_2(t)\|_{L^2}\lesssim\varepsilon^{5-\eta},$$

where ν_2 is defined in Lemma 14.

In particular, letting t go to infinity, we infer

$$W=\varepsilon\widehat{v}_0+\varepsilon^3\widehat{\nu}_2^\infty+\mathscr{O}\left(\varepsilon^{5-\eta}\right)\quad\text{in }L^2.$$

As $u_+^\varepsilon=\mathscr{F}^{-1}\left(We^{-i\Phi}\right)$, and we have seen at the end of Section 5 that $\|\Phi\|_{L^\infty}=\mathscr{O}(\varepsilon^4)$, we infer

$$u_+^\varepsilon=\varepsilon v_0+\varepsilon^3\nu_2+\mathscr{O}\left(\varepsilon^{5-\eta}\right)\quad\text{in }L^2,$$

thus completing the proof of Theorem 2.

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The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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