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On the small scale nonlinear theory of operator spaces

Sur la théorie non linéaire à petite échelle dans les espaces d'opérateurs

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Abstract. We initiate the study of the small scale geometry of operator spaces. The authors have previously shown that a map between operator spaces which is completely coarse (that is, the sequence of its amplifications is equi-coarse) must be \mathbb{R} -linear. We obtain a generalization of the aforementioned result to completely coarse maps defined on the unit ball of an operator space. By relaxing the condition to a small scale one, we prove that there are many non-linear examples of maps which are completely Lipschitz in small scale. We define a geometric parameter for homogeneous Hilbertian operator spaces which imposes restrictions on the existence of such maps.

Résumé. Nous commençons l'étude de la géométrie à petite échelle des espaces d'opérateurs. Les auteurs ont précédemment montré qu'une application entre espaces d'opérateurs qui est complètement grossière (c'est-à-dire que la séquence de ses amplifications est équi-grossière) doit être \mathbb{R} -linéaire. Nous obtenons une généralisation du résultat susmentionné aux applications complètement grossières définies sur la boule unité d'un espace d'opérateurs. En assouplissant la condition à une petite échelle, nous prouvons qu'il existe de nombreux exemples non linéaires d'applications qui sont complètement Lipschitz à petite échelle. Nous définissons un paramètre géométrique pour les espaces d'opérateurs hilbertiens homogènes qui impose des restrictions sur l'existence de telles applications.

Keywords. Operator spaces, Coarse geometry, Embeddings.

Mots-clés. Espaces d'opérateurs, Géométrie grossière, Injections.

2020 Mathematics Subject Classification. 47L25, 46L07, 46B80.

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1. Introduction

Although the interest of functional analysts in the nonlinear theory of Banach spaces has increased significantly in the past few decades (e.g. [1, 15, 18, 23]), researchers have only recently started to develop the nonlinear theory of their noncommutative counterpart, i.e., of operator spaces (see [5–8]). As of now, this study was restricted to constructing a large scale geometry for such spaces. The goal of the current article is to initiate the treatment of the small scale geometry of operator spaces.

Before describing our main results, we start this introduction with a paragraph recalling the basics for the non-expert: an *operator space* is a Banach subspace of the space of bounded operators on a given Hilbert space H , which we denote by $\mathcal{B}(H)$. Given $n \in \mathbb{N}$ and a set X , we denote the space of n -by- n matrices with entries in X by $M_n(X)$ — if X is either a vector space or an algebra, $M_n(X)$ inherits a canonical vector space or algebra structure, respectively. Since $M_n(\mathcal{B}(H))$ is canonically isomorphic to $\mathcal{B}(H^{\oplus n})$ (where $H^{\oplus n}$ denotes the Hilbert sum of n copies of H), each $M_n(\mathcal{B}(H))$ is endowed with the canonical norm given by this isomorphism. Given an operator space $X \subseteq \mathcal{B}(H)$, the inclusions $M_n(X) \subseteq M_n(\mathcal{B}(H))$ then induce norms on each $M_n(X)$. The n -*amplification* of a map $f: X \rightarrow Y$ between operator spaces is the map $f_n: M_n(X) \rightarrow M_n(Y)$ given by

$$f_n([x_{ij}]) = [f(x_{ij})] \text{ for all } [x_{ij}] \in M_n(X).$$

If f is linear, so is each f_n and $\|f_n\|_n$ denotes its operator norm. The *completely bounded norm* of f , abbreviated as the *cb-norm* of f , is given by

$$\|f\|_{\text{cb}} = \sup_{n \in \mathbb{N}} \|f_n\|$$

and f is called *completely bounded* if $\|f\|_{\text{cb}} < \infty$. Completely bounded maps play the role bounded maps play in Banach space theory and are used to define *complete isomorphisms* between operator spaces.

Our approach to study the small scale geometry of operator spaces comes from a strengthening of the main result of [6]; which, as the reader will see below, is something in between the large and the small scale geometry of operator spaces. We start recalling the concept of coarse maps in the category of operator spaces:

Definition 1. *Let X and Y be operator spaces and $B \subseteq X$. A map $f: B \rightarrow Y$ is called completely coarse if for all $r > 0$ there is $s > 0$ such that*

$$\|[x_{ij}] - [y_{ij}]\|_{M_n(X)} \leq r \text{ implies } \|f_n([x_{ij}]) - f_n([y_{ij}])\|_{M_n(Y)} \leq s$$

for all $n \in \mathbb{N}$ and all $[x_{ij}], [y_{ij}] \in M_n(B)$.¹

The main result of [6] showed that, despite its nonlinear definition, completely coarse maps are essentially already linear. Precisely, the following version of the Mazur–Ulam theorem holds for completely coarse maps between operator spaces:

Theorem 2 ([6, Theorem 1.1]). *Let X and Y be operator spaces. Any completely coarse map $f: X \rightarrow Y$ with $f(0) = 0$ must be \mathbb{R} -linear.*

In this paper, we take the techniques developed in [6] further and show that a much stronger result remains valid. Throughout these notes, if X is a Banach space, B_X denotes its closed unit ball.

Theorem 3. *Let X and Y be operator spaces. Any completely coarse map $f: B_X \rightarrow Y$ with $f(0) = 0$ must be the restriction of an \mathbb{R} -linear map.*

¹In classic coarse geometry, a map is called *coarse* if this holds for $n = 1$.

The previous result exhausts any possible attempt to build a large scale nonlinear theory for operator spaces in the “naive” way: ideally, if we were to merge operator space theory with the nonlinear theory for Banach spaces, there could be an interesting theory which would capture aspects of the large scale geometry of operator spaces by simply considering maps $f: X \rightarrow Y$, or at least maps $f: B_X \rightarrow Y$, and their amplifications. However, this does not mean that it is not possible to obtain a nontrivial large scale geometry of operator spaces. Indeed, as shown in [5, 7, 8], there are several interesting things to be said if one takes a more sophisticated approach: for instance, instead of considering a *single* map f from either X or B_X to Y , one can consider *sequences* of maps $(f^n: X \rightarrow Y)_n$ or even of maps $(f^n: n \cdot B_X \rightarrow Y)_n$.

Motivated by Theorem 3, we start the study of the *small* scale structure of operator spaces. For that, we want to take into consideration not only the behavior of maps on sets of small diameter, but also their amplifications on such sets. Notice that this is not dealt with in Theorem 3 since $\lim_{n \rightarrow \infty} \text{diam}(M_n(B_X)) = \infty$.² For this reason, we will restrict our maps to the unit balls of $M_n(X)$ in order to guarantee the diameters of the sets are uniformly bounded. Before presenting our main findings, we start by recalling the definition of the modulus of uniform continuity of a map. Let (X, d) and (Y, ∂) be metric spaces, and $f: X \rightarrow Y$ be a map. The *modulus of uniform continuity* of f is the map $\omega_f: [0, \infty) \rightarrow [0, \infty]$ given by

$$\omega_f(t) = \sup \{ \partial(f(x), f(y)) \mid d(x, y) \leq t \}.$$

Definition 4. Let X and Y be operator spaces, and $f: B_X \rightarrow Y$ be a map.

- (1) The small scale modulus of uniform continuity of f is the map $\omega_f^{\text{ss}}: [0, \infty) \rightarrow [0, \infty]$ given by

$$\omega_f^{\text{ss}}(t) = \sup_{n \in \mathbb{N}} \omega_{f_n \upharpoonright B_{M_n(X)}}(t) \text{ for all } t \geq 0.$$

- (2) We say that f is completely Lipschitz in small scale if there is $L > 0$ such that

$$\omega_f^{\text{ss}}(t) \leq Lt \text{ for all } t \geq 0.$$

In contrast with Theorem 3, the next result shows that the property of a map being completely Lipschitz in small scale does not force the map to be the restriction of an \mathbb{R} -linear map. In fact, the next theorem provides a large class of non- \mathbb{R} -linear maps which are completely Lipschitz in small scale.

Theorem 5. Any polynomial p in one complex variable is completely Lipschitz in small scale as a map $B_{\mathbb{C}} \rightarrow \mathbb{C}$. More generally, if $A \subseteq \mathcal{B}(H)$ is an operator algebra (with its induced operator space structure) then $p: B_A \rightarrow A$ is completely Lipschitz in small scale.

Theorem 5 can be further generalized since it is obtained by looking at the compositions of m -linear maps with completely bounded maps. For brevity, we refer the reader to Section 3 and Theorem 16 for further details.

Knowing that there are plenty of interesting non- \mathbb{R} -linear maps which are completely Lipschitz in small scale, we then turn to study what the existence of such maps can tell us about the operator spaces involved. For that, we need the embedding notion given by maps which are completely Lipschitz in small scale. Recall, if $f: (X, d) \rightarrow (Y, \partial)$ is a map between metric spaces, then the *compression modulus* of f is the map $\rho_f: [0, \infty) \rightarrow [0, \infty]$ given by

$$\rho_f(t) = \inf \{ \partial(f(x), f(y)) \mid d(x, y) \geq t \}.$$

²This is why we say Theorem 3 is somewhat in between the scope of large and small scale geometric analysis.

Definition 6. Let X and Y be operator spaces, and $f: B_X \rightarrow Y$ be a map.

(1) The small scale compression modulus $\rho_f^{ss}: [0, \infty) \rightarrow [0, \infty]$ is given by

$$\rho_f^{ss}(t) = \inf_{n \in \mathbb{N}} \rho_{f_n \upharpoonright B_{M_n(X)}}(t) \text{ for all } t \geq 0.$$

(2) We say that f is a completely Lipschitz in small scale embedding if it is completely Lipschitz in small scale and there is $L \geq 1$ such that

$$\rho_f^{ss}(t) \geq \frac{1}{L}t \text{ for all } t \geq 0.$$

With our notion of nonlinear small scale embeddability being established, we now describe a linear property which is preserved under such notion. Recall that an operator space X is called *Hilbertian* if it is (linearly) isometric to a Hilbert space and *homogeneous* if for every linear map $u: X \rightarrow X$ we have $\|u\|_{cb} = \|u\|$. Then, if X is a homogeneous Hilbertian operator space with $\dim(E) \geq n$, we can unambiguously define

$$\kappa_n(X) = \left\| \left\| \begin{bmatrix} e_1 & 0 & \dots & 0 \\ e_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ e_n & 0 & \dots & 0 \end{bmatrix} \right\|_{M_n(X)} \right\|.$$

where $\{e_1, \dots, e_n\}$ is an arbitrary orthonormal set in X . Our main results about rigidity of operator spaces with respect to nonlinear embeddings will be based on the asymptotic behavior of $\kappa_n(X)$. If $(a_n)_n$ and $(b_n)_n$ are positive real numbers we use the common notation that $a_n \simeq b_n$ meaning that there is $L \geq 1$ such that $a_n/L \leq b_n \leq La_n$ for all $n \in \mathbb{N}$.

The following is our main theorem about the preservation of the linear geometry of operator spaces by completely Lipschitz in small scale embeddings.

Theorem 7. Let X and Y be homogeneous Hilbertian operator spaces. If there is a completely Lipschitz in small scale embedding $B_X \rightarrow Y$, then $\kappa_n(X) \simeq \kappa_n(Y)$.

Theorem 7 is obtained in two stages, one giving the lower estimate and other the upper (see Theorems 32 and 35, respectively). For each of these weaker results, the hypotheses are also much weaker.

In order to obtain applications of Theorem 7, it is important to compute, or at least estimate, $\kappa_n(X)$ for some operator spaces. Our main source of examples comes from interpolating operator spaces. We refer the reader to Section 4 and the references therein for precise definitions. Here, we simply mention that if X and Y are homogeneous Hilbertian operator spaces and $\theta \in [0, 1]$, $(X, Y)_\theta$ denotes the θ -interpolation operator space of X and Y . We compute the following (R and C denote the *row* and the *column* operator spaces, respectively, see Section 4):

- $\kappa_n((R, C)_\theta) = n^{\theta/2}$ (Corollary 25),
- $\kappa_n((\min(\ell_2), \max(\ell_2))_\theta) = n^{\theta/2}$ (Corollary 25),
- $\kappa_n((R \cap C, R + C)_\theta) = n^{\theta/2}$ (Corollary 26), and
- $\kappa_n(\Phi) \simeq \sqrt{n}$, where Φ is the Fermionic operator space (Proposition 27).

In particular, the computations above allow us to conclude the following:

Corollary 8. Let $\theta, \gamma \in [0, 1]$,

- $X \in \{(R, C)_\theta, (\min(\ell_2), \max(\ell_2))_\theta, (R \cap C, R + C)_\theta\}$, and
- $Y \in \{(R, C)_\gamma, (\min(\ell_2), \max(\ell_2))_\gamma, (R \cap C, R + C)_\gamma\}$.

If there is a completely Lipschitz in small scale embedding $f: B_X \rightarrow Y$, then $\theta = \gamma$.

Note that due to the very definition of κ_n , Theorems 7, 32, and 35 deal exclusively with maps $B_X \rightarrow Y$ where X and Y are homogeneous Hilbertian operator spaces. In Section 6, we use local techniques to push things beyond the Hilbertian setting and prove a result similar to the lower bound in Theorem 7 in the case where X is not Hilbertian (see Theorem 38). We defer the detailed statement to Section 6 to avoid introducing various technical definitions here, and for the moment we only state a corollary of it to illustrate the kind of results we obtain. Recall that from Dvoretzky’s theorem for operator spaces (see [25, Corollary 1.5]), for any infinite-dimensional operator space X there is an infinite-dimensional homogeneous Hilbertian operator space Z which is completely isometric to a subspace of an ultrapower of X (see [27, Section 2.8] for more details on ultraproducts of operator spaces). We call such a space a *Dvoretzky space* for X .

Corollary 9. *Let X be an infinite-dimensional operator space, and let Z be a Dvoretzky space for X . Let Y be a homogeneous Hilbertian operator space such that $\kappa_n(Y) \gtrsim n^c$ for some $c \in [0, 1/2]$. If there is a completely Lipschitz in small scale embedding $B_X \rightarrow Y$, then $\kappa_n(Z) \gtrsim n^{c/(1+2c)}$.*

The conclusion in Corollary 9 is weaker than the lower bound in Theorem 7, which is not surprising since the assumption on Z is weaker. Moreover, let us emphasize that it is significantly weaker: while the finite-dimensional subspaces of Z are uniformly isomorphic to subspaces of X because ultrapowers of a Banach space are finitely representable in the original space [17, Proposition 6.1], in the operator space setting the corresponding statement with *complete* isomorphisms does not hold [13, p. 88].

2. Revisiting completely coarse maps

In this section, we prove Theorem 3. The next lemma gives a sufficient condition for a map $B_X \rightarrow Y$ to be the restriction of an \mathbb{R} -linear map.

Lemma 10. *Let X and Y be normed \mathbb{R} -vector spaces, and let $f : B_X \rightarrow Y$ be a bounded function such that $f(0) = 0$ and*

$$f\left(\frac{1}{2}(x + z)\right) = \frac{1}{2}(f(x) + f(z))$$

for all $x, z \in B_X$. Then f is the restriction of an \mathbb{R} -linear function $X \rightarrow Y$.

Proof. Firstly, for computational reasons, it will be useful to assume that f is defined on $2 \cdot B_X$. This is not an issue since, replacing f with $f(\frac{\cdot}{2})$, we can assume that f is defined on the whole $2 \cdot B_X$ and it still satisfies the assumptions of the lemma for all $x, z \in 2 \cdot B_X$. Moreover, since $f(0) = 0$, we must have

$$f\left(\frac{1}{2}x\right) = \frac{1}{2}f(x) \text{ for all } x \in 2 \cdot B_X. \tag{1}$$

Therefore, in order to show that $f : 2 \cdot B_X \rightarrow Y$ is the restriction of an \mathbb{R} -linear function, it is enough that $f \upharpoonright B_X$ is so.

Claim 11. *For all $x_1, \dots, x_n \in B_X$, we have*

$$f\left(\frac{x_1 + \dots + x_n}{n}\right) = \frac{1}{n}(f(x_1) + \dots + f(x_n)).$$

Proof. This follows from induction on n . For $n = 1$, the result is trivial; suppose then it holds for some $n \in \mathbb{N}$. By (1), we have

$$f\left(\frac{x+z}{2}\right) = \frac{1}{2}f(x) + \frac{1}{2}f(z) = f\left(\frac{x}{2}\right) + f\left(\frac{z}{2}\right) \tag{2}$$

for all $x, z \in 2 \cdot B_X$. Analogously, the induction hypothesis also implies that

$$f\left(\frac{x_1 + \dots + x_n}{n}\right) = f\left(\frac{x_1}{n}\right) + \dots + f\left(\frac{x_n}{n}\right) \tag{3}$$

for all $x_1, \dots, x_n \in B_X$ (notice that the induction hypothesis does not allow us to conclude this holds for all elements in $2 \cdot B_X$ though).

Notice that if $x_1, \dots, x_{n+1} \in B_X$, then

$$\frac{2x_1}{n+1} \in B_X \text{ and } \frac{2(x_2 + \dots + x_{n+1})}{n+1} \in 2 \cdot B_X.$$

Therefore, by (2) and (3), we have

$$\begin{aligned} f\left(\frac{x_1 + \dots + x_{n+1}}{n+1}\right) &= f\left(\frac{2x_1/(n+1) + 2(x_2 + \dots + x_{n+1})/(n+1)}{2}\right) \\ &= f\left(\frac{x_1}{n+1}\right) + f\left(\frac{x_2 + \dots + x_{n+1}}{n+1}\right) \\ &= f\left(\frac{x_1}{n+1}\right) + f\left(\frac{nx_2/(n+1) + \dots + nx_{n+1}/(n+1)}{n}\right) \\ &= f\left(\frac{x_1}{n+1}\right) + f\left(\frac{x_2}{n+1}\right) + \dots + f\left(\frac{x_{n+1}}{n+1}\right) \end{aligned} \tag{4}$$

for all $x_1, \dots, x_{n+1} \in B_X$. In particular, if all x_1, \dots, x_{n+1} are the same, say $x = x_i$ for all $i \in \{1, \dots, n\}$, this shows that

$$f\left(\frac{x}{n+1}\right) = \frac{1}{n+1} f(x) \text{ for all } x \in B_X.$$

Revisiting (4) with this extra information, we conclude that

$$f\left(\frac{x_1 + \dots + x_{n+1}}{n+1}\right) = \frac{1}{n+1} (f(x_1) + \dots + f(x_{n+1}))$$

for all $x_1, \dots, x_{n+1} \in B_X$ as desired. □

The previous claim together with the fact that $f(0) = 0$ gives

$$f(qx) = qf(x) \text{ for all } q \in [0, 1] \cap \mathbb{Q} \text{ and all } x \in B_X, \tag{5}$$

which also implies that

$$f(qx) = qf(x) \text{ for all } q \in [0, \infty) \cap \mathbb{Q} \text{ and all } x \in B_X \text{ with } qx \in B_X. \tag{6}$$

For each $x \in X \setminus \{0\}$, pick $r_x \in [2, 3)$ such that $\|r_x x\|_X \in \mathbb{Q}$. Define a map $F: X \rightarrow Y$ by letting

$$F(x) = \begin{cases} r_x \|x\|_X f\left(\frac{x}{r_x \|x\|_X}\right), & x \neq 0, \\ 0, & x = 0. \end{cases} \tag{7}$$

It follows immediately from (6) that F is an extension of $f \upharpoonright B_X$. We are left to notice that F is \mathbb{R} -linear. For additivity, notice that if $x, z \in \frac{1}{2} \cdot B_X$, then $x + z \in B_X$ and, by (6), we must have

$$f(x + z) = f\left(2 \frac{x+z}{2}\right) = 2f\left(\frac{x+z}{2}\right) = f(x) + f(z). \tag{8}$$

Fix $x, z \in X$ and pick $M > 1$ large enough so that

$$\frac{x}{Mr_{x+z}\|x+z\|_X}, \frac{z}{Mr_{x+z}\|x+z\|_X} \in \frac{1}{2} \cdot B_X$$

and

$$\frac{r_x \|x\|_X}{Mr_{x+z}\|x+z\|_X}, \frac{r_z \|z\|_X}{Mr_{x+z}\|x+z\|_X} \leq 1.$$

Then, (5) and (8) together imply

$$\begin{aligned} \frac{1}{M} \cdot F(x+z) &= \frac{r_{x+z}\|x+z\|_X}{M} f\left(\frac{x+z}{r_{x+z}\|x+z\|_X}\right) \\ &= r_{x+z}\|x+z\|_X f\left(\frac{x+z}{Mr_{x+z}\|x+z\|_X}\right) \\ &= r_{x+z}\|x+z\|_X \left(f\left(\frac{x}{Mr_{x+z}\|x+z\|_X}\right) + f\left(\frac{z}{Mr_{x+z}\|x+z\|_X}\right) \right) \\ &= \frac{1}{M}(F(x) + F(z)). \end{aligned}$$

So, F is additive.

We now show that $F(tx) = tF(x)$ for all $t \geq 0$ and $x \in X$. For that, fix $x \in X$ and define a map $g_x: \mathbb{R} \rightarrow X$ by letting

$$g_x = F(tx) - tF(x) \text{ for all } t \in \mathbb{R}.$$

As f is bounded, g_x is bounded on bounded sets. On the other hand, since F is additive, we have

$$g_x(t+1) = F(tx+x) - (t+1)F(x) = F(tx) - tF(x) = g(t)$$

for all $t \in \mathbb{R}$, i.e., g_x is 1-periodic. Therefore, g_x must be bounded. However, as F is additive, so is g_x . Since a bounded additive function must be zero, the result follows. \square

The following is elementary and it is the operator space version of [20, Lemma 1.4]. This will be used in the proof of Theorem 3 below.

Proposition 12. *Let X and Y be operator spaces and $K \subseteq X$ be convex. Then a map $f: K \rightarrow Y$ is completely coarse if and only if there is $C > 0$ so that*

$$\|f([x_{ij}]) - f([y_{ij}])\|_{M_n(Y)} \leq C\| [x_{ij}] - [y_{ij}] \|_{M_n(X)} + C$$

for all $n \in \mathbb{N}$ and all $[x_{ij}] \in M_n(K)$. \square

Before presenting the proof of Theorem 3, we recall the concept of Hadamard matrices: a *Hadamard matrix* is a square matrix whose entries are either 1 or -1 and whose rows are mutually orthogonal. To notice that Hadamard matrices of arbitrarily large size exist, we define matrices $A_{2^k} \in M_{2^k}(\mathbb{C})$ inductively by letting

$$A_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \text{ and } A_{2^{k+1}} = \begin{pmatrix} A_{2^k} & A_{2^k} \\ A_{2^k} & -A_{2^k} \end{pmatrix}$$

for all $k > 1$.

Proof of Theorem 3. Our goal to conclude that $f: B_X \rightarrow Y$ is the restriction of an \mathbb{R} -linear map is to use Lemma 10. Hence, we fix distinct $x, z \in B_X$ and show that

$$f\left(\frac{1}{2}(x+z)\right) = \frac{1}{2}(f(x) + f(z)).$$

For convenience, let

$$x_0 = \frac{x+z}{2} \text{ and } h = \frac{x-z}{2}.$$

so $x_0, h, x_0 + h$, and $x_0 - h$ are still in B_X , and we are left to show that

$$f(x_0) = \frac{1}{2}(f(x_0 + h) + f(x_0 - h)). \tag{9}$$

As $x \neq z$, $h \neq 0$ and we can use Hahn–Banach to pick $\varphi \in X^*$ with $\varphi(h) = 1$. We then set

$$y_0 = \frac{f(x_0 - h) - f(x_0 + h)}{2}$$

and define a map $g: X \rightarrow Y$ by letting

$$g(x) = f(x) + \varphi(x)y_0 \text{ for all } x \in B_X.$$

Since f and φ are completely coarse, so is g . By Proposition 12, there exists a constant $C > 0$ such that for any $n \in \mathbb{N}$ and $[x_{ij}]_{ij}, [y_{ij}]_{ij} \in M_n(B_X)$, we have

$$\| [g(x_{ij}) - g(y_{ij})]_{ij} \|_{M_n(Y)} \leq C \| [x_{ij} - y_{ij}]_{ij} \|_{M_n(X)} + C. \tag{10}$$

Let $(A_{2^k})_{k=1}^\infty$ be the Hadamard matrices and, for each $k \in \mathbb{N}$, write $A_{2^k} = [a_{i,j}^k]_{i,j=1}^{2^k}$. Since each $a_{i,j}^k$ is either 1 or -1 , we have that each $x_0 + ha_{ij}^k$ belongs to B_X . In particular, if $\mathbb{1}_{2^k}$ denotes the $2^k \times 2^k$ scalar matrix whose entries are all 1,

$$\mathbb{1}_{2^k} \otimes x_0 + A_{2^k} \otimes h = \left[x_0 + ha_{ij}^k \right]_{i,j=1}^{2^k} \in M_{2^k}(B_X)$$

and

$$\mathbb{1}_{2^k} \otimes x_0 = [x_0]_{i,j=1}^{2^k} \in M_{2^k}(B_X)$$

for all $k \in \mathbb{N}$. Hence, by (10),

$$\| [g(x_0 + ha_{ij}^k) - g(x_0)]_{i,j=1}^{2^k} \|_{M_{2^k}(Y)} \leq C \| A_{2^k} \otimes h \|_{M_{2^k}(X)} + C.$$

Notice that, by the formula of g , $g(x_0 + h) = g(x_0 - h)$. Therefore

$$\| g(x_0 + h) - g(x_0) \|_Y \cdot \| \mathbb{1}_{2^k} \|_{M_{2^k}} \leq C \cdot \| h \|_X \cdot \| A_{2^k} \|_{M_{2^k}} + C,$$

which yields

$$\| g(x_0 + h) - g(x_0) \|_Y 2^k \leq C \| h \|_X \sqrt{2^k} + C.$$

Letting k tend to infinity, we can conclude that $g(x_0 + h) = g(x_0)$. Unfolding definitions, this means that

$$f(x_0) + \varphi(x_0)y_0 = f(x_0 + h) + \varphi(x_0 + h)y_0,$$

which, rearranging the terms and using that $\varphi(h) = 1$, give $f(x_0) = f(x_0 + h) + y_0$. By the definition of y_0 , this implies

$$f(x_0) = f(x_0 + h) + \frac{f(x_0 - h) - f(x_0 + h)}{2} = \frac{1}{2} (f(x_0 + h) + f(x_0 - h)).$$

This shows that (9) holds and we are done. □

Remark 13. The results of [6] are stated for both real and complex operator spaces. Since we make heavy use of complex interpolation in the present work, for simplicity we have written the whole paper in terms of complex operator spaces. Nevertheless, note that the proof of Theorem 3 yields the same result for real operator spaces.

3. Nonlinear small scale maps

In this section, we produce examples of maps $B_X \rightarrow Y$ which are completely Lipschitz in small scale but are not the restriction of \mathbb{R} -linear maps. This culminates in Theorem 5 and, more generally, in Theorem 16.

We start recalling some standard notation. Given $m \in \mathbb{N}$ and Banach spaces X_1, \dots, X_m , the sum $\bigoplus_{k=1}^m X_k$ will always denote the Banach space obtained by considering the vector space $\bigoplus_{k=1}^m X_k$ endowed with the ℓ_∞ -sum. If moreover each X_k is an operator space, then $\bigoplus_{k=1}^m X_k$ is also an operator space endowed with the ℓ_∞ -sum operator space structure. Since m -linear maps will play an important role in what follows, we quickly recall some basic terminology. Given Banach spaces X_1, \dots, X_m, Y , the *norm of an m -linear map* $Q: \bigoplus_{k=1}^m X_k \rightarrow Y$ is the infimum of all $L > 0$ such that

$$\| Q(x^{(1)}, \dots, x^{(m)}) \|_Y \leq L \| x^{(1)} \|_{X_1} \cdots \| x^{(m)} \|_{X_m}$$

for all $(x^{(1)}, \dots, x^{(m)}) \in \bigoplus_{k=1}^m X_k$. This infimum is denoted by $\|Q\|$ and we say that Q is a *bounded m -linear map* if $\|Q\| < \infty$. Notice that, if X_1, \dots, X_m, Y are moreover operator spaces, then each n -amplification $Q_n: \bigoplus_{k=1}^m M_n(X_k) \rightarrow M_n(Y)$ is also m -linear and hence $\|Q_n\|$ is well defined.

Definition 14. Let $m \in \mathbb{N}$, X_1, \dots, X_m, Y be operator spaces, and $Q: \bigoplus_{k=1}^m X_k \rightarrow Y$ be an m -linear map. We say that Q is *completely controlled* if³

$$\|Q\|_{cc} = \sup_n \|Q_n\| < \infty.$$

Proposition 15. Let $m \in \mathbb{N}$, H be a Hilbert space, and $X_1, \dots, X_m \subseteq \mathcal{B}(H)$ be operator spaces. The product map $P: \bigoplus_{k=1}^m X_k \rightarrow \mathcal{B}(H)$ given by

$$P(x^{(1)}, \dots, x^{(m)}) = x^{(1)} \cdots x^{(m)} \text{ for all } (x^{(1)}, \dots, x^{(m)}) \in \bigoplus_{k=1}^m X_k$$

is a completely controlled m -linear map with $\|P\|_{cc} \leq 1$.

Proof. The m -linearity of P is straightforward. To notice that P is completely controlled, the crucial tool is the following generalization of Schur’s inequality [30, Satz III] which follows from [9, Theorem 2.3]: if $n \in \mathbb{N}$ and $[x_{ij}], [z_{ij}] \in M_n(\mathcal{B}(H))$, then

$$\|[x_{ij} z_{ij}]\|_{M_n(\mathcal{B}(H))} \leq \|[x_{ij}]\|_{M_n(\mathcal{B}(H))} \|[z_{ij}]\|_{M_n(\mathcal{B}(H))}.$$

By a straightforward induction, this implies that

$$\begin{aligned} \left\| P_n([x_{ij}^{(1)}], \dots, [x_{ij}^{(m)}]) \right\|_{M_n(\mathcal{B}(H))} &= \left\| [x_{ij}^{(1)}] \cdots [x_{ij}^{(m)}] \right\|_{M_n(\mathcal{B}(H))} \\ &\leq \left\| [x_{ij}^{(1)}] \right\|_{M_n(X_1)} \cdots \left\| [x_{ij}^{(m)}] \right\|_{M_n(X_m)} \end{aligned}$$

for all $n \in \mathbb{N}$ and all $([x_{ij}^{(1)}], \dots, [x_{ij}^{(m)}]) \in \bigoplus_{k=1}^m M_n(X_k)$. This shows that $\|P\|_{cc} \leq 1$. □

The following is our main result to construct nontrivial examples of non- \mathbb{R} -linear maps which are completely Lipschitz in small scale.

Theorem 16. Let $m \in \mathbb{N}$, X, X_1, \dots, X_m, Y be operator spaces, $T: X \rightarrow \bigoplus_{k=1}^m X_k$ be a completely bounded operator, and $Q: \bigoplus_{k=1}^m X_k \rightarrow Y$ be a completely controlled m -linear map. Then $Q \circ T: B_X \rightarrow Y$ is completely Lipschitz in small scale.

We start with a lemma about small scale behavior of m -linear maps on Banach spaces.

Lemma 17. Let $m \in \mathbb{N}$, X_1, \dots, X_m, Y be Banach spaces, and $Q: \bigoplus_{k=1}^m X_k \rightarrow Y$ be a bounded m -linear map. Then, for all $(x^{(1)}, \dots, x^{(m)}), (z^{(1)}, \dots, z^{(m)}) \in B_{\bigoplus_{k=1}^m X_k}$, we have

$$\|Q(x^{(1)}, \dots, x^{(m)}) - Q(z^{(1)}, \dots, z^{(m)})\|_Y \leq m \|Q\| \|(x^{(1)}, \dots, x^{(m)}) - (z^{(1)}, \dots, z^{(m)})\|_{\bigoplus_{k=1}^m X_k}.$$

Proof. We proceed by induction on $m \in \mathbb{N}$. If $m = 1$, Q is a bounded linear operator and the result is immediate. Suppose it holds for $m - 1$, with $m \geq 2$, and let us show it is also valid for m . Fix $x^{(1)}, \dots, x^{(m)}, z^{(1)}, \dots, z^{(m)}$ in $B_{\bigoplus_{k=1}^m X_k}$. Then, as Q is m -linear and as each $x^{(k)}$ has norm at most 1, it follows that

$$\begin{aligned} \|Q(x^{(1)}, \dots, x^{(m-1)}, x^{(m)}) - Q(x^{(1)}, \dots, x^{(m-1)}, z^{(m)})\|_Y &= \|Q(x^{(1)}, \dots, x^{(m-1)}, x^{(m)} - z^{(m)})\|_Y \\ &\leq \|Q\| \|x^{(m)} - z^{(m)}\|_{X_m}. \end{aligned}$$

Let $B: \bigoplus_{k=1}^{m-1} X_k \rightarrow Y$ be given by

$$B(w^{(1)}, \dots, w^{(m-1)}) = Q(w^{(1)}, \dots, w^{(m-1)}, z^{(m)})$$

³We chose to control (no pun intended) the automatic instinct of calling such m -linear map *completely bounded* since this definition already exists for m -linear maps and it is not the one given above (see, for instance, [9]).

for all $(w^{(1)}, \dots, w^{(m-1)}) \in \bigoplus_{k=1}^{m-1} X_k$. So, B is an $(m - 1)$ -linear map and, as $\|z^{(m)}\|_{X_m} \leq 1$, we have $\|B\| \leq \|Q\|$. Therefore, the induction hypothesis gives that

$$\begin{aligned} & \|B(x^{(1)}, \dots, x^{(m-1)}) - B(z^{(1)}, \dots, z^{(m-1)})\|_Y \\ & \leq (m - 1)\|Q\| \|(x^{(1)}, \dots, x^{(m-1)}) - (z^{(1)}, \dots, z^{(m-1)})\|_{\bigoplus_{k=1}^{m-1} X_k}. \end{aligned}$$

The lemma then follows by the triangle inequality. \square

Proof of Theorem 16. Since each amplification Q_n is an m -linear map with norm at most $\|Q\|_{cc}$, Lemma 17 gives that

$$\begin{aligned} & \left\| Q_n([x_{ij}^{(1)}], \dots, [x_{ij}^{(m)}]) - Q_n([z_{ij}^{(1)}], \dots, [z_{ij}^{(m)}]) \right\|_{M_n(Y)} \\ & \leq m\|Q\|_{cc} \left\| ([x_{ij}^{(1)}], \dots, [x_{ij}^{(m)}]) - ([z_{ij}^{(1)}], \dots, [z_{ij}^{(m)}]) \right\|_{\bigoplus_{k=1}^m M_n(X_k)} \end{aligned}$$

for all $n \in \mathbb{N}$ and all $([x_{ij}^{(1)}], \dots, [x_{ij}^{(m)}]), ([z_{ij}^{(1)}], \dots, [z_{ij}^{(m)}]) \in B_{\bigoplus_{k=1}^m M_n(X_k)}$. By the m -linearity of Q_n , if $([x_{ij}^{(1)}], \dots, [x_{ij}^{(m)}])$ and $([z_{ij}^{(1)}], \dots, [z_{ij}^{(m)}])$ have norm at most $\|T\|_{cb}$ instead, we obtain the a similar inequality with the factor $\|T\|_{cb}^m$ added to it on the right-hand side. Therefore, we conclude that

$$\begin{aligned} & \left\| (Q \circ T)_n([x_{ij}]) - (Q \circ T)_n([z_{ij}]) \right\|_{M_n(Y)} = \left\| Q_n(T_n([x_{ij}])) - Q_n(T_n([z_{ij}])) \right\|_{M_n(Y)} \\ & \leq m\|Q\|_{cc} \|T\|_{cb}^m \left\| [T(x_{ij})] - [T(z_{ij})] \right\|_{\bigoplus_{k=1}^m M_n(X_k)} \\ & \leq m\|Q\|_{cc} \|T\|_{cb}^{m+1} \left\| [x_{ij}] - [z_{ij}] \right\|_{M_n(X)} \end{aligned}$$

for all $n \in \mathbb{N}$ and all $[x_{ij}], [z_{ij}] \in B_{M_n(X)}$. So,

$$\omega_{Q \circ T}^{ss}(t) \leq m\|Q\|_{cc} \|T\|_{cb}^{m+1} t$$

for all $t \geq 0$. \square

Proof of Theorem 5. Let $m \in \mathbb{N}$ and let p^m be the complex polynomial given by $p^m(x) = x^m$ for all $x \in \mathbb{C}$. Let A be an operator algebra. Then, by Proposition 15, the map $(a_1, \dots, a_m) \in \bigoplus_{k=1}^m A \mapsto a_1 \cdots a_m \in A$ is completely controlled. Since the map $a \in A \mapsto (a, \dots, a) \in \bigoplus_{k=1}^m A$ is completely bounded, it follows from Theorem 16 that p^m is completely Lipschitz in small scale. As any polynomial is a linear combination of polynomials of the form p^m , $m \in \mathbb{N}$, the result follows. \square

The reader familiar with m -linear maps on operator spaces knows that other notions of “complete boundedness” are also frequently studied. More precisely, if $Q: \bigoplus_{k=1}^m X_k \rightarrow Y$ is an m -linear map between operator spaces, Q may be *completely bounded* or *jointly completely bounded*. Those are technical definitions which, for brevity, we chose to omit here, see [10, p. 281] and [2, Section 1.5.11] for the precise definitions of each of them, respectively.⁴ We finish this section briefly relating these notions.

Proposition 18. *Let $m \in \mathbb{N}$, X_1, \dots, X_m, Y be operator spaces, and $Q: \bigoplus_{k=1}^m X_k \rightarrow Y$ be a jointly completely bounded m -linear map. Then Q is completely controlled with $\|Q\|_{cc} \leq \|Q\|_{jcb}$, where $\|Q\|_{jcb}$ denotes the norm of joint complete boundedness of Q (see [2, Section 1.5.11]).*

Proof. For simplicity we will write the proof in the case $m = 2$, the general case is analogous. Let $n \in \mathbb{N}$. Since Q is jointly completely bounded, for every $[x_{ij}^{(1)}] \in M_n(X_1)$ and $[x_{ij}^{(2)}] \in M_n(X_2)$ we have

$$\left\| \left[Q \left(x_{ij}^{(1)}, x_{kl}^{(2)} \right) \right] \right\|_{M_{n^2}(Y)} \leq \|Q\|_{jcb} \left\| [x_{ij}^{(1)}] \right\|_{M_n(X_1)} \left\| [x_{ij}^{(2)}] \right\|_{M_n(X_2)}$$

⁴The reader should be warned that some references such as [14] use the terminology completely bounded (resp. multiplicatively bounded) for the multilinear mappings that are nowadays generally called jointly completely bounded (resp. completely bounded).

The desired conclusion now follows from Ruan’s axioms by observing that $[Q(x_{ij}^{(1)}, x_{ij}^{(2)})]$ is an $n \times n$ principal submatrix of the $n^2 \times n^2$ matrix $[Q(x_{ij}^{(1)}, x_{kl}^{(2)})]$, that is, there exists a coordinate partial isometry $P \in M_{n,n^2}(\mathbb{C})$ such that $[Q(x_{ij}^{(1)}, x_{ij}^{(2)})] = P[Q(x_{ij}^{(1)}, x_{kl}^{(2)})]P^*$. \square

Since completely bounded m -linear maps are jointly completely bounded, Proposition 18 shows that many of the multilinear maps that have previously been studied in operator space theory are completely controlled. Also, in the case of maximal operator spaces there is an abundance of completely controlled maps (see [27, Chapter 3] for the definition of the minimal and maximal operator space structures).

Corollary 19. *Let $m \in \mathbb{N}$, X_1, \dots, X_m be Banach spaces, and Y be an operator space. Then any bounded m -linear map $Q: \bigoplus_{k=1}^m X_k \rightarrow Y$ is completely controlled as a map $\bigoplus_{k=1}^m \max(X_k) \rightarrow Y$.*

Proof. By Proposition 18, it suffices to show that Q is jointly completely bounded. Since jointly completely bounded m -linear maps correspond to completely bounded linear maps on the projective operator space tensor product (see [14, Proposition 7.1.2]), and $\max(X_1) \widehat{\otimes} \dots \widehat{\otimes} \max(X_m) = \max(X_1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi X_m)$, where $\widehat{\otimes}_\pi$ is the Banach space projective tensor product (see [2, Proposition 1.5.12]), the conclusion follows from the fact that bounded linear maps whose domain is a maximal operator space are automatically completely bounded. \square

The reader familiar with the theory of polynomials on vector spaces will already have recognized that in the situation of Theorem 16, if $X = X_1 = \dots = X_m$ and T is the diagonal map $T(x) = (x, x, \dots, x)$, the composition $Q \circ T$ is precisely a polynomial on X . Therefore, one would expect that the available literature on polynomials on operator spaces might already provide us with more examples of maps which are completely Lipschitz in small scale. However, this is a subject that has not yet been developed much: the only significant works in this regard appear to be [11, 12]. Corollary 19 is closely related to [11, Proposition 9.3], and we next list other examples of completely Lipschitz in small scale maps that follow from the aforementioned two papers.

Corollary 20. *Let $m \in \mathbb{N}$ and Y an operator space. Then any bounded m -linear map $Q: \bigoplus_{k=1}^m \ell_\infty \rightarrow Y$ is completely controlled.*

Proof. This follows from [11, Proposition 9.5], since the Schur multilinear mappings defined in that paper are easily seen to be completely controlled and thus Theorem 16 yields the desired result. \square

For our last example, the operator space OH mentioned in it is the Hilbert operator space of G. Pisier, see [27, Chapter 7] for its definition.

Corollary 21. *Let $(e_j)_{j=1}^\infty$ be an orthonormal basis for OH, Y an operator space, and $(y_j)_{j=1}^\infty$ a norm null sequence in Y . Let $m \geq 2$ be a natural number and define $P: \text{OH} \rightarrow Y$ by*

$$P\left(\sum_{n=1}^\infty x_n e_n\right) = \sum_{n=1}^\infty x_n^m y_n.$$

Then P is completely Lipschitz in small scale.

Proof. This follows from [12, Proposition 4.1], since the completely bounded polynomials defined in that paper are constructed as restrictions to the diagonal of jointly completely bounded multilinear maps. Thus, Proposition 18 and Theorem 16 yield the desired result. \square

4. Basics about $\kappa_n(X)$ and examples

In this section, we prove basic properties about the κ_n ’s and compute $\kappa_n(X)$ for many homogeneous Hilbertian operator spaces X . For its definition, see Section 1.

We start recalling the definition of the row and column operator spaces. Given $i, j \in \mathbb{N}$, we let $e_{i,j}$ denote the operator on ℓ_2 whose matrix representation has 1 in the (i, j) -th entry and zero elsewhere. The *row* and the *column operator spaces* are then defined to be

$$R = \overline{\text{span}}\{e_{1,j} \mid j \in \mathbb{N}\} \text{ and } C = \overline{\text{span}}\{e_{i,1} \mid i \in \mathbb{N}\}.$$

In particular, both R and C are homogeneous Hilbertian operator spaces.

Example 22. If R and C are the row and column operator spaces, respectively, then $\kappa_n(R) = 1$ and $\kappa_n(C) = \sqrt{n}$ for all $n \in \mathbb{N}$. Moreover, it is clear that these spaces minimize and maximize κ_n , respectively, i.e., for any operator space E we have $1 \leq \kappa_n(E) \leq \sqrt{n}$ for all $n \in \mathbb{N}$. The lower bound is obvious since each of the canonical projections $M_n(E) \rightarrow E$ is completely contractive and the upper bound follows equally as easily by considering some representation $E \subseteq \mathcal{B}(H)$ and computing $b(x)$, where b is the n -by- n E -valued matrix in the definition of $\kappa_n(E)$ and x is an arbitrary normalized vector in $H^{\oplus n}$.

Proposition 23. For any homogeneous Hilbertian operator space X and any $n \in \mathbb{N}$, we have $\kappa_n(X)\kappa_n(X^*) \geq \sqrt{n}$.

Proof. Let $\{e_j\}_{j=1}^n$ be an arbitrary orthonormal set in X and $\{e_k^*\}_{k=1}^n$ be an orthonormal set in X^* which is biorthogonal to $\{e_j\}_{j=1}^n$. Then the product $\kappa_n(X)\kappa_n(X^*)$ dominates the norm in M_{n^2} of the matrix pairing between

$$\begin{bmatrix} e_1 & 0 & \dots & 0 \\ e_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ e_n & 0 & \dots & 0 \end{bmatrix} \text{ and } \begin{bmatrix} e_1^* & 0 & \dots & 0 \\ e_2^* & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ e_n^* & 0 & \dots & 0 \end{bmatrix},$$

which is exactly \sqrt{n} since, as a matrix in M_{n^2} , it has n entries equal to 1 in the first column and all other entries 0. □

Interpolation spaces will provide a good source of examples of operator spaces whose κ_n 's can be estimated. For simplicity, we now set some notation: if X_0 and X_1 are homogeneous Hilbertian operator spaces and $\theta \in [0, 1]$, we let

$$X_\theta = (X_0, X_1)_\theta,$$

where the interpolation above is taken with respect to some isometric identification $X_0 \simeq X_1$ (since both spaces are homogeneous Hilbertian, the specific identification is irrelevant). Due to its technical definition, we refer the reader to [27, Section 2.7] for the definition of interpolation operator spaces.

Proposition 24. Consider the interpolation spaces $(X_\theta)_{\theta \in [0,1]}$ of a given pair (X_0, X_1) of infinite dimensional homogeneous Hilbertian operator spaces. The following holds.

- (1) $\kappa_n(X_\theta) \leq \kappa_n(X_0)^{1-\theta} \kappa_n(X_1)^\theta$ for all $\theta \in [0, 1]$ and all $n \in \mathbb{N}$.
- (2) If $X_0^* \equiv X_1$, then $\kappa_n(X_\theta) \geq \sqrt{n} / (\kappa_n(X_1)^{1-\theta} \kappa_n(X_0)^\theta)$.

Proof.

(1). It is standard in interpolation theory that

$$\|b\|_{M_n(X_\theta)} \leq \|b\|_{M_n(X_0)}^{1-\theta} \|b\|_{M_n(X_1)}^\theta$$

for any $b \in M_n(X_\theta)$ (see [2, Section 1.2.30]). So, the inequality is immediate.

(2). Since X_0 is reflexive and using that $X_0^* = X_1$, we have that

$$X_\theta^* = (X_0, X_1)_\theta^* \equiv (X_0^*, X_1^*)_\theta \equiv (X_1, X_0)_\theta \equiv (X_0, X_1)_{1-\theta} \tag{11}$$

([27, Theorem 2.7.4]). It then follows from (2) that $\kappa_n(X_\theta^*) \leq \kappa_n(X_0)^\theta \kappa_n(X_1)^{1-\theta}$. Therefore, since Proposition 23 gives $\kappa_n(X_\theta)\kappa_n(X_\theta^*) \geq \sqrt{n}$, the result follows. □

Corollary 25. *The following holds for $\theta \in [0, 1]$ and $n \in \mathbb{N}$.*

- (1) $\kappa_n((R, C)_\theta) = n^{\theta/2}$.
- (2) $\kappa_n((\min(\ell_2), \max(\ell_2))_\theta) = n^{\theta/2}$.

Proof.

(1). It is immediate that $\kappa_n(R) = 1$ and $\kappa_n(C) = \sqrt{n}$ (Example 22). Therefore, since $R^* \equiv C$ ([27, p. 41]), the upper and lower bounds given by Proposition 24 imply that $\kappa_n((R, C)_\theta) = n^{\theta/2}$.

(2). Since the identity $R \rightarrow \min(\ell_2)$ is completely contractive and $\kappa_n(R) = 1$, this implies $\kappa_n(\min(\ell_2)) = 1$. Similarly, since the identity $\max(\ell_2) \rightarrow C$ is completely contractive, $\kappa_n(\max(\ell_2)) \geq \sqrt{n}$ and equality must then hold as $\kappa_n(\max(\ell_2)) \leq \sqrt{n}$ (Example 22). Therefore, as $\min(\ell_2)^* = \max(\overline{\ell_2})$ ([27, p. 72]) and as $\max(\overline{\ell_2})$ is completely isometric to $\max(\ell_2)$, the result then follows from Proposition 24 again. □

As our next result shows, Corollary 25(2) can be considerably generalized. Recall that, if X is a homogeneous Hilbertian operator space, then Riesz representation gives us a linear isometry $X \rightarrow \overline{X^*}$ and this allows us to construct the interpolation spaces $(X, \overline{X^*})_\theta$. In order to present a consequence of this generalization, we recall the definitions of $R \cap C$ and $R + C$. Let $r: \ell_2 \rightarrow R$ and $c: \ell_2 \rightarrow C$ be canonical isometries. The operator space $R \cap C$ is the Banach space ℓ_2 together with the operator space structure given by the isometric inclusion

$$x \in \ell_2 \mapsto (r(x), c(x)) \in R \oplus C \subseteq \mathcal{B}(\ell_2) \oplus \mathcal{B}(\ell_2).$$

The operator space $R + C$ is the quotient $(R \oplus_1 C) / \Delta$, where $\Delta = \{(r(x), -c(x)) \mid x \in \ell_2\}$ ([27, p. 194]).

Corollary 26. *Let X be a homogeneous Hilbertian space such that the identity $R \rightarrow X$ has cb-norm 1. Then, $\kappa_n((X, \overline{X^*})_\theta) = n^{\theta/2}$ for all $\theta \in [0, 1]$ and all $n \in \mathbb{N}$. In particular, $\kappa_n((R \cap C, R + C)_\theta) = n^{\theta/2}$ for all $\theta \in [0, 1]$ and all $n \in \mathbb{N}$.*

Proof. Since the identity $R \rightarrow X$ has cb-norm 1, its adjoint $X^* \rightarrow C$ also has cb-norm 1. So, the proof follows exactly as the one of Proposition 25(2). The last statement follows since the identity $R \rightarrow R \cap C$ is clearly completely contractive and $(R \cap C)^* \equiv R + C$ ([27, p. 194]). □

For our next example, we recall the definition of Fermionic operator spaces. Let H be a Hilbert space and $(v_i)_{i \in I}$ be a family of operators in $\mathcal{B}(H)$ such that

$$v_i v_j + v_j v_i = 0 \text{ and } v_i v_j^* + v_j^* v_i = \delta_{i,j} \text{Id}_H$$

for all $i, j \in I$, where $(\delta_{i,j})_{i,j \in I}$ are the Kronecker deltas. Then, the *Fermionic operator space associated to I* is

$$\Phi(I) = \overline{\text{span}}\{v_i \mid i \in I\}.$$

It turns out the space above does not depend on $(v_i)_{i \in I}$ per se but only on I . We refer to [27, Theorem 9.3.1] for a proof of that. For shortness, if $I = \{1, \dots, n\}$, we write Φ_n for $\Phi(\{1, \dots, n\})$.

Proposition 27. *For any infinite set I , we have that $\kappa_n(\Phi(I)) \simeq \sqrt{n}$.*

Proof. The upper bound $\kappa_n(\Phi) \leq \sqrt{n}$ is immediate since it holds for any homogeneous Hilbertian operator space (Example 22). Let C_n denote ℓ_2^n with the column operator space structure, i.e., $C_n = \text{span}\{e_{1,i} \mid i \in \{1, \dots, n\}\}$. It is shown in [27, Equation (10.22)] that the identities $\Phi_n \rightarrow C_n$ have uniformly bounded cb-norms. Therefore, $\kappa_n(\Phi(I)) \gtrsim \kappa_n(C)$. Since, $\kappa_n(C) = \sqrt{n}$, the result follows. □

In the first version of this paper, based on the examples for which we have been able to calculate $\kappa_n(X)$, we asked whether the inequality in Proposition 23 is always an equality, and also asked about the possible growth rates for the sequence $(\kappa_n(X))_{n=1}^\infty$. We thank the referee for providing us with the proofs of the following two propositions, which completely answer the aforementioned questions.

Proposition 28. *For any homogeneous Hilbertian operator space X and any $n \in \mathbb{N}$, we have $\kappa_n(X)\kappa_n(X^*) = \sqrt{n}$.*

Proof. Note that, using \otimes to denote the injective tensor product of operator spaces, and letting $(e_i)_i$ be an orthonormal basis for X ,

$$\kappa_n(X) = \left\| \sum_{i=1}^n e_{i,1} \otimes e_i \right\|_{M_n(X)} = \left\| \sum_{i=1}^n e_{i,1} \otimes e_i \right\|_{C_n \otimes X} = \|\text{Id} : R_n \rightarrow X_n\|_{\text{cb}}$$

where we have used the standard identification $C_n \otimes X \equiv \text{CB}(C_n^*, X)$, and $\text{Id} : R_n \rightarrow X_n$ is the canonical isometry sending an orthonormal basis of R_n to one of X_n . Therefore,

$$\kappa_n(X)\kappa_n(X^*) = \|\text{Id} : R_n \rightarrow X_n\|_{\text{cb}} \|\text{Id} : X_n \rightarrow C_n\|_{\text{cb}}.$$

On the other hand, by [27, Theorem 7.8], there exist $u : R_n \rightarrow X_n$ and $v : X_n \rightarrow C_n$ such that $vu = \text{Id}$ and $\|u\|_{\text{cb}} \|v\|_{\text{cb}} = \sqrt{n}$. Next we use Zhang’s averaging technique [32, 33] (see also [27, Proposition 10.1] for a very similar proof using this type of argument). By homogeneity and using the singular value decomposition of an $n \times n$ matrix, we can assume that u is a diagonal operator with diagonal entries $\lambda_1, \dots, \lambda_n > 0$. Thus v must also be diagonal, with reciprocal diagonal entries. If $S : R_n \rightarrow R_n$ and $T : X_n \rightarrow X_n$ denote the “cyclical shift” isometries given by $Se_{1,i} = e_{1,i+1}$ and $Te_i = e_{i-1}$ (where the indices are taken modulo n), note that TuS is diagonal with diagonal entries $\lambda_2, \dots, \lambda_n, \lambda_1$. Once again by homogeneity, $\|TuS\|_{\text{cb}} = \|u\|_{\text{cb}}$. It now follows from applying similar arguments with higher powers of S and T , together with the triangle inequality, that

$$\|u\|_{\text{cb}} \geq \frac{1}{n} \left\| \sum_{i=1}^n T^i u S^i \right\|_{\text{cb}} = \frac{\lambda_1 + \dots + \lambda_n}{n} \|\text{Id} : R_n \rightarrow X_n\|_{\text{cb}},$$

and analogously

$$\|v\|_{\text{cb}} \geq \frac{\lambda_1^{-1} + \dots + \lambda_n^{-1}}{n} \|\text{Id} : X_n \rightarrow C_n\|_{\text{cb}}.$$

Applying the Arithmetic Mean-Geometric Mean inequality we thus conclude

$$\sqrt{n} = \|u\|_{\text{cb}} \|v\|_{\text{cb}} \geq \|\text{Id} : R_n \rightarrow X_n\|_{\text{cb}} \|\text{Id} : X_n \rightarrow C_n\|_{\text{cb}} \geq \|\text{Id} : R_n \rightarrow C_n\|_{\text{cb}} = \sqrt{n},$$

yielding the desired conclusion. □

Proposition 29.

- (1) *Let X be a homogeneous Hilbertian operator space. Then the function $n \in \mathbb{N} \mapsto n^{-1/2}\kappa_n(X) \in \mathbb{R}$ is decreasing.*
- (2) *Let $\phi : \mathbb{N} \rightarrow \mathbb{R}$ be an increasing function satisfying that $\phi(1) = 1$ and $n \mapsto n^{-1/2}\phi(n)$ is decreasing. Then there exists an (infinite dimensional) Hilbertian homogeneous operator space X such that for every $n \in \mathbb{N}$ we have $\kappa_n(X) = \phi(n)$. (Note that these conditions are optimal, because of part (1) and the clear fact that the sequence of the κ_n ’s is increasing and takes the value 1 at 1).*

Proof.

(1). Let $X \subseteq \mathcal{B}(H)$ be a completely isometric embedding. Then, once again denoting by $(e_i)_i$ an orthonormal basis for X ,

$$\kappa_n(X)^2 = \left\| \sum_{i=1}^n e_{i,1} \otimes e_i \right\|_{M_n(\mathcal{B}(H))}^2 = \left\| \sum_{i,j=1}^n (e_{j,1}^* \otimes e_j^*)(e_{i,1} \otimes e_i) \right\|_{M_n(\mathcal{B}(H))} = \left\| \sum_{i=1}^n e_i^* e_i \right\|_{\mathcal{B}(H)}. \quad (12)$$

Therefore,

$$\kappa_{n+1}(X)^2 = \left\| \sum_{i=1}^{n+1} e_i^* e_i \right\|_{\mathcal{B}(H)} = \frac{1}{n} \left\| \sum_{j=1}^{n+1} \sum_{i \neq j} e_i^* e_i \right\|_{\mathcal{B}(H)} \leq \frac{1}{n} \sum_{j=1}^{n+1} \left\| \sum_{i \neq j} e_i^* e_i \right\|_{\mathcal{B}(H)} = \frac{n+1}{n} \kappa_n(X)^2,$$

from where it follows that $(n+1)^{-1/2}\kappa_{n+1}(X) \leq n^{-1/2}\kappa_n(X)$.

(2). For each $n \in \mathbb{N}$, let \mathcal{C}_n be the set of all contractions of rank at most n on ℓ_2 . We define an operator space X by setting, for each $m \in \mathbb{N}$ and each $x \in M_m(\ell_2)$,

$$\|x\|_{M_m(X)} = \max \left\{ \|x\|_{M_m(R)}, \sup_{n \geq 1} \sup_{P \in \mathcal{C}_n} n^{-1/2} \phi(n) \|P_m x\|_{M_m(C)} \right\}.$$

It is clear that these are norms satisfying Ruan’s axioms, and that at level $k = 1$ the norm agrees with the usual norm on ℓ_2 . The resulting operator space is also homogeneous, since for a linear map $u : \ell_2 \rightarrow \ell_2$ we have $\|u_m x\|_{M_m(R)} \leq \|u\| \|x\|_{M_m(R)}$ (by homogeneity of R) and $\sup_{P \in \mathcal{C}_n} \|P_m u_m x\|_{M_m(C)} \leq \|u\| \sup_{Q \in \mathcal{C}_n} \|Q_m x\|_{M_m(C)}$ (because the rank of Pu is at most that of P). Recall that for $y = \sum_{i=1}^m e_{i,1} \otimes e_i$ we have $\|y\|_{M_m(R)} = 1$ and $\|y\|_{M_m(C)} = \sqrt{m}$. For any contraction P on ℓ_2 , since $P_m y = \sum_{i=1}^m e_{i,1} \otimes P e_i$ a calculation analogous to (12) together with the triangle inequality shows that we have $\|P_m y\|_{M_m(C)} \leq \sqrt{m}$, and therefore for $n \geq m$ we have $\sup_{P \in \mathcal{C}_n} \|P_m y\|_{M_m(C)} = \sqrt{m}$. In the case $n < m$, for any $P \in \mathcal{C}_n$ we have, using once again the standard identification $C_m \otimes C_m \cong \text{CB}(C_m^*, C_m)$

$$\begin{aligned} \|P_m y\|_{M_m(C)} &= \left\| \sum_{i=1}^m e_{i,1} \otimes P e_i \right\|_{M_m \otimes C_m} = \left\| \sum_{i=1}^m e_{i,1} \otimes P e_{i,1} \right\|_{C_m \otimes C_m} \\ &= \|P : R_m \rightarrow C_m\|_{\text{cb}} = \|P\|_{\text{HS}} \leq \sqrt{\text{rank}(P)} \|P\| \leq \sqrt{n}, \end{aligned}$$

where we have used that for a linear map $R_m \rightarrow C_m$ the completely bounded norm agrees with the Hilbert–Schmidt norm [27, Equation (1.5)]. Moreover, the inequality above is achieved by taking P to be the orthogonal projection onto a subspace generated by an n -element subset of the canonical basis. Therefore, we conclude that

$$\kappa_m(X) = \|y\|_{M_m(X)} = \sup_{n \geq 1} n^{-1/2} \phi(n) \sqrt{\min\{n, m\}} = \phi(m)$$

as desired. □

5. Small scale rigidity of $\kappa_n(E)$

In this section, we now show how the κ_n ’s impact the existence of small scale Lipschitz maps between operator spaces. The results herein will culminate in Theorem 7. For that, in order to obtain the equivalence $\kappa_n(X) \simeq \kappa_n(Y)$, we prove the inequalities \lesssim and \gtrsim separately, see Theorems 32 and 35, respectively. We emphasize this here since each of these partial results have weaker hypotheses than Theorem 7.

We start with a lemma relating the κ_n ’s with a similar quantity computed with respect to weakly null sequences instead of orthonormal sets.

Lemma 30. *Let X be an infinite dimensional homogeneous Hilbertian operator space, $b \geq a > 0$, and let $(x_m)_m$ be a weakly null sequence in X such that $\|x_m\| \in [a, b]$ for all $m \in \mathbb{N}$. Then, for all $\varepsilon > 0$, there is an infinite $\mathbb{M} \subseteq \mathbb{N}$ such that*

$$(a - \varepsilon) \kappa_n(X) \leq \left\| \begin{bmatrix} x_{m_1} & 0 & \dots & 0 \\ x_{m_2} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_{m_n} & 0 & \dots & 0 \end{bmatrix} \right\|_{M_n(X)} \leq (b + \varepsilon) \kappa_n(X).$$

for all $m_1 < \dots < m_n \in \mathbb{M}$.

Proof. Let $(e_n)_n$ be an orthonormal sequence in X and $\varepsilon > 0$, without loss of generality, assume $\varepsilon < a$. By going to a subsequence if necessary, a standard gliding-hump argument allows us to assume that the assignment $x_m \mapsto e_m$ defines an isomorphism $T : \overline{\text{span}}\{x_m \mid m \in \mathbb{N}\} \rightarrow \overline{\text{span}}\{e_m \mid m \in \mathbb{N}\}$ such that $\|T\| \leq 1/(a - \varepsilon)$ and $\|T^{-1}\| \leq b + \varepsilon$. As X is a homogeneous space, we must also have $\|T\|_{\text{cb}} \leq 1/(a - \varepsilon)$ and $\|T^{-1}\|_{\text{cb}} \leq b + \varepsilon$, so the result follows. □

We say that a map $f: B_X \rightarrow Y$ is *completely bounded in small scale* if there is $M > 0$ such that

$$\|f_n([x_{ij}])\|_{M_n(X)} \leq M \text{ for all } n \in \mathbb{N} \text{ and all } [x_{ij}] \in B_{M_n(X)}.$$

In order to obtain restrictions for the existence of certain maps, we must also demand the maps to satisfy some nontrivial lower estimates. The next definition is an “operator space/small scale” version of the the compression exponent of a metric space X into another space Y introduced in [16].

Definition 31. Let X and Y be operator spaces. We denote by $\alpha_Y^{ss}(X)$ the infimum of all $\alpha \geq 1$ for which there is completely bounded in small scale $f: B_X \rightarrow Y$ and $L \geq 1$ such that

$$\|f(x) - f(y)\|_Y \geq \frac{1}{L} \|x - y\|_X^\alpha$$

for all $x, y \in B_X$.⁵

Theorem 32. Given arbitrary homogeneous Hilbertian operator spaces X and Y , we have that $\kappa_n(X)^{\alpha_Y^{ss}(X)} \gtrsim \kappa_n(Y)$.

Proof. Let $f: B_X \rightarrow Y$ be completely bounded in small scale. Suppose $\alpha, L \geq 1$ are such that

$$\|f(x) - f(y)\|_Y \geq \frac{1}{L} \|x - y\|_X^\alpha \text{ for all } x, y \in B_X. \tag{13}$$

As f is completely bounded in small scale, fix $M > 0$ such that

$$\|f_n(a)\| \leq M \text{ for all } a \in B_{M_n(X)}. \tag{14}$$

Fix $n \in \mathbb{N}$ and let $(e_j)_j$ be an orthonormal sequence in X . Appealing to Rosenthal’s ℓ_1 -theorem (see [29, The Main Theorem]), by going to a subsequence if necessary, we can assume $(f(e_j/\kappa_n(X)))_j$ is weakly Cauchy. In particular, $(f(e_{2j-1}/\kappa_n(X)) - f(e_{2j}/\kappa_n(X)))_j$ is weakly null. Moreover, (13) implies that

$$\left\| f\left(\frac{e_{2j-1}}{\kappa_n(X)}\right) - f\left(\frac{e_{2j}}{\kappa_n(X)}\right) \right\|_Y \geq \frac{1}{L\kappa_n(X)^\alpha}$$

for all $j \in \mathbb{N}$. Hence, going to a further subsequence if necessary, Lemma 30 allows us to assume that

$$\left\| \begin{bmatrix} f(e_1/\kappa_n(X)) - f(e_2/\kappa_n(X)) & 0 & \dots & 0 \\ f(e_3/\kappa_n(X)) - f(e_4/\kappa_n(X)) & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ f(e_{2n-1}/\kappa_n(X)) - f(e_{2n}/\kappa_n(X)) & 0 & \dots & 0 \end{bmatrix} \right\|_{M_n(Y)} \geq \frac{\kappa_n(Y)}{2L\kappa_n(X)^\alpha}.$$

Let $c_n \in M_n(X)$ be the operator in $\mathcal{B}(\ell_2)$ whose $(j, 1)$ -coordinate is $e_{2j-1}/\kappa_n(X)$, for all $j \in \{1, \dots, n\}$, and all other coordinates are zero, and let $d_n \in M_n(X)$ be the operator in $\mathcal{B}(\ell_2)$ whose $(j, 1)$ -coordinate is $e_{2j}/\kappa_n(X)$, for all $j \in \{1, \dots, n\}$, and all other coordinates are zero. So, $\|c_n\|_{M_n(X)} = \|d_n\|_{M_n(X)} = 1$ and (14) gives

$$\|f_n(c_n) - f_n(d_n)\|_{M_n(Y)} \leq 2M.$$

As

$$\|f_n(c_n) - f_n(d_n)\|_{M_n(X)} = \left\| \begin{bmatrix} f(e_1/\kappa_n(X)) - f(e_2/\kappa_n(X)) & 0 & \dots & 0 \\ f(e_3/\kappa_n(X)) - f(e_4/\kappa_n(X)) & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ f(e_{2n-1}/\kappa_n(X)) - f(e_{2n}/\kappa_n(X)) & 0 & \dots & 0 \end{bmatrix} \right\|_{M_n(Y)},$$

⁵We point out that the compression exponent of a metric space X into another space Y (see [16]) considers the supremum of all $\alpha \leq 1$ for which a similar inequality holds. This difference comes from the fact that the compression exponent deals with large scale geometry, while the exponent $\alpha_Y^{ss}(X)$ is supposed to capture small scale behavior.

the arbitrariness of $n \in \mathbb{N}$ implies that

$$\frac{\kappa_n(Y)}{2L\kappa_n(X)^\alpha} \leq 2M \text{ for all } n \in \mathbb{N}.$$

This finishes the proof. □

Corollary 33. *Let $\theta, \gamma \in [0, 1]$,*

- $X \in \{(R, C)_\theta, (\min(\ell_2), \max(\ell_2))_\theta, (R \cap C, R + C)_\theta\}$, and
- $Y \in \{(R, C)_\gamma, (\min(\ell_2), \max(\ell_2))_\gamma, (R \cap C, R + C)_\gamma\}$.

Then, $\alpha_Y^{ss}(X) \geq \gamma/\theta$.

Proof. Corollaries 25 and 26 show that $\kappa_n(X) = n^{\theta/2}$ and $\kappa_n(Y) = n^{\gamma/2}$. Hence, Theorem 32 implies that $n^{\alpha_Y^{ss}(X)\theta/2} \gtrsim n^{\gamma/2}$, so, $\alpha_Y^{ss}(X) \geq \gamma/\theta$. □

The next definition considers another approach to obtain lower bounds for the small scale distortion of maps between operator spaces. Similar definitions have already been studied by C. Rosenthal and the first named author in the context of Banach spaces under the names of *uncollapsed* and *almost uncollapsed* maps, see [4, 28].

Definition 34. *Let X and Y be operator spaces. We call a map $f: B_X \rightarrow Y$ completely almost uncollapsed if there are $\varepsilon > 0$ and $t \in (0, 1)$ such that*

$$\|[x_{ij}] - [z_{ij}]\|_{M_n(X)} = t \text{ implies } \|f_n([x_{ij}]) - f_n([z_{ij}])\|_{M_n(Y)} \geq \varepsilon$$

for all $n \in \mathbb{N}$ and all $[x_{ij}], [z_{ij}] \in B_{M_n(X)}$.

Theorem 35. *Let X and Y be homogeneous Hilbertian operator spaces. If there is a Lipschitz map $f: B_X \rightarrow Y$ which is completely almost uncollapsed, then $\kappa_n(X) \lesssim \kappa_n(Y)$.*

Proof. The proof resembles the one of Theorem 32 but with the arguments for the upper and lower estimates replacing each other. For this reason, we start this proof letting $X, Y, (c_n)_n$ and $(d_n)_n$ be as in there.

Since f is completely almost uncollapsed, fix $t \in (0, 1)$ and $\varepsilon > 0$ such that

$$\|[x_{ij}] - [z_{ij}]\|_{M_n(X)} = t \text{ implies } \|f_n([x_{ij}]) - f_n([z_{ij}])\|_{M_n(Y)} \geq \varepsilon$$

for all $n \in \mathbb{N}$ and all $[x_{ij}], [z_{ij}] \in B_{M_n(X)}$. Since X is a homogeneous Hilbertian space, it immediately follows that $\|c_n - d_n\| = \sqrt{2}$ for all $n \in \mathbb{N}$. Therefore, replacing each of the c_n 's and d_n 's by $(t/\sqrt{2})c_n$ and $(t/\sqrt{2})d_n$, respectively, we can assume that

$$\|c_n - d_n\| = t \text{ for all } n \in \mathbb{N}.$$

Our choice of t and ε then give that

$$\|f_n(c_n) - f_n(d_n)\|_{M_n(Y)} \geq \varepsilon \text{ for all } n \in \mathbb{N}. \tag{15}$$

On the other hand, letting $L = \text{Lip}(f)$, we have

$$\left\| f\left(\frac{t}{\sqrt{2}\kappa_n(X)} e_{2j-1}\right) - f\left(\frac{t}{\sqrt{2}\kappa_n(X)} e_{2j}\right) \right\|_Y \leq \frac{L}{\kappa_n(X)} \text{ for all } j, n \in \mathbb{N}.$$

Hence, proceeding as in the proof of Theorem 32 and passing to a subsequence if necessary, Lemma 30 gives that

$$\|f_n(c_n) - f_n(d_n)\|_{M_n(Y)} \leq \frac{2L\kappa_n(Y)}{\kappa_n(X)} \text{ for all } n \in \mathbb{N}. \tag{16}$$

Equations (15) and (16) together then imply that $\kappa_n(X) \lesssim \kappa_n(Y)$ as desired. □

Corollary 36. *Let $\theta, \gamma \in [0, 1]$,*

- $X \in \{(R, C)_\theta, (\min(\ell_2), \max(\ell_2))_\theta, (R \cap C, R + C)_\theta\}$, and
- $Y \in \{(R, C)_\gamma, (\min(\ell_2), \max(\ell_2))_\gamma, (R \cap C, R + C)_\gamma\}$.

If there is a Lipschitz map $f: B_X \rightarrow Y$ which is completely almost uncollapsed, then $\theta \leq \gamma$.

Proof. Corollaries 25 and 26 show that $\kappa_n(X) = n^{\theta/2}$ and $\kappa_n(Y) = n^{\gamma/2}$. Hence, Theorem 35 implies that $n^{\theta/2} \lesssim n^{\gamma/2}$, so, $\theta \leq \gamma$. \square

Proof of Theorem 7. Suppose there is a completely Lipschitz in small scale map $f: B_X \rightarrow Y$ which is completely almost uncollapsed. In particular, f is completely bounded in small scale and $\alpha_Y^{\text{ss}}(X) = 1$. Therefore, Theorem 32 gives that $\kappa_n(Y) \lesssim \kappa_n(X)$. On the other hand, Theorem 35 implies $\kappa_n(X) \lesssim \kappa_n(Y)$. \square

Proof of Corollary 8. This follows from Corollaries 33 and 36. \square

6. A foray into the non Hilbertian setting

In this final section, we go beyond the homogeneous Hilbertian case and provide lower bounds for compression exponents $\alpha_Y^{\text{ss}}(X)$ for non-Hilbertian operator spaces X .

Theorem 38 below is in a sense a localized version of Theorem 32. The only significant difference in its proof is that instead of using tools such as Rosenthal's ℓ_1 -theorem to extract a subsequence from an infinite sequence, we will extract a subsequence of a finite sequence using the following slight generalization of the original Bourgain–Tzafriri restricted invertibility theorem [3, Theorem 1.2] (we point out that the version below follows e.g. from [31, Theorem 2]).

Theorem 37. *There exists a universal constant $D > 0$ such that whenever $T: \ell_2^n \rightarrow \ell_2^n$ is a linear map with $\|Te_j\| \geq 1$ for each $1 \leq j \leq n$, where $\{e_j\}_{j=1}^n$ is the canonical basis of ℓ_2^n , then there exists a subset $\sigma \subseteq \{1, 2, \dots, n\}$ of cardinality $|\sigma| \geq Dn / \|T\|^2$ such that for any choice of scalars $\{a_j\}_{j \in \sigma}$ we have*

$$\left\| \sum_{j \in \sigma} a_j Te_j \right\| \geq D \left(\sum_{j \in \sigma} |a_j|^2 \right)^{1/2}.$$

For a homogeneous Hilbertian operator space Z and $n \in \mathbb{N}$, we denote by Z_n an n -dimensional subspace of Z (they are all completely isometric by homogeneity, so there is no ambiguity). Note that by Example 22, in the statement of the following theorem the interval $[0, 1/2]$ covers all possible values of the constant c .

Theorem 38. *Let Z and Y be homogeneous Hilbertian operator spaces. Suppose that $\kappa_n(Y) \gtrsim n^c$ for some $c \in [0, 1/2]$. If X is an operator space for which there exist a constant $A \geq 1$ and a sequence of injective linear maps $\varphi^n: Z_n \rightarrow X$ such that $\|(\varphi^n)_n\| \cdot \|(\varphi^n)^{-1}\| \leq A$, then $\kappa_n(Z) \alpha_Y^{\text{ss}}(X) \gtrsim n^{c/(1+2c)}$.*

Proof. Without loss of generality, let us assume that for each $n \in \mathbb{N}$ we have $\|(\varphi^n)_n\| = 1$ and $\|(\varphi^n)^{-1}\| \leq A$. Suppose there is a map $f: B_X \rightarrow Y$ which is completely bounded in small scale and numbers $\alpha, L \geq 1$ such that

$$\|f(x) - f(y)\|_Y \geq \frac{1}{L} \|x - y\|_X^\alpha \text{ for all } x, y \in B_X. \quad (17)$$

As f is completely bounded in small scale, fix $M > 0$ such that

$$\|f_n(a)\|_{M_n(Y)} \leq M \text{ for all } a \in B_{M_n(X)}. \quad (18)$$

Fix $n \in \mathbb{N}$ and let $(e_j)_{j=1}^{2n}$ be an orthonormal basis for Z_{2n} . For $1 \leq j \leq 2n$, let $\tilde{e}_j = \varphi^{2n}(e_j)$, and note that $\|\tilde{e}_j\| \leq \|e_j\| = 1$. Note that (17) implies that for $1 \leq j \leq n$,

$$\begin{aligned} \left\| f\left(\frac{\tilde{e}_{2j-1}}{\kappa_n(Z)}\right) - f\left(\frac{\tilde{e}_{2j}}{\kappa_n(Z)}\right) \right\|_Y &\geq \frac{1}{L} \left\| \frac{\tilde{e}_{2j-1}}{\kappa_n(Z)} - \frac{\tilde{e}_{2j}}{\kappa_n(Z)} \right\|_X^\alpha = \frac{1}{L\kappa_n(Z)^\alpha} \|\tilde{e}_{2j-1} - \tilde{e}_{2j}\|_X^\alpha \\ &\geq \frac{1}{L\kappa_n(Z)^\alpha A^\alpha} \|e_{2j-1} - e_{2j}\|_Z^\alpha \geq \frac{1}{L\kappa_n(Z)^\alpha A^\alpha}. \end{aligned}$$

Let $c_n \in M_n(X)$ be the matrix whose $(j, 1)$ -entry is $\tilde{e}_{2j-1}/\kappa_n(Z)$, for $1 \leq j \leq n$, and all other entries are zero, and let $d_n \in M_n(X)$ be the matrix in $M_n(X)$ whose $(j, 1)$ -entry is $\tilde{e}_{2j}/\kappa_n(Z)$, for all $1 \leq j \leq n$, and all other entries are zero. Since $\|(\varphi^{2n})_{2n}\| = 1$, note that $\|c_n\|_{M_n(X)} \leq \kappa_n(Z)/\kappa_n(Z) = 1$. Analogously, $\|d_n\|_{M_n(X)} \leq 1$. Therefore, by (18), we conclude that

$$\|f_n(c_n) - f_n(d_n)\|_{M_n(Y)} \leq 2M.$$

Letting $y_j = f(\tilde{e}_{2j-1}/\kappa_n(Z)) - f(\tilde{e}_{2j}/\kappa_n(Z))$ for $1 \leq j \leq n$, the previous inequality means that

$$\left\| \begin{bmatrix} y_1 & 0 & \dots & 0 \\ y_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ y_n & 0 & \dots & 0 \end{bmatrix} \right\|_{M_n(Y)} \leq 2M. \tag{19}$$

Now, for any $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{C}$ it follows from Ruan’s axioms that

$$\left\| \sum_{j=1}^n \lambda_j y_j \right\|_Y \leq \|[\lambda_1 \ \lambda_2 \ \dots \ \lambda_n]\| \left\| \begin{bmatrix} y_1 & 0 & \dots & 0 \\ y_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ y_n & 0 & \dots & 0 \end{bmatrix} \right\|_{M_n(Y)} \left\| \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\| \leq 2M \left(\sum_{j=1}^n |\lambda_j|^2 \right)^{1/2},$$

that is, the operator $T : Y_n \rightarrow Y$ which sends the j -th element of the canonical basis to y_j has norm at most $2M$. Since $\|y_j\|_Y \geq L^{-1}\kappa_n(Z)^{-\alpha}A^{-\alpha}$ for each $1 \leq j \leq n$, it follows from Theorem 37 that there is a universal constant D such that there is a subset $\sigma = \{\sigma_1, \dots, \sigma_m\} \subseteq \{1, 2, \dots, n\}$ of cardinality $m \geq \frac{Dn}{4M^2} L^{-2}\kappa_n(Z)^{-2\alpha}A^{-2\alpha}$ such that the operator T , when restricted to the coordinate subspace corresponding to σ , is invertible and the norm of the inverse is at most $D^{-1}L\kappa_n(Z)^\alpha A^\alpha$. By homogeneity, the cb-norm of the inverse of said restriction is also bounded by this same number. Therefore,

$$\left\| \begin{bmatrix} y_{\sigma_1} & 0 & \dots & 0 \\ y_{\sigma_2} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ y_{\sigma_m} & 0 & \dots & 0 \end{bmatrix} \right\|_{M_m(Y)} \geq DL^{-1}\kappa_n(Z)^{-\alpha}A^{-\alpha}\kappa_m(Y), \tag{20}$$

and thus we have from (19) and (20) that

$$2M \gtrsim \frac{\kappa_m(Y)}{(\kappa_n(Z)A)^\alpha} \gtrsim \frac{(n(\kappa_n(Z)A)^{-2\alpha})^c}{(\kappa_n(Z)A)^\alpha} = \frac{n^c}{(\kappa_n(Z)A)^{\alpha(1+2c)}},$$

where the implied constants are independent of both n and α . It then follows that $(\kappa_n(Z)A)^{\alpha(1+2c)} \gtrsim n^c$, so $(\kappa_n(Z)A)^\alpha \gtrsim n^{c/(1+2c)}$ from where the desired result follows. \square

Remark 39. If the space Z in Theorem 38 satisfies $\kappa_n(Z) \lesssim n^d$ for some constant d , we get the lower bound $\alpha_Y^{\text{ss}}(X) \geq \frac{c}{d(1+2c)}$. However, this bound is trivial when $d \geq 1/4$: since $c \in [0, 1/2]$ by Example 22, we get $1 \geq \frac{c}{d(1+2c)}$. In particular, Theorem 38 gives no information when $Z = \text{OH} = (R, C)_{1/2}$.

Remark 40. By [22, Theorem 3.3], the sequence of maps $\varphi^n : Z_n \rightarrow X$ in the hypotheses of Theorem 38 is guaranteed to exist whenever X has weak cotype $(2, Z^*)$: we even get the stronger condition $\|\varphi^n\|_{\text{cb}} \|(\varphi^n)^{-1}\| \leq A$ for some constant A . Such maps are called *complete semi-isomorphisms* in the literature, see [24, Section 3]. Since we will not need the aforementioned notion of weak cotype in this paper, the reader is directed to [22, Section 2] for the definition.

As a first example of the consequences one can obtain from Theorem 38, we state one that easily follows from our previous calculations of κ_n 's in specific cases.

Corollary 41. Let $\theta, \gamma \in [0, 1]$,

- $Z \in \{(R, C)_\theta, (\min(\ell_2), \max(\ell_2))_\theta, (R \cap C, R + C)_\theta\}$, and
- $Y \in \{(R, C)_\gamma, (\min(\ell_2), \max(\ell_2))_\gamma, (R \cap C, R + C)_\gamma\}$.

If X is an operator space for which there exist a constant $A \geq 1$ and a sequence of injective linear maps $\varphi^n : Z_n \rightarrow X$ such that $\|(\varphi^n)_n\| \cdot \|(\varphi^n)^{-1}\| \leq A$, (in particular, if X has weak cotype $(2, Z^*)$), then $\alpha_Y^{\text{ss}}(X) \geq \frac{\gamma}{(1+\gamma)\theta}$.

Now we present an example for some specific operator spaces X , namely the Schatten classes S_p . See [26, Chapter 1] for the definition of their operator space structure.

Corollary 42. Let $1 < p \leq 2$ and $p' \in [2, \infty)$ be such that $1/p + 1/p' = 1$. Let Y be a homogeneous Hilbertian space such that $\kappa_n(Y) \gtrsim n^c$ for some $c \in [0, 1/2]$. Then $\alpha_Y^{\text{ss}}(S_p) \geq \frac{2p'c}{(1+2c)}$. Moreover, if $p = 1$ and $c > 0$, then $\alpha_Y^{\text{ss}}(S_1) = \infty$.

Proof. By Item (1) of the first theorem in p. 222 of [21], S_p has cotype $(2, (R \cap C, R + C)_{1/p})$, which implies weak cotype $(2, (R \cap C, R + C)_{1/p'})$. Note that $(R \cap C, R + C)_{1/p'} = Z^*$ for $Z = (R \cap C, R + C)_{1/p}$ by (11), so by Remark 40 we can apply Theorem 38 to get $\kappa_n(Z) \alpha_Y^{\text{ss}}(S_p) \gtrsim n^{c/(1+2c)}$. But $\kappa_n(Z) = n^{1/2p'}$ by Corollary 26, yielding the conclusion. \square

Furthermore, we next show that Theorem 38 can always be applied to non-Hilbertian operator spaces.

Corollary 43. Let X be an infinite-dimensional operator space, and let Z be a Dvoretzky space for X . Let Y be a homogeneous Hilbertian operator space such that $\kappa_n(Y) \gtrsim n^c$ for some $c \in [0, 1/2]$. Then $\kappa_n(Z) \alpha_Y^{\text{ss}}(X) \gtrsim n^{c/(1+2c)}$. In particular, if X is a minimal operator space and $c > 0$ then $\alpha_Y^{\text{ss}}(X) = \infty$.

Proof. By the definition of a Dvoretzky space, Z is contained in an ultrapower of X completely isometrically. Therefore, approximating the finite dimensional pieces of this containment by elements in X , it follows straightforwardly that, for all $\varepsilon > 0$ and all $n \in \mathbb{N}$, X contains a copy of the n -dimensional subspaces of Z by maps whose n -amplifications are isomorphic embeddings with distortion at most $1 + \varepsilon$. In particular, for any $n \in \mathbb{N}$ and $\varepsilon > 0$, we can find an injective linear map $\varphi^n : Z_n \rightarrow X$ such that $\|(\varphi^n)^{-1}\| \leq 1$ and $\|(\varphi^n)_n\| \leq 1 + \varepsilon$; here $Z_n \subseteq Z$ is a subspace of dimension n . Therefore, the hypotheses of Theorem 38 are satisfied which yields the conclusion.

If X is minimal, since both ultraproducts and subspaces of minimal operator spaces are also minimal, then the only possible (separable) Dvoretzky space for X is $\min(\ell_2)$. Since $\kappa_n(\min(\ell_2)) = 1$ by Corollary 25, we conclude $\alpha_Y^{\text{ss}}(X) = \infty$. \square

Proof of Corollary 9. Just as in the proof of Theorem 7, the existence of a completely Lipschitz in small scale embedding $B_X \rightarrow Y$ implies $\alpha_Y^{\text{ss}}(X) = 1$. Thus, the desired conclusion follows from Corollary 43. \square

As we have seen in the proof of Corollary 43 above, for any infinite-dimensional operator space X there is a homogeneous Hilbertian Z which satisfies the condition in Theorem 38. The opposite is also true: for a given homogeneous Hilbertian operator space Z , it is not difficult to find nonhomogeneous and not Hilbertian operator spaces X satisfying the desired condition (and not containing Z). For example, take the ℓ_p -sum $(\bigoplus_{n=1}^{\infty} Z_n)_{\ell_p}$ for $p \in (1, \infty) \setminus \{2\}$, which obviously contains completely isometric copies of the Z_n but is not homogeneous since it also contains a completely 1-complemented copy of ℓ_p , which is not homogeneous [19, p. 137].

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References

- [1] F. Baudier, G. Lancien and T. Schlumprecht, “The coarse geometry of Tsirelson’s space and applications”, *J. Am. Math. Soc.* **31** (2018), no. 3, pp. 699–717.
- [2] D. Blecher and C. Le Merdy, *Operator algebras and their modules—an operator space approach*, London Mathematical Society Monographs. New Series, Clarendon Press, 2004, pp. x+387. Oxford Science Publications.
- [3] J. Bourgain and L. Tzafriri, “Invertibility of “large” submatrices with applications to the geometry of Banach spaces and harmonic analysis”, *Isr. J. Math.* **57** (1987), no. 2, pp. 137–224.
- [4] B. M. Braga, “On weaker notions of nonlinear embeddings between Banach spaces”, *J. Funct. Anal.* **274** (2018), no. 11, pp. 3149–3169.
- [5] B. M. Braga, “Towards a theory of coarse geometry of operator spaces”, *Isr. J. Math.* **259** (2024), no. 2, pp. 527–558.
- [6] B. M. Braga and J. A. Chávez-Domínguez, “Completely coarse maps are \mathbb{R} -linear”, *Proc. Am. Math. Soc.* **149** (2021), no. 3, pp. 1139–1149.
- [7] B. M. Braga, J. A. Chávez-Domínguez and T. Sinclair, “Lipschitz geometry of operator spaces and Lipschitz-free operator spaces”, *Math. Ann.* **388** (2024), no. 1, pp. 1053–1090.
- [8] B. M. Braga and T. Oikhberg, “Coarse geometry of operator spaces and complete isomorphic embeddings into ℓ_1 and c_0 -sums of operator spaces”, *Math. Z.* **304** (2023), no. 3, article no. 52 (19 pages).
- [9] E. Christensen, “On the complete boundedness of the Schur block product”, *Proc. Am. Math. Soc.* **147** (2019), no. 2, pp. 523–532.
- [10] E. Christensen, E. Effros and A. Sinclair, “Completely bounded multilinear maps and C^* -algebraic cohomology”, *Invent. Math.* **90** (1987), no. 2, pp. 279–296.
- [11] A. Defant and D. Wiesner, “Polynomials in operator space theory”, *J. Funct. Anal.* **266** (2014), no. 9, pp. 5493–5525.
- [12] S. Dineen and C. Radu, “Completely bounded polynomials between operator spaces”, *Math. Scand.* **107** (2010), no. 2, pp. 249–266.

- [13] E. Effros, M. Junge and Z.-J. Ruan, “Integral mappings and the principle of local reflexivity for noncommutative L^1 -spaces”, *Ann. Math.* **151** (2000), no. 1, pp. 59–92.
- [14] E. Effros and Z.-J. Ruan, *Operator spaces*, London Mathematical Society Monographs. New Series, Clarendon Press, 2000, pp. xvi+363.
- [15] G. Godefroy and N. J. Kalton, “Lipschitz-free Banach spaces”, *Stud. Math.* **159** (2003), no. 1, pp. 121–141. Dedicated to Professor Aleksander Pełczyński on the occasion of his 70th birthday.
- [16] E. Guentner and J. Kaminker, “Exactness and uniform embeddability of discrete groups”, *J. Lond. Math. Soc.* **70** (2004), no. 3, pp. 703–718.
- [17] S. Heinrich, “Ultraproducts in Banach space theory”, *J. Reine Angew. Math.* **313** (1980), pp. 72–104.
- [18] W. B. Johnson, J. Lindenstrauss and G. Schechtman, “Banach spaces determined by their uniform structures”, *Geom. Funct. Anal.* **6** (1996), no. 3, pp. 430–470.
- [19] M. Junge, *Factorization Theory for Spaces of Operators*, 1996. Habilitation Thesis, Kiel.
- [20] N. Kalton, “The nonlinear geometry of Banach spaces”, *Rev. Mat. Complut.* **21** (2008), no. 1, pp. 7–60.
- [21] H. Lee, “Type and cotype of operator spaces”, *Stud. Math.* **185** (2008), no. 3, pp. 219–247.
- [22] H. Lee, “Weak type $(2, H)$ and weak cotype $(2, H)$ of operator spaces”, *Houston J. Math.* **35** (2009), no. 4, pp. 1171–1201.
- [23] M. Mendel and A. Naor, “Metric cotype”, *Ann. Math.* **168** (2008), no. 1, pp. 247–298.
- [24] T. Oikhberg and H. Rosenthal, “Extension properties for the space of compact operators”, *J. Funct. Anal.* **179** (2001), no. 2, pp. 251–308.
- [25] G. Pisier, “Dvoretzky’s theorem for operator spaces”, *Houston J. Math.* **22** (1996), no. 2, pp. 399–416.
- [26] G. Pisier, *Non-commutative vector valued L_p -spaces and completely p -summing maps*, Astérisque, vol. 247, Société Mathématique de France, 1998, pp. vi+131.
- [27] G. Pisier, *Introduction to Operator Space Theory*, London Mathematical Society Lecture Note Series, Cambridge University Press, 2003, pp. viii+478.
- [28] C. Rosendal, “Equivariant geometry of Banach spaces and topological groups”, *Forum Math. Sigma* **5** (2017), article no. e22 (62 pages).
- [29] H. Rosenthal, “A characterization of Banach spaces containing l^1 ”, *Proc. Natl. Acad. Sci. USA* **71** (1974), pp. 2411–2413.
- [30] J. Schur, “Bemerkungen zur Theorie der beschränkten Bilinearformen mit unendlich vielen Veränderlichen”, *J. Reine Angew. Math.* **140** (1911), pp. 1–28.
- [31] D. Spielman and N. Srivastava, “An elementary proof of the restricted invertibility theorem”, *Isr. J. Math.* **190** (2012), pp. 83–91.
- [32] C. Zhang, *Representation and geometry of operator spaces*, PhD thesis, University of Houston, 1995.
- [33] C. Zhang, “Completely bounded Banach-Mazur distance”, *Proc. Edinb. Math. Soc., II. Ser.* **40** (1997), no. 2, pp. 247–260.