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Smooth Stable Foliations of Anosov Diffeomorphisms

Foliations stables et lisses des difféomorphismes d'Anosov

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Abstract. In this paper, we focus on the rigidity of C^{2+}-smooth codimension-one stable foliations of Anosov diffeomorphisms. Specifically, we show that if the regularity of these foliations is slightly bigger than 2, then they will have the same smoothness of the diffeomorphisms.

Résumé. Dans cet article, nous nous concentrons sur la rigidité des foliations stables de codimension un des difféomorphismes d'Anosov en C^{2+}. Plus précisément, nous montrons que si la régularité de ces foliations est légèrement supérieure à 2, alors elles auront la même régularité que les difféomorphismes.

Keywords. Anosov diffeomorphism, stable foliation, rigidity.

Mots-clés. difféomorphisme d'Anosov, foliation stables, rigidité.

2020 Mathematics Subject Classification. 37C05, 37C15, 37D20.

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1. Introduction

Let f be a C^r-smooth ($r \geq 1$) Anosov diffeomorphism of a smooth closed Riemannian manifold M, i.e, there exists a Df-invariant splitting $TM = E^s_f \oplus E^u_f$ such that Df is contracting on E^s_f and expanding on E^u_f, uniformly. It is well known that the distributions E^s_f and E^u_f are Hölder continuous and uniquely integrable to foliations $\mathscr{F}^s_f$ and $\mathscr{F}^u_f$, respectively with C^r-smooth leaves varing continuously with respect to C^r-topology. However, the regularity of these foliations may not be C^r. Indeed, if the diffeomorphism f is C^r for $r \geq 2$, the foliation $\mathscr{F}^s_f$ (or symmetrically $\mathscr{F}^u_f$) is absolutely continuous and if we further assume that it is codimension-one, then it is C^1-smooth [2, 7, 15] but could not be C^2 in general [2].

In 1991, Flaminio and Katok [3] proposed the following conjecture.

Conjecture 1 ([3]). *If the foliations $\mathscr{F}_f^s$ and $\mathscr{F}_f^u$ of a C^k-smooth Anosov diffeomorphism $f: M \to M$ are both C^2-smooth, then f is $C^{\max\{2,k\}}$ conjugate to a hyperbolic automophism of an infra-nilmainfold.*

This conjecture can be divided into two parts which are still open. One is the famous conjecture of Smale [20] which expects to classify the Anosov diffeomorphisms in the sense of topology, i.e, every Anosov diffeomorphism is topologically conjugate to a hyperbolic automophism of an infra-nilmainfold. The other one is a rigidity issue, i.e, whether the smooth foliation leads to a higher regularity of the conjugacy or not, since the conjugacy between two Anosov diffeomorphisms is usually Hölder continuous only.

The topological classification conjecture has some evidences [4, 13, 14]. For instance, when E_f^s (or E_f^u) is codimension-one, then M is a torus. Moreover, under the assumption of M is a nilmanifold, f is conjugate to a hyperbolic algebraic model. These are also why researches of rigidity usually focus on the toral Anosov diffeomorphisms. The rigidity issue has been extensively and deeply studied under some restriction of Lyapunov exponent [6, 12, 18]. However we know few of the rigidity on smooth foliation. Indeed, as far as authors know, there are only partial answer in [3, 5, 8, 12].

In [3], Flaminio and Katok proved that a volume-preserving Anosov diffeomorphism f of 2-torus $\mathbb{T}^2$ with $C^r(r \geq 2)$ stable and unstable foliations is C^r conjugate to a linear one. Moreover, they obtained a similar result for an Anosov diffeomorphism f of $\mathbb{T}^4$ preserving a symplectic form with C^∞-smooth stable and unstable foliations. However de la Llave [12, Theorem 6.3] constructed a counterexample on $\mathbb{T}^d(d \geq 4)$, precisely, for any $k \in \mathbb{N}$ there exist hyperbolic automorphism $A: \mathbb{T}^d \to \mathbb{T}^d$, Anosov diffeomorphism $f: \mathbb{T}^d \to \mathbb{T}^d$ and a C^k-conjugacy h between f with A such that h is not C^{k+1}-smooth.

As a corollary of [3], C^2-regularity of hyperbolic foliation on $\mathbb{T}^2$ implies higher-regularity of itself. In a same sense of such bootstrap of foliation, Katok and Hurder [8] proved that for a $C^r(r \geqslant 5)$ volume-preserving Anosov diffeomorphism f of $\mathbb{T}^2$, if distributions $E_f^{s\backslash u}$ are $C^{1,\omega}$, i.e, the derivatives are respectively of class $\omega(s) = o(s|\log(s)|)$, then $\mathscr{F}_f^{s/u}$ are actually C^{r-3}-smooth and f is C^{r-3}-conjugate to a toral hyperbolic automorphism. Similarly, Ghys [5] showed that for a C^r $(r \geq 2)$ Anosov diffeomorphism f of $\mathbb{T}^2$, if the stable foliation $\mathscr{F}_f^s$ is $C^{1+\text{Lip}}$-smooth, then it is actually C^r-smooth.

Our aim in this paper is getting higher regularity of codimension-one hyperbolic foliations under the assumption of more or less C^2-smoothness, see Theorem 2 and Theorem 6. In particular, we get some rigidity results on $\mathbb{T}^2$. Let us give two notations. We denote by $\lambda_f^u(x)$ the sum of Lyapunov exponents (if it exists) of f on the unstable subbundle at the point x, namely,

$$\lambda_f^u(x) = \lim_{n\to+\infty} \frac{1}{n} \log\left|\det\left(Df^n|_{E_f^u(x)}\right)\right|.$$

For $r > 1$, let $r_* = \begin{cases} r - 1 + \text{Lip}, & r \in \mathbb{N} \\ r, & r \notin \mathbb{N} \text{ or } r = +\infty. \end{cases}$

Theorem 2. *Let $f: \mathbb{T}^d \to \mathbb{T}^d$ $(d \geq 2)$ be a C^r $(r > 2)$ Anosov diffeomorphism with the $(d-1)$-dimensional $C^{2+\varepsilon}$-smooth ($\varepsilon > 0$) stable foliation $\mathscr{F}_f^s$. Then $\mathscr{F}_f^s$ is C^{r_*}-smooth and $\lambda_f^u(p) \equiv \lambda_A^u$, for all periodic points p of f, where A is the linearization of f.*

Remark 3. Here we briefly explain why we just get C^{r_*}-smoothness. In this paper, the regularity of foliation is given by foliation chart, see Section 2 for precise definition. Instead of the regularity of local chart, we will first prove that the foliation has C^r-smooth holonomy. However, the regularity of foliation may be lower than its holonomy, e.g., see [16, Section 6].

In particular, we have the following corollary linking the regularity of foliation with Lyapunov exponents of its transversal.

Corollary 4. *Let f be a $C^r(r>2)$ Anosov diffeomorphism of $\mathbb{T}^d$ $(d\geq 2)$ with the $(d-1)$-dimensional stable foliation $\mathscr{F}_f^s$ and linearization $A:\mathbb{T}^d\to\mathbb{T}^d$. Then the followings are equivalent:*

(1) *There exists small $\varepsilon>0$ such that $\mathscr{F}_f^s$ is $C^{2+\varepsilon}$-smooth;*
(2) *For all periodic points p of f, $\lambda_f^u(p)\equiv\lambda_A^u$;*
(3) *The foliation $\mathscr{F}_f^s$ is C^{r_*}-smooth.*

Remark 5. By the same way of proving "(2) ⇒ (3)" in Corollary 4, one can get an interesting result for non-invertible Anosov maps with codimension-one unstable foliations. Namely, these are nearly as smooth as the corresponding map is. Concretely, for a C^r $(r>1)$ non-invertible Anosov endomorphism $f:\mathbb{T}^d\to\mathbb{T}^d$ with one-dimensional stable bundle, if there exists unstable foliation $\mathscr{F}_f^u$ of f, then $\mathscr{F}_f^u$ is C^{r_*}-smooth. Indeed, by [1], the existence of $\mathscr{F}_f^u$ implies $\lambda_f^s(p)=\lambda_A^s$ for all $p\in \mathrm{Per}(f)$. Then the proof of Corollary 4 leads to C^{r_*} regularity of $\mathscr{F}_f^u$.

We mention in advance that our method to prove Theorem 2 is different from that of [3, 5, 8]. Indeed, we will consider a circle diffeomorphism induced by the codimension-one foliation and apply KAM theory (see Theorem 14) to it. Hence the regularity $C^{2+\varepsilon}$ of foliation is in fact a condition of the induced circle diffeomorphism for using KAM. Particularly, when $\mathbb{T}^d$ is restricted to be $\mathbb{T}^2$, we can lower the regularity of our assumption to be $C^{1+\mathrm{AC}}$, i.e, the derivative of foliation charts are absolutely continuous.

Theorem 6. *Let $f:\mathbb{T}^2\to\mathbb{T}^2$ be a $C^r(r\geqslant 2)$ Anosov diffeomorphism with the $C^{1+\mathrm{AC}}$-smooth stable foliation $\mathscr{F}_f^s$. Then $\mathscr{F}_f^s$ is C^{r_*}-smooth and $\lambda_f^u(p)\equiv\lambda_A^u$, for all periodic point p of f, where A is the linearization of f.*

By combining our result and a rigidity result of R. de la Llave [12] which says that constant periodic Lyapunov exponents implies smooth conjugacy on $\mathbb{T}^2$, we have following two direct corollaries.

Corollary 7. *Let f be a $C^r(r\geqslant 2)$ Anosov diffeomorphism of $\mathbb{T}^2$. If the stable and unstable foliations of f are both $C^{1+\mathrm{AC}}$, then f is C^{r_*} conjugate to its linearization. In particular, f preserves a smooth volume-measure.*

Corollary 8. *Let f be a $C^r(r\geqslant 2)$ volume-preserving Anosov diffeomorphism of $\mathbb{T}^2$. If the stable foliation of f is $C^{1+\mathrm{AC}}$, then f is C^{r_*} conjugate to its linearization.*

2. Preliminaries

As usual, a foliation $\mathscr{F}$ with dimension l of a closed Riemannian manifold $M=M^d$ is C^r-smooth, if there exists a set of C^r local charts $\{(\phi_i,U_i)\}_{i=1}^k$ of M such that $\phi_i:D^l\times D^{d-l}\to U_i$ satisfies

$$\phi_i\big(D^l\times\{y\}\big)\subset\mathscr{F}\big(\phi_i(0,y)\big),\quad \forall y\in D^{d-l},$$

where D^l and D^{d-l} are open disks with dimension l and $d-l$ respectively. The chart (ϕ_i,U_i) is called a C^r *foliation chart*.

Let f be a C^r-smooth Anosov diffeomorphism of a d-torus $\mathbb{T}^d$, i.e., there is a Df-invariant splitting

$$T\mathbb{T}^d=E_f^s\oplus E_f^u,$$

and constants $C,\lambda>1$, such that for all $n>0$, $x\in\mathbb{T}^d$ and $v^{s/u}\in E_f^{s/u}$

$$\|D_xf^n(v^s)\|\leq C\lambda^{-n}\|v^s\|,\quad\text{and}\quad \|D_xf^n(v^u)\|\geq C\lambda^n\|v^u\|.$$

Note that $f_*:\pi_1(\mathbb{T}^d)\to\pi_1(\mathbb{T}^d)$ also induces a hyperbolic automorphism $A:\mathbb{T}^d\to\mathbb{T}^d$[4] which is called the *linearization* of f. Denote the A-invariant hyperbolic splitting by

$$T\mathbb{T}^d=E^s_A\oplus E^u_A.$$

Let $i\in\{f,A\}$. Denote the foliations tangent to E^s_i and E^u_i by $\mathscr{F}^s_i$ and $\mathscr{F}^u_i$ respectively. It is known that the leaf $\mathscr{F}^\sigma_f(x)$ $(\sigma=s,u)$ is an immersed C^r submanifold of $\mathbb{T}^d$.

Since f and A are always conjugate [4], we denote the conjugacy by $h:\mathbb{T}^d\to\mathbb{T}^d$, namely, h is a homeomorphism such that

$$h\circ f=A\circ h$$

By the topological character of (un)stable foliation, i.e.,

$$\mathscr{F}^s_f(x)=\{y\in\mathbb{T}^d:d\big(f^n(x),f^n(y)\big)\longrightarrow 0,n\longrightarrow+\infty\},$$

h preserves the foliations, that is for all $x\in\mathbb{T}^d$,

$$h\big(\mathscr{F}^u_f(x)\big)=\mathscr{F}^u_A\big(h(x)\big),\quad h\big(\mathscr{F}^s_f(x)\big)=\mathscr{F}^s_A\big(h(x)\big).$$

It is convenient to look at the foliations on the universal cover $\mathbb{R}^d$. Let $\pi:\mathbb{R}^d\to\mathbb{T}^d$ be the natural projection. Denote by F,A and $H:\mathbb{R}^d\to\mathbb{R}^d$ the lifts of f,A and $h:\mathbb{T}^d\to\mathbb{T}^d$ respectively. For convenience, we can assume that $H(0)=0$. We denote the lift of $\mathscr{F}^\sigma_i$ $(\sigma=s/u, i=f/A)$ on $\mathbb{R}^d$ by $\widetilde{\mathscr{F}}^\sigma_i$ which are also the stable/unstable foliation of the lift F/A. Recall that $H\big(\widetilde{\mathscr{F}}^\sigma_f\big)=\widetilde{\mathscr{F}}^\sigma_A$, $\sigma=s/u$ and hence $\widetilde{\mathscr{F}}^s_f$ and $\widetilde{\mathscr{F}}^u_f$ admit the Global Product Structure just like $\widetilde{\mathscr{F}}^s_A$ and $\widetilde{\mathscr{F}}^u_A$, i.e., each pair of leaves $\widetilde{\mathscr{F}}^s_f(x)$ and $\widetilde{\mathscr{F}}^u_f(y)$ transversally intersects at exactly one point. Then we can define the holonomy map Hol^s_i $(i=f/A)$ induced by the foliation $\widetilde{\mathscr{F}}^s_i$ as

$$\mathrm{Hol}^s_{i,x,y}:\widetilde{\mathscr{F}}^u_i(x)\longrightarrow\widetilde{\mathscr{F}}^u_i(y),\quad \mathrm{Hol}^s_{i,x,y}(z)=\widetilde{\mathscr{F}}^s_i(z)\cap\widetilde{\mathscr{F}}^u_i(y).$$

Note that the holonomy map Hol^s_f and foliation $\widetilde{\mathscr{F}}^s_f$ are both absolutely continuous[15]. As mentioned before, the regularity of foliation may be lower than one of its holonomy. However, we still have the following lemma. We refer to [16, Section 6] for more details about the next lemma and also the counterexample of foliations whose regularity is strictly lower than the holonomy.

Lemma 9 ([16]). *Let $f:\mathbb{T}^d\to\mathbb{T}^d$ be a C^r-smooth $(r\ge 1)$ Anosov diffeomorphisms. Then*

(1) *If the holonomy maps Hol^s_f of $\widetilde{\mathscr{F}}^s_f$ are uniformly C^r-smooth, i.e., for any $x,y,z\in\mathbb{R}^d$ the holonomy map $\mathrm{Hol}^s_{f,x,y}$ is C^r and its derivatives (with respect to z) of order $\le r$ vary continuously with respect to (x,y,z), then the foliation $\mathscr{F}^s_f$ is C^{r_*}-smooth.*

(2) *If $\mathscr{F}^s_f$ is a C^k-smooth $(k\le r)$ foliation, then the holonomy maps Hol^s_f of $\widetilde{\mathscr{F}}^s_f$ are uniformly C^k-smooth.*

Remark 10. Note that the second item of Lemma 9 is trivial. And the first item is just an application of Journé's lemma [9] which asserts that the regularity of a diffeomorphism can be obtained from the uniformly regularity of its restriction on two tranverse foliations with uniformly smooth leaves. Indeed, considering a point $x\in\mathbb{R}^d$ and let $\alpha:D^l\to\widetilde{\mathscr{F}}^s_f(x)$ and $\beta:D^{d-l}\to\widetilde{\mathscr{F}}^u_f(x)$ be two C^r-parameterization such that $x=\big(\alpha(0),\beta(0)\big)$. Then

$$\phi(a,b):=\mathrm{Hol}^s_{f,\alpha(0),\alpha(a)}\big(\beta(b)\big)$$

gives us a foliation chart whose derivatives along D^l and D^{d-l} are both C^r. Hence by Journé's lemma, it is a C^{r_*}-foliation chart.

On the other hand, the regularity of holonomy induced by $\widetilde{\mathscr{F}}^s_f$ can be given by the smoothness of the conjugacy H restricted on the transversal direction.

Lemma 11. *Assume that the conjugacy $H:\mathbb{R}^d\to\mathbb{R}^d$ is uniformly C^r-smooth along the unstable foliation $\widetilde{\mathscr{F}}^u_f$, then the holonomy Hol^s_f is uniformly C^r-smooth.*

Proof. For given $x,y\in\mathbb{R}^d$ and $z\in\widetilde{\mathscr{F}}^u_f(x)$, the holonomy map satisfies

$$\mathrm{Hol}^s_{f,x,y}(z)=H^{-1}\circ\mathrm{Hol}^s_{A,H(x),H(y)}\circ H(z),$$

since H preserves the foliations. Note that the holonomies Hol^s_A induced by $\widetilde{\mathscr{F}}^s_A$ are actually translations. Therefore the holonomies Hol^s_f has the same regularity as $H|_{\widetilde{\mathscr{F}}^u}$. □

Combining Lemma 9 and Lemma 11, we can get Theorem 2 and Theorem 6 by proving that H is C^r-smooth along the unstable leaves. Precisely, we will prove the following property.

Proposition 12. *Let $f:\mathbb{T}^d\to\mathbb{T}^d$ be a C^r-smooth Anosov diffeomorphism with the $(d-1)$-dimensional C^k-smooth $(k<r)$ stable foliation $\mathscr{F}^s_f$. If one of the followings holds,*

(1) *$r>2$ and $k>2$;*
(2) *$d=2, r\geq 2$ and $k=1+\mathrm{AC}$.*

Then the conjugacy H between lifts F and A is uniformly C^r-smooth along each unstable leaf.

We will prove this proposition in Section 3. Before that, we note that to get C^r-regularity of $H|_{\widetilde{\mathscr{F}}^u_f}$, we can just prove a lower one. Indeed, by an enlightening work of de la Llave[12], one can get the C^r-smoothness from the absolute continuity, see [12, Lemma 4.1, Lemma 4.5 and Lemma 4.6]. Here we state it for convenience.

Lemma 13 ([12]). *Let $f:\mathbb{T}^d\to\mathbb{T}^d$ be a C^r-smooth Anosov diffeomorphism with one-dimensional unstable foliation $\mathscr{F}^u_f$. If one of the followings holds,*

(1) *The conjugaies H and H^{-1} are absolutely continuous restricted on the unstable foliation $\widetilde{\mathscr{F}}^u_f$ and $\widetilde{\mathscr{F}}^u_A$, respectively.*
(2) *For all periodic point p of f, $\lambda^u_f(p)\equiv\lambda^u_A$;*

Then the restriction $H|_{\widetilde{\mathscr{F}}^u_f}$ is uniformly C^r-smooth.

Now we can finish the proof of our main theorems.

Proof of Theorem 2, Corollary 4 and Theorem 6. Let f satisfy the condition of Theorem 2 or Theorem 6. By Proposition 12, $H|_{\widetilde{\mathscr{F}}^u_f}$ is smooth, so is $h|_{\mathscr{F}^u_f}$. Hence $\lambda^u_f(p)\equiv\lambda^u_A$, for all periodic point p of f. Combining with Lemma 9 and Lemma 11, we can get these two theorems immediately.

Note that "(1) ⇒ (2)" of Corollary 4 is given by Theorem 6, "(2) ⇒ (3)" is guaranteed by the case (2) of Lemma 13 and "(3) ⇒ (1)" is trivial. □

We will get the absolute continuity of H restricted on unstable leaves by applying following KAM theory. Specifically, we need two linearization theorems of circle diffeomorphisms given by Katznelson–Ornstein [10] and Khanin–Teplinsky [11]. For convenience, we state the condition (K.O. condition) in the work of Katznelson–Ornstein [10] here. Let $R_\alpha:\mathbb{R}\to\mathbb{R}$ be the translation on $\mathbb{R}$ such that $R_\alpha(x)=x+\alpha, x\in\mathbb{R}$. Denote the induced rigid rotation on S_1 by $\overline{R_\alpha}$. Let a_n be the coefficients of the continued fraction expansion of α and q_n be the denominators, so that

$$\alpha=\cfrac{1}{a_1+\frac{1}{a_2+\cdots}}\quad\text{and}\quad q_{n+1}=a_nq_n+q_{n-1}.$$

Let $T:S^1\to S^1$ be a $C^{1+\mathrm{AC}}$ diffeomorphism topologically conjugate to $\overline{R_\alpha}$. Let $K_n=\|\log DT^{q_n}\|_\infty$. We say that the pair (T,α) satisfies the *K.O. condition*, if $\Sigma_{n\geq 1}(a_nK_n)^2<\infty$.

Theorem 14 ([10, 11]). *Let T be an orientation-preserving circle diffeomorphism with irrational rotation number α which is algebraic. Then one has the following two properties*

(1) *If T is $C^{2+\epsilon}$-smooth, then T is $C^{1+\epsilon-\delta}$-smoothly conjugate to $\overline{R_\alpha}$ for any $\delta > 0$.*
(2) *If the pair (T, α) satisfies the K.O. condition, then T is absolutely continuously conjugate to $\overline{R_\alpha}$.*

Remark 15. In particular, the conditions: T is $C^{1+\mathrm{AC}}$, $T''/T' \in L^p$ for some $p > 1$ and $\deg(\alpha) = 2$ imply the K.O. condition. Indeed, by [10, Theorem 3.1], the assumption $T''/T' \in L^p$ $(p > 1)$ implies that $\Sigma_{n\geq 1}(K_n)^2 < \infty$. Combining this with the fact [19] that $\deg(\alpha) = 2$ if and only if α has a periodic simple continued fraction expansion, we get the K.O. condition.

3. Absolutely continuous rotation induced by smooth foliation

In this section, we will obtain our main result Proposition 12. To prove it, as mentioned before, we can consider the circle diffeomorphism induced by the codimenison-one foliation $\widetilde{\mathscr{F}}^s_f$ and show it is smooth conjugate to a rigid rotation given by $\widetilde{\mathscr{F}}^s_A$. We use the same notations as Section 2.

For reducing the action of $\widetilde{\mathscr{F}}^s_f$ on $\widetilde{\mathscr{F}}^u_f(0)$ to action on S^1, one can apply the $\mathbb{Z}^d$-actions. By the Global Product Structure, the following map T^i_n is well defined. For $n \in \mathbb{Z}^d$ and $i \in \{f, A\}$,

$$T^n_i : \widetilde{\mathscr{F}}^u_i(0) \longrightarrow \widetilde{\mathscr{F}}^u_i(0),$$
$$T^n_i(x) = \widetilde{\mathscr{F}}^u_i(0) \cap \widetilde{\mathscr{F}}^s_i(x+n), \quad \forall x \in \widetilde{\mathscr{F}}^u_i(0).$$

Note that for each $n \in \mathbb{Z}^d$, $T^n_i(x) = \mathrm{Hol}^s_{i,n,0} \circ R_n(x)$ where $R_n(x) = x + n$.

Proposition 16. *Assume that $\mathscr{F}^s_f$ is a C^k-smooth foliation. Then for each $n \in \mathbb{Z}^d$, T^n_i is C^k-smooth, $i \in \{f, A\}$. Moreover, one has:*

(1) *$T^n_i \circ T^m_i = T^{n+m}_i$ for all $n, m \in \mathbb{Z}^d$;*
(2) *$H \circ T^n_f = T^n_A \circ H$ for all $n \in \mathbb{Z}^d$.*

Proof. The regularity of $T^n_i = \mathrm{Hol}^s_{i,n,0} \circ R_n(x)$ is directly from the C^k-holonomy, since the holonomy is smoother than the foliation (see Lemma 9). And $\{T^n_i\}_{n\in\mathbb{Z}^d}$ is commutative by the fact that the holonomy maps are commutative with the $\mathbb{Z}^d$-actions on $\mathbb{R}^d$, i.e.,

$$R_m \circ \mathrm{Hol}^s_{i,x,y}(z) = \mathrm{Hol}^s_{i,R_m(x),R_m(y)} \circ R_m, \quad \forall m \in \mathbb{Z}^d, \forall x, y \in \mathbb{R}^d \text{ and } \forall z \in \widetilde{\mathscr{F}}^u_i(x),$$

since $\widetilde{\mathscr{F}}^{s/u}_i(x+m) = \widetilde{\mathscr{F}}^{s/u}_i(x) + m$ for all $x \in \mathbb{R}^d$ and $m \in \mathbb{Z}^d$. Hence

$$\begin{aligned} T^n_i \circ T^i_m &= \mathrm{Hol}^s_{i,n,0} \circ R_n \circ \mathrm{Hol}^s_{i,m,0} \circ R_m \\ &= \mathrm{Hol}^s_{i,n,0} \circ \mathrm{Hol}^s_{i,n+m,n} \circ R_{n+m} = \mathrm{Hol}^s_{i,n+m,0} \circ R_{n+m} = T^{n+m}_i. \end{aligned}$$

Recall that we can assume $H(0) = 0$. Note that H preserves the foliation and satisfies $H \circ R_m = R_m \circ H$, for all $m \in \mathbb{Z}^d$. Hence

$$H \circ T^n_f = H \circ \mathrm{Hol}^s_{f,n,0} \circ R_n = \mathrm{Hol}^s_{A,n,0} \circ H \circ R_n = \mathrm{Hol}^s_{A,n,0} \circ R_n \circ H = T^n_A \circ H.$$

This completes the proof of proposition. □

Now we are going to prove Proposition 12. Let $\{e_i\}_{i=1}^d$ be an orthonormal basis of $\mathbb{R}^d$. We will reduce a pair of conjugate $\mathbb{Z}^d$-actions, for instance $(T^{e_1}_f, T^{e_1}_A)$, to be a pair of conjugate circle diffeomorphisms and show the conjugacy is absolutely continuous by applying the KAM theory (Theorem 14). This method has a similar spirit with one used by Rodriguez Hertz, F. in [17].

Proof of Proposition 12. We pick two unit vector of the normal orthogonal basis, for example e_1, e_d. Assume that $\mathscr{F}^s_f$ is a C^k-smooth codimension-one foliation, where k satisfies the condition of proposition. Firstly, we use $T^{e_1}_f$ to construct a C^k circle diffeomorphism. We still denote the translation on $\mathbb{R}$ by $R_\alpha(x) = x + \alpha$ and the natural projection by $\pi : \mathbb{R} \to S^1$ for short.

Claim 17. *There exists a C^k diffeomorphism $h_f : \mathbb{R} \to \widetilde{\mathscr{F}}^u_f(0)$ satisfying the followings*

(1) $h_f \circ R_1 = T^{e_d}_f \circ h_f$;
(2) $T_f \circ R_1 = R_1 \circ T_f$, *where* $T_f \triangleq h^{-1}_f \circ T^{e_1}_f \circ h_f : \mathbb{R} \to \mathbb{R}$.

Consequently, $T^{e_1}_f$ induces a C^k diffeomorphism $\overline{T_f}$ on S^1 such that $\pi \circ T_f = \overline{T_f} \circ \pi$.

Proof of Claim 17. We would like to define the conjugacy h_f locally and extend it to $\mathbb{R}$ by $T^{e_d}_f$. More specifically, let $\gamma : (-\varepsilon, \varepsilon) \to \widetilde{\mathscr{F}}^u_f(0)$ be a C^r diffeomorphism onto the image and ε be small enough such that $T^{e_d}_f\big(\gamma(-\varepsilon,\varepsilon)\big) \cap \gamma(-\varepsilon,\varepsilon) = \emptyset$. This can be done by the C^r leaf $\widetilde{\mathscr{F}}^u_f(0)$ and the C^k diffeomorphism $T^{e_d}_f|_{\widetilde{\mathscr{F}}^u_f(0)}$ with $T^{e_d}_f(0) \neq 0$.

Then, we can define a C^k diffeomorphism onto the image, $h_0 : (-\epsilon, 1] \to \widetilde{\mathscr{F}}^u_f(0)$ such that

$$h_0(x) = \begin{cases} \gamma(x), & x \in (-\varepsilon, \varepsilon); \\ T^{e_d}_f \circ \gamma(x-1), & x \in (1-\epsilon, 1]; \\ \varphi(x), & x \in [\varepsilon, 1-\varepsilon]. \end{cases} \tag{1}$$

where φ is a C^k diffeomorphism onto the image and can be chosen arbitrarily. Let

$$h_f : \mathbb{R} \longrightarrow \widetilde{\mathscr{F}}^u_f(0),$$
$$h_f(x) \triangleq (T^{e_d}_f)^{[x]} \circ h_0(x - [x]), \quad \forall x \in \mathbb{R}.$$

where $[x]$ stands for the integer part of x. By the construction, one can verify that h_f and T_f are both C^k diffeomorphisms and $h_f \circ R_1 = T^{e_d}_f \circ h_f$ directly. And $T_f \circ R_1 = R_1 \circ T_f$ is guaranteed by the commutativity of T^n_f, see Proposition16. Indeed,

$$\begin{aligned} T_f \circ R_1 &= h^{-1}_f \circ T^{e_1}_f \circ h_f \circ R_1 = h^{-1}_f \circ T^{e_1}_f \circ T^{e_d}_f \circ h_f \\ &= h^{-1}_f \circ T^{e_d}_f \circ T^{e_1}_f \circ h_f = (h^{-1}_f \circ T^{e_d}_f \circ h_f) \circ (h^{-1}_f \circ T^{e_1}_f \circ h_f) = R_1 \circ T_f. \end{aligned}$$

Then we obtain the desired diffeomorphism h_f and hence a C^k diffeomorphism $\overline{T_f} : S^1 \to S^1$. □

Claim 18. *There exists a C^∞ diffeomorphism $h_A : \mathbb{R} \to \widetilde{\mathscr{F}}^u_A(0)$ satisfying the following*

(1) $h_A \circ R_1 = T^{e_d}_A \circ h_A$;
(2) $R_\alpha = h^{-1}_A \circ T^{e_1}_A \circ h_A$, *where α is an irrational algebraic number.*

In particular, R_α induces a rotation $\overline{R_\alpha}$ on S^1. Moreover if $d = 2$ (the $\mathbb{T}^2$ case), one has $\deg(\alpha) = 2$.

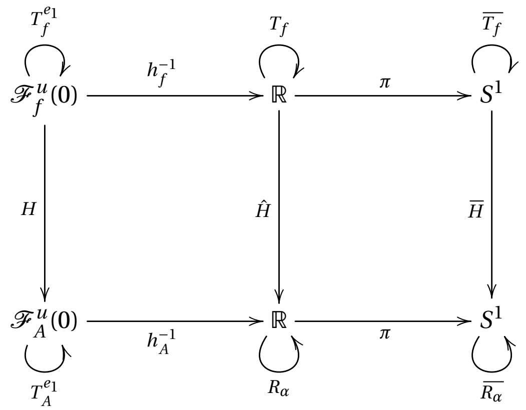
Proof of Claim 18. Let $h_A : \mathbb{R} \to \widetilde{\mathscr{F}}^u_A(0)$ be the linear map such that $h_A(0) = 0 \in \mathbb{R}^d$ and $h_A(1) = T^{e_d}_A(0)$. Then, $h^{-1}_A \circ T^{e_1}_A \circ h_A$ is actually a translation $R_\alpha(x) = x + \alpha, x \in \mathbb{R}$. By elementary calculate, $\alpha = x_1/x_d$ where $\vec{v} = (x_1, \dots, x_d)$ (given under the an orthonormal basis $\{e_i\}^d_{i=1}$) is an eigenvector of A in $\widetilde{\mathscr{F}}^s_A(0)$. Then α is an irrational algebraic number. Indeed, the irrational eigenvectors of A implie that there is at least a pair of irrationally related coordinates $(x_i, x_j), (i \neq j)$ of $\vec{v}$ which we may assume that is (x_1, x_d) and by the fact that the set of algebraic numbers is a field. Moreover, $\alpha = x_1/x_d$ is a quadratic irrational in the case of $d = 2$. □

Since $H\circ T_f^{e_1} = T_A^{e_1}\circ H$ (see Proposition 16), H also induces a conjugacy $\overline{H}: S^1 \to S^1$ from $\overline{T_f}$ to $\overline{R_\alpha}$. Indeed, let $\widehat{H} \triangleq h_A^{-1}\circ H\circ h_f : \mathbb{R}\to\mathbb{R}$. Then by Proposition 16, Claim 17 and Claim 18, one has

(1) $\widehat{H}\circ R_1 = R_1\circ\widehat{H}$;

(2) $R_\alpha\circ\widehat{H} = \widehat{H}\circ T_f$.

In particular, $\widehat{H}:\mathbb{R}\to\mathbb{R}$ induces $\overline{H}: S^1\to S^1$ with $\pi\circ\widehat{H} = \overline{H}\circ\pi$. Moreover, $\overline{H}\circ\overline{T_f} = \overline{R_\alpha}\circ\overline{H}$. Namely, we have the following commutative diagram:



By Theorem14 and Remark 15, $\overline{H}^{\pm 1}$ is absolutely continuous, so is $\widehat{H}^{\pm 1}:\mathbb{R}\to\mathbb{R}$. Note that in the case of $\mathbb{T}^2$ and $k = 1+\mathrm{AC}$, one has that both T_f, T_f^{-1} are $C^{1+\mathrm{AC}}$-smooth. It follows that there is $C > 1$ such that $|T_f''(x)| < C$ and $|T_f'(x)| > \frac{1}{C}$ for Lebesgue-almost everywhere $x \in S^1$. Hence $T_f''/T_f' \in L^2$.

Thus $H = h_A\circ\widehat{H}\circ h_f^{-1}$ and H^{-1} are also absolutely continuous along unstable leaves. Finally, by Lemma 13, H is C^r-smooth restricted on unstable leaves. □

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Declaration of interests

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