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The mod 2 Margolis homology of the Dickson algebra

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In memory of Nguyễn Thị Thanh Bình

Abstract. We completely compute the mod 2 Margolis homology of the Dickson algebra D_n, i.e. the homology of D_n with the differential to be the Milnor operation Q_j, for every n and j. The motivation for this problem is that, the Margolis homology of the Dickson algebra plays a key role in study of the Morava K-theory $K(j)^*(BS_m)$ of the symmetric group on m letters S_m.

We show that Pengelley–Sinha's conjecture on $H_*(D_n;Q_j)$ for $n \le j$ is true if and only if $n = 1$ or 2. For $3 \le n \le j$, our result proves that this conjecture turns out to be false since the occurrence of some "critical elements" $h_{s_1,\dots,s_k}$'s of degree $(2^{j+1}-2^n)+\sum_{i=1}^k(2^n-2^{s_i})$ in this homology for $0 < s_1 < \cdots < s_k < n$ and $k > 1$.

Résumé. Dans cette note on calcule entièrement l'homologie de Margolis modulo 2 de l'algèbre de Dickson D_n, i.e. l'homologie de D_n en choisissant pour différentielles les opérations de Milnor Q_j, pour tous n et j. La motivation pour cette étude est le rôle clé joué par cette homologie dans l'étude de la K-théorie de Morava $K(j)^*(BS_m)$ du groupe symétrique S_m en m lettres.

Nous montrons que la conjecture de Pengelley–Sinha sur $H_*(D_n;Q_j)$ pour $n \le j$ est vraie si et seulement si $n = 1,2$. Pour $3 \le n \le j$ notre résultat montre que la conjecture est fausse à cause de l'occurence d'éléments « critiques » $h_{s_1,\dots,s_k}$ de degré $(2^{j+1}-2^n)+\sum_{i=1}^k(2^n-2^{s_i})$ dans cette homologie pour $0 < s_1 < \cdots < s_k < n$ et $k > 1$.

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Let $\mathscr{A}$ be the mod 2 Steenrod algebra, genenated by the cohomology operations Sq^j with $j \ge 0$ and subject to the Adem relation with $Sq^0 = 1$. Further $\mathscr{A}$ is a Hopf algebra, whose coproduct is given by the formula $\Delta(Sq^j) = \sum_{i=0}^j Sq^i \otimes Sq^{j-i}$.

Let $\mathscr{A}_*$ be the Hopf algebra, which is dual to $\mathscr{A}$. Let $\xi_j = (Sq^{2^j} \cdots Sq^2 Sq^1)^*$ be the Milnor element of degree $2^{j+1}-1$ in $\mathscr{A}_*$, for $j \ge 0$, where the duality is taken with respect to the admissible basis of $\mathscr{A}$. According to Milnor [4], as an algebra, $\mathscr{A}_* \cong \mathbb{F}_2[\xi_0,\xi_1,\dots,\xi_j,\dots]$, the polynomial algebra in infinitely many generators $\xi_0,\xi_1,\dots,\xi_j,\dots$.

Let Q_j, for $j \ge 0$, be the Milnor operation (see [4]) of degree $(2^{j+1}-1)$ in $\mathscr{A}$, which is dual to ξ_j with respect to the basis of $\mathscr{A}_*$ consisting of all monomials in the generators $\xi_0,\xi_1,\dots,\xi_j,\dots$.

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Remarkably, Q_j is a differential, that is $Q_j^2 = 0$ for every j. In fact, $Q_0 = Sq^1$, $Q_j = [Q_{j-1}, Sq^{2^j}]$, the commutator of Q_{j-1} and Sq^{2^j} in the Steenrod algebra $\mathscr{A}$, for $j > 0$.

In the article, we compute the Margolis homology of the Dickson algebra D_n, i.e. the homology of D_n with the differential to be the Milnor operation Q_j.

The real goal that we persue is to compute the Morava K-theory $K(j)^*(BS_m)$ of the symmetric group S_m on m letters. It was well known that, the Milnor operation is the first non-zero differential, $Q_j = d_{2^{j+1}-1}$, in the Atiyah–Hirzebruch spectral sequence for computing $K(j)^*(X)$, the Morava K-theory of a space X. So, the Q_j-homology of $H^*(X)$ is the $E_{2^{j+1}}$-page in the Atiyah–Hirzebruch spectral sequence for $K(j)^*(X)$. (See e.g. Yagita [10, §2], although the fact was well known before this article.)

A key step in the determination of the symmetric group's cohomology is to apply the Quillen restiction from this cohomology to the cohomologies of all elementary abelian subgroups of the symmetric group. For $m = 2^n$ and the "generic" elementary abelian 2-subgroup $(\mathbb{Z}/2)^n$ of the symmetric group S_{2^n}, the image of the restriction $H^*(BS_{2^n}) \to H^*(B(\mathbb{Z}/2)^n)$ is exactly the Dickson algebra D_n (see Mùi [5, Thm. II.6.2]). So, the $E_{2^{j+1}}$-page in the Atiyah–Hirzebruch spectral sequence for $K(j)^*(BS_{2^n})$ maps to the Margolis homology $H_*(D_n; Q_j)$. This is why the Margolis homology of the Dickson algebra is taken into account.

Let us study the range n Dickson algebra of invariants

$$D_n = \mathbb{F}_2[x_1, \ldots, x_n]^{\mathrm{GL}(n,\mathbb{F}_2)},$$

where each generator x_i is of degree 1, and the general linear group $\mathrm{GL}(n,\mathbb{F}_2)$ acts canonically on $\mathbb{F}_2[x_1, \ldots, x_n]$. Following Dickson [1], let us consider the determinant

$$[e_1, \ldots, e_n] = \det \begin{pmatrix} x_1^{2^{e_1}} & \cdots & x_n^{2^{e_1}} \\ \vdots & \ddots & \vdots \\ x_1^{2^{e_n}} & \cdots & x_n^{2^{e_n}} \end{pmatrix}$$

for non-negative integers $e_1, \ldots, e_n$. Then $\omega[e_1, \ldots, e_n] = \det(\omega)[e_1, \ldots, e_n]$, for $\omega \in \mathrm{GL}(n,\mathbb{F}_2)$ (see [1]). Set

$$L_{n,s} = [0, 1, \ldots, \widehat{s}, \ldots, n], \quad (0 \le s \le n),$$

where $\widehat{s}$ means s being omitted, and $L_n = L_{n,n}$. The Dickson invariant $c_{n,s}$ of degree $2^n - 2^s$ is originally defined as follows:

$$c_{n,s} = L_{n,s}/L_n, \quad (0 \le s < n).$$

Dickson proved in [1] that D_n is a polynomial algebra on the Dickson invariants

$$D_n = \mathbb{F}_2[c_{n,0}, \ldots, c_{n,n-1}].$$

To be explicit, the Dickson invariant can be expressed as in Hưng–Peterson [3, §2]:

$$c_{n,s} = \sum_{i_1 + \cdots + i_n = 2^n - 2^s} x_1^{i_1} \cdots x_n^{i_n}, \quad (0 \le s < n).$$

where the sum is over all sequences $i_1, \ldots, i_n$ with i_k either 0 or a power of 2.

We are interested in the following element of the Dickson algebra D_n:

$$A_{j,n,s} = [0, \ldots, \widehat{s}, \ldots, n-1, j]/L_n,$$

for $0 \le s < n \le j$. By convention, $A_{j,n,-1} = 0$.

In this article, when j and n are fixed, the elements $c_{n,s}$ and $A_{j,n,s}$ will respectively be denoted by c_s and A_s for abbreviation.

Lemma 1. *For* $0 \le j$, $0 \le s < n$,

$$Q_j(c_s) = \begin{cases} c_0, & 0 \le j < n-1,\ j = s-1, \\ 0, & 0 \le j < n-1,\ j \ne s-1, \\ c_0 c_s, & j = n-1, \\ c_0\big(c_s A_{n-1}^2 + A_{s-1}^2\big), & 0 \le s < n \le j. \end{cases}$$

The action of the Steenrod algebra on the Dickson one is basically computed in [2]. Related and partial results concerning the lemma can be seen in [7–9].

The next two theorems are stated in Sinha [6]. Their proofs are straightforward from Lemma 1.

Theorem 2. *For* $0 \le j < n-1$,

$$H_*(D_n, Q_j) \cong \mathbb{F}_2[c_{j+1}^2] \otimes \mathbb{F}_2[c_1, \ldots, \widehat{c}_{j+1}, \ldots, c_{n-1}],$$

where $\widehat{c}_{j+1}$ *means* c_{j+1} *being omitted.*

Let $\mathbb{F}_2[c_1,\ldots,c_{n-1}]_{\mathrm{ev}}$ be the $\mathbb{F}_2$-submodule of $\mathbb{F}_2[c_1,\ldots,c_{n-1}]$ generated by all the monomials $c_1^{i_1}\cdots c_{n-1}^{i_{n-1}}$ with $i_1 + \cdots + i_{n-1}$ even.

Theorem 3.

$$H_*(D_n; Q_{n-1}) \cong \mathbb{F}_2[c_1, \ldots, c_{n-1}]_{ev}.$$

Proposition 4. *For* $0 \le s_1, \ldots, s_k < n \le j$,

$$Q_j(c_{s_1}\cdots c_{s_k}) = c_0\left(k c_{s_1}\cdots c_{s_k} A_{n-1}^2 + \sum_{i=1}^{k} c_{s_1}\ldots \widehat{c}_{s_i}\ldots c_{s_k} A_{s_i-1}^2\right),$$

where $\widehat{c}_{s_i}$ *means* c_{s_i} *being omitted.*

Conjecture 5 (D. Pengelley – D. Sinha, see [6]). *For* $n \le j$,

$$H_*(D_n; Q_j) \cong D_n^2 / \big(Q_j(c_0), Q_j(c_0c_1), \ldots, Q_j(c_0c_{n-1})\big).$$

Let D_n^{odd} be the $\mathbb{F}_2$-submodule of D_n spanned by all monomials $c_0^{i_0}\cdots c_{n-1}^{i_{n-1}}$ with at least one of the exponents $i_0,\ldots,i_{n-1}$ odd. Note clearly that D_n^{odd} is not a Q_j-submodule of D_n, but $\mathrm{Im}\, Q_j \cap D_n^{\mathrm{odd}}$ is, since Q_j vanishes on this module.

Pengelley–Sinha's conjecture is equivalent to the equality: $\mathrm{Ker}\, Q_j = \big(\mathrm{Im}\, Q_j \cap D_n^{\mathrm{odd}}\big) \oplus D_n^2$. In other words, there is no class in $H_*(D_n; Q_j)$ represented by an element in D_n^{odd}. The following two theorems show that Pengelley–Sinha's conjecture is true for $n = 1$ or 2 and every j.

Theorem 6. *For* $n = 1$, $0 \le j$,

$$H_*(D_1; Q_j) \cong \mathbb{F}_2[c_0^2]/(c_0^{2^{j+1}}).$$

In particular, $H_*(D_1; Q_0) = \mathbb{F}_2$ (this is also a special case of Theorem 3), $H_*(D_1; Q_1) = \Lambda(c_0^2)$, where $\Lambda(c_0^2)$ denotes the $\mathbb{F}_2$-exterior algebra on c_0^2.

Theorem 7. *Denote* $\overline{\Lambda(c_0^2)} = \Lambda(c_0^2)/(\mathbb{F}_2 \cdot 1)$. *If* $n = 2$, $0 \le j$,

$$H_*(D_2; Q_j) \cong \begin{cases} \mathbb{F}_2[c_1^2], & \text{for } j = 0, 1, \\ \overline{\Lambda(c_0^2)} \oplus \mathbb{F}_2[c_1^2], & \text{for } j = 2, \\ \mathbb{F}_2[c_0^2, c_1^2]/(c_0^2 A_0^2, c_0^2 A_1^2), & \text{for } j > 2, \end{cases}$$

where $A_0 = (x_1^2 x_2^{2^j} + x_1^{2^j} x_2^2)/(x_1 x_2^2 + x_1^2 x_2)$, $A_1 = (x_1 x_2^{2^j} + x_1^{2^j} x_2)/(x_1 x_2^2 + x_1^2 x_2)$.

The cases $j = 0, 1$ in the previous theorem are special cases of Theorems 2 and 3.

Proposition 8. *Pengelley–Sinha's Conjecture for* $n \le j$ *is true if and only if* $1 \le n \le 2$.

How can we adjust Pengelley–Sinha's conjecture to make a correct one in the problem for $3 \le n \le j$? The critical elements $h_{s_1,\dots,s_k}$'s, defined below in the Margolis homology of the Dickson algebra D_n, are the main ingredient in our correction of Pengelley–Sinha's conjecture for $3 \le n \le j$.

Note that, c_0^2 divides $Q_j(c_0 c_{s_1} \cdots c_{s_k})$ in D_n, if $s_1, \dots, s_k$ are pairwise distinct.

Definition 9. *For $n \le j$, $0 \le s_1, \dots, s_k < n$, and $s_1, \dots, s_k$ pairwise distinct:*

$$h_{s_1,\dots,s_k} = \frac{1}{c_0^2} Q_j(c_0 c_{s_1} \cdots c_{s_k}).$$

To be more explicit, under the hypotheses of the definition:

$$h_{s_1,\dots,s_k} = (k+1) c_{s_1} \cdots c_{s_k} A_{n-1}^2 + \sum_{i=1}^{k} (c_{s_1} \dots \widehat{c}_{s_i} \dots c_{s_k}) A_{s_i-1}^2.$$

Note that, $h_{s_1,\dots,s_k} \in D_n^{\mathrm{odd}}$ if $k > 1$, and that $h_{s_1,\dots,s_k}$ depends also on n and j.

Lemma 10.

(i) $A_s = \begin{cases} 0 \mod (c_0, \dots, c_r), & 0 \le s \le r < n, \\ \neq 0 \mod (c_0, \dots, c_r), & 0 \le r < s < n. \end{cases}$

(ii) *A_r and A_s are coprime in D_n for $0 \le r \neq s < n$.*

(iii) *If $n \le j$, then $h_{s_1,\dots,s_k} \in \operatorname{Ker} Q_j$ and $[h_{s_1,\dots,s_k}] \neq 0$ in the Margolis homology $H_*(D_n; Q_j)$ for $0 < s_1 < \cdots < s_k < n$, $1 < k$.*

The lemma is based on the following inductive formula, in which the complete notations $A_{j,n,s}$ and $c_{n,s}$ are used instead of the simplified ones A_s and c_s:

$$A_{j,n,s} = A_{j-1,n-1,s-1}^2 + A_{j-1,n,n-1}^2 c_{n-1,s} \frac{L_n}{L_{n-1}},$$

for $0 \le s < n \le j$. Here, by convention, $A_{n-1,n,n-1} = 1$, $c_{n-1,n-1} = 1$.

Note that Q_j is a (total) derivation, that is $Q_j(ab) = Q_j(a)b + aQ_j(b)$. We study the s-th partial derivation for $0 < s \le n$, and its "inverse", the so-called integral on a direction. These notions will play key roles in the remaining part of the article.

Definition 11. *Let $s_1, \dots, s_k$ be pairwise distinct, with $0 \le s_1, \dots, s_k < n$, and $R \in D_n$. The s-th partial derivation is the morphism defined for $0 \le s \le n$ by*

$$\partial_s(c_{s_1} \cdots c_{s_k} R^2) = \begin{cases} c_0 c_{s_1} \cdots c_{s_k} A_{n-1}^2 R^2, & k \text{ odd}, s = n, \\ c_0 c_{s_1} \cdots \widehat{c}_{s_i} \cdots c_{s_k} A_{s_i-1}^2 R^2, & s = s_i, \\ 0, & \text{otherwise}. \end{cases}$$

Since $A_{-1} = 0$, it yields $\partial_0 = 0$. If $\partial_s(c_{s_1} \cdots c_{s_k}) \neq 0$, then s should be one of the indices $s_1, \dots, s_k$ or n. Obviously, $\operatorname{Im} \partial_s \subset c_0 A_{s-1}^2 D_n$. Proposition 4 leads to:

Lemma 12. *Let $s_1, \dots, s_k$ be pairwise distinct, with $0 \le s_1, \dots, s_k < n \le j$, and $R \in D_n$. Then*

$$Q_j(c_{s_1} \cdots c_{s_k} R^2) = \sum_{s=1}^{n} \partial_s(c_{s_1} \cdots c_{s_k}) R^2.$$

Definition 13. *The integral on the r-th direction $I_r : c_0 A_{r-1}^2 D_n \to D_n^{\mathrm{odd}}$, for $0 < r \le n$, is the morphism given by:*

$$I_r(c_0 c_{s_1} \cdots c_{s_k} A_{r-1}^2 R^2) = \begin{cases} c_{s_1} \cdots c_{s_k} R^2, & k \text{ odd}, r = n, \\ c_{s_1} \cdots c_{s_k} c_r R^2, & r \neq s_1, \dots, s_k, n, \\ 0, & \text{otherwise}. \end{cases}$$

where $s_1, \dots, s_k$ are pairwise distinct, $0 \le s_1, \dots, s_k < n$, $0 \le k$, and $R \in D_n$.

Lemma 14. *Let $s_1,\dots,s_k$ be pairwise distinct, with $0 \le s_1,\dots,s_k < n$, $0 < s \le n$, and $R \in D_n$. Then*

(i) $I_s\partial_s(c_{s_1}\cdots c_{s_k}R^2) = \begin{cases} c_{s_1}\cdots c_{s_k}R^2, & \textit{either } k \textit{ odd, } s = n, \textit{ or } s \in \{s_1,\dots,s_k\},\\ 0, & \textit{otherwise.}\end{cases}$

(ii) $\partial_s I_s(c_0c_{s_1}\cdots c_{s_k}A_{s-1}^2R^2) = \begin{cases} c_0c_{s_1}\cdots c_{s_k}A_{s-1}^2R^2, & \textit{either } k \textit{ odd, } s = n, \textit{ or } s \neq s_1,\dots,s_k,n,\\ 0, & \textit{otherwise.}\end{cases}$

Let $hc_0^2D_n^2$ and $h\overline{D}_n^2$ be the submodules of D_n generated by the generators $\{h_{s_1,\dots,s_k} \mid 0 < s_1 < \cdots < s_k < n, 1 < k\}$ over $c_0^2D_n^2$ and $\overline{D}_n^2 = \mathbb{F}_2[c_1^2,\dots,c_{n-1}^2]$ respectively. Let $h_0D_n^2$ be the submodule of D_n generated by $\{h_{0,s_2,\dots,s_k} \mid 0 = s_1 < s_2 < \cdots < s_k < n, 1 < k\}$ over D_n^2.

Theorem 15. *For $3 \le n \le j$,*

$$\operatorname{Ker} Q_j \cap D_n^{odd} = \left(\operatorname{Im} Q_j \cap D_n^{odd}\right) + h\overline{D}_n^2,$$

where $\operatorname{Im} Q_j \cap D_n^{odd} = h_0D_n^2 \oplus hc_0^2D_n^2$, and $h_0D_n^2 \cap h\overline{D}_n^2 = \{0\}$.

The exponent of c_0 in each Dickson monomial of $h_0D_n^2$ is odd, whereas the exponent of c_0 in every Dickson monomial of $hc_0^2D_n^2$ or of $h\overline{D}_n^2$ is even. It yields $h_0D_n^2 \cap hc_0^2D_n^2 = \{0\}$ and $h_0D_n^2 \cap h\overline{D}_n^2 = \{0\}$.

The smallest natural number n such that there exists a sequence $0 < s_1 < \cdots < s_k < n$ with $k > 1$ is $n = 3$.

Remark 16. The sum in Theorem 15 is not a direct sum. This is a consequence of the fact that the critical elements are not linear independent over D_n^2.

Let $S = (s_1,\dots,s_k)$ be a sequence with $0 < s_1 < \cdots < s_k < n$ and $k > 2$. It is remarkable that

$$H_S = kh_{s_1,\dots,s_k}A_{n-1}^2 + \sum_{i=1}^{k} h_{s_1,\dots,\widehat{s_i},\dots,s_k}A_{s_i-1}^2 = 0.$$

Let $\pi : D_n^2 \to \mathbb{F}_2[c_1^2,\dots,c_{n-1}^2]$ be the projection, whose kernel is $c_0^2D_n^2$. We denote $\pi(Z^2)$ by $\overline{Z}^2$ for abbreviation. So $Z^2 + \overline{Z}^2 \in c_0^2D_n^2$ for $Z^2 \in D_n^2$. The equality $H_S = 0$ implies

$$\begin{aligned} H_S + \overline{H}_S &= kh_{s_1,\dots,s_k}(A_{n-1}^2 + \overline{A}_{n-1}^2) + \sum_{i=1}^{k} h_{s_1,\dots,\widehat{s_i},\dots,s_k}(A_{s_i-1}^2 + \overline{A}_{s_i-1}^2)\\ &= kh_{s_1,\dots,s_k}\overline{A}_{n-1}^2 + \sum_{i=1}^{k} h_{s_1,\dots,\widehat{s_i},\dots,s_k}\overline{A}_{s_i-1}^2 = \overline{H}_S. \end{aligned}$$

The left hand side belongs to $hc_0^2D_n^2 \subset \left(\operatorname{Im} Q_j \cap D_n^{\mathrm{odd}}\right)$ (it is in D_n^{odd} as $k-1 > 1$), while the right hand side is in $h\overline{D}_n^2$ with at most one "coefficient" $\overline{A}_{s_i-1}^2$ being zero. (The zero-coefficient occurs when $s_1 = 1$, since $\overline{A}_{s_i-1}^2 \neq 0$ for $s_i > 1$ by Lemma 10.) Therefore, $\left(\operatorname{Im} Q_j \cap D_n^{\mathrm{odd}}\right) \cap h\overline{D}_n^2 = hc_0^2D_n^2 \cap h\overline{D}_n^2 \neq \{0\}$.

The following main result of the article is a consequence of the preceding one and the equalities: $Q_j(c_0) = c_0^2A_{n-1}^2$, $Q_j(c_0c_s) = c_0^2A_{s-1}^2$ $(0 < s < n)$.

Theorem 17. *For $3 \le n \le j$,*

$$H_*(D_n; Q_j) = \frac{D_n^2}{(c_0^2A_0^2,\dots,c_0^2A_{n-1}^2)} \oplus \frac{h\overline{D}_n^2}{hc_0^2D_n^2 \cap h\overline{D}_n^2}.$$

Example 18. For $j = n \ge 3$, we have $A_s = c_s$ for $0 \le s < n$. So the critical element, which also depends on n and j, is explicitly given by

$$h_{s_1,\dots,s_k} = (k+1)c_{s_1}\cdots c_{s_k}c_{n-1}^2 + \sum_{i=1}^{k}(c_{s_1}\dots\widehat{c}_{s_i}\dots c_{s_k})c_{s_i-1}^2,$$

for $0 < s_1 < \cdots < s_k < n$, $1 < k$. Theorem 17 yields

$$H_*(D_n; Q_n) = \frac{D_n^2}{(c_0^4, c_0^2 c_1^2, \ldots, c_0^2 c_{n-1}^2)} \oplus \frac{h\overline{D}_n^2}{hc_0^2 D_n^2 \cap h\overline{D}_n^2}$$
$$= \overline{\Lambda(c_0^2)} \oplus \mathbb{F}_2[c_1^2, \ldots, c_{n-1}^2] \oplus \frac{h\overline{D}_n^2}{hc_0^2 D_n^2 \cap h\overline{D}_n^2}.$$

For $k > 2$, by Remark 16,

$$h_{s_2,\ldots,s_k} c_0^2 = k h_{1,s_2,\ldots,s_k} c_{n-1}^2 + \sum_{i=2}^{k} h_{1,s_2,\ldots,\widehat{s_i}\ldots,s_k} c_{s_i-1}^2$$

is a nonzero element in $hc_0^2 D_n^2 \cap h\overline{D}_n^2$ for $1 = s_1 < \cdots < s_k < n$, while

$$k h_{s_1,s_2,\ldots,s_k} c_{n-1}^2 + \sum_{i=1}^{k} h_{s_1,\ldots,\widehat{s_i}\ldots,s_k} c_{s_i-1}^2 = 0$$

is a linear relationship of the critical elements over $\overline{D}_n^2$ for $1 < s_1 < \cdots < s_k < n$.

The contains of this note will be published in detail elsewhere.

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