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Factorisation des opérateurs de Hankel, inclusion des images des opérateurs de Toeplitz et de Hankel sur l'espace de Hardy à valeurs vectorielles

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Abstract. Using Douglas theorem on factorization and range inclusion of bounded linear operators, we give the factorization of Hankel operators, range inclusion of Hankel and Toeplitz operators defined on vector-valued Hardy spaces.

Résumé. En utilisant le théorème de Douglas sur la factorisation impliquée par l'inclusion des images de deux opérateurs linéaires bornés, nous donnons la factorisation des opérateurs de Hankel quand il y a inclusion des images de deux opérateurs de Hankel ou de Toeplitz définis sur des espaces de Hardy à valeur vectorielle.

Keywords. Toeplitz operators, Hankel operators, Hardy space, inner functions, Douglas Lemma, factorization, majorization.

Mots-clés. Opérateurs de Toeplitz, opérateurs de Hankel, espace de Hardy, fonctions intérieures, lemme de Douglas, factorisation, majorisation.

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1. Introduction and Preliminaries

Let $\mathscr{E}$ be a Hilbert space (here all Hilbert spaces are separable and over $\mathbb{C}$) and T be a bounded linear operator on $\mathscr{E}$ ($T \in \mathscr{B}(\mathscr{E})$ in short). The subspaces $\mathscr{R}(T) = \{Tx : x \in \mathscr{E}\}$ and $\mathscr{N}(T) = \{x \in \mathscr{E} : Tx = 0\}$ denotes the range space and the null space, respectively. A linear operator $T \in \mathscr{B}(\mathscr{E})$ is said to be bounded below if there exists a constant $M \geq 0$ such that $\|Tx\| \geq M\|x\|$ for every $x \in \mathcal{E}$, $T \in \mathcal{B}(\mathcal{E})$ is said to normal if $T^*T = TT^*$, hyponormal if $TT^* \leq T^*T$, and isometry if $T^*T = I$.

Let $\mathbb{D}$ be the open unit disc in $\mathbb{C}$, and let $\mathcal{E}$ be a Hilbert space. The $\mathcal{E}$-valued *Hardy space* $H^2_{\mathcal{E}}(\mathbb{D})$ ($H^2(\mathbb{D})$ if $\mathcal{E} = \mathbb{C}$) over $\mathbb{D}$ is the Hilbert space of all $\mathcal{E}$-valued analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $a_n \in \mathcal{E}$, $z \in \mathbb{D}$ such that

$$\|f\| = \left(\sum_{n=0}^{\infty} \|a_n\|^2\right)^{\frac{1}{2}} < \infty.$$

Let $\mathcal{E}_*$ be another Hilbert space. Then denote by $H^\infty_{\mathcal{B}(\mathcal{E},\mathcal{E}_*)}(\mathbb{D})$ ($H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D})$ if $\mathcal{E}_* = \mathcal{E}$) the set of $\mathcal{B}(\mathcal{E},\mathcal{E}_*)$-valued bounded analytic functions on $\mathbb{D}$ which is a Banach space with the norm defined by

$$\|\Theta\|_\infty = \sup\{\|\Theta(z)\| : |z| < 1\}.$$

An operator-valued analytic function $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E},\mathcal{E}_*)}(\mathbb{D})$ is said to be inner if $\Theta(z)^*\Theta(z) = I_{\mathcal{E}}$ a.e. $z \in \mathbb{T}$.

1.1. *Toeplitz and Hankel Operators*

In this section we give definitions of Toeplitz and Hankel operators and their basic properties. Let $\mathcal{E}$ be any Hilbert space. Let $L^2_{\mathcal{E}}(\mathbb{T})$, where $\mathbb{T}$ denotes the unit circle in the complex plane, denote the Hilbert space of all square $\mathcal{E}$-valued Lebesgue integrable functions on $\mathbb{T}$, that is

$$L^2_{\mathcal{E}}(\mathbb{T}) = \left\{ f : \mathbb{T} \longrightarrow \mathcal{E} \text{ measurable} : \|f\|^2 = \int_{\mathbb{T}} \|f(z)\|^2_{\mathcal{E}}\, \mathrm{d}m(z) < \infty \right\},$$

where m represent the normalized Lebesgue measure on $\mathbb{T}$. The Hardy space $H^2_{\mathcal{E}}(\mathbb{D})$ can also be identified (via radial limits) with a subspace of $\mathcal{E}$-valued functions in $L^2_{\mathcal{E}}(\mathbb{T})$, which we will also denote by $H^2_{\mathcal{E}}(\mathbb{D})$. This subspace consists of functions f for which $\widehat{f}(n) = 0$ for all $n < 0$, where $\widehat{f}(n)$ denotes the n-th Fourier coefficient of f.

Given another Hilbert space $\mathcal{E}_*$, we denote by $L^\infty_{\mathcal{B}(\mathcal{E},\mathcal{E}_*)}(\mathbb{T})$ ($L^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{T})$ if $\mathcal{E}_* = \mathcal{E}$) the set of $\mathcal{B}(\mathcal{E},\mathcal{E}_*)$-valued bounded functions on $\mathbb{T}$.

Let $\mathcal{E}_*$ and $\mathcal{E}$ be Hilbert spaces. For $\Phi \in L^\infty_{\mathcal{B}(\mathcal{E},\mathcal{E}_*)}(\mathbb{T})$, the *Laurent operator* $L_\Phi : L^2_{\mathcal{E}}(\mathbb{T}) \to L^2_{\mathcal{E}_*}(\mathbb{T})$ is defined by $(L_\Phi f)(z) = \Phi(z)f(z)$, $z \in \mathbb{T}$. In this case, L_Φ is bounded and $\|L_\Phi\| = \|\Phi\|_\infty$. The *Toeplitz operator* $T_\Phi : H^2_{\mathcal{E}}(\mathbb{D}) \to H^2_{\mathcal{E}_*}(\mathbb{D})$ with (operator-valued) symbol Φ is defined by

$$T_\Phi = P_{H^2_{\mathcal{E}_*}(\mathbb{D})} L_\Phi|_{H^2_{\mathcal{E}}(\mathbb{D})},$$

where $P_{H^2_{\mathcal{E}_*}(\mathbb{D})}$ (in short P) is the orthogonal projection of $L^2_{\mathcal{E}_*}(\mathbb{T})$ onto $H^2_{\mathcal{E}_*}(\mathbb{D})$. It is well known that $\|T_\Phi\| = \|\Phi\|_\infty$ (cf. [1, Theorem 1.7, p. 112]).

The Toeplitz operator $T_\Phi = A$ is characterized by the operator equation

$$T_z^* A T_z = A.$$

Let $\mathcal{E}$ be a Hilbert space, and let $\Phi \in H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D})$. The *analytic Toeplitz operator* $T_\Phi : H^2_{\mathcal{E}}(\mathbb{D}) \to H^2_{\mathcal{E}}(\mathbb{D})$ with symbol Φ is defined by

$$(T_\Phi f)(z) = \Phi(z)f(z) \qquad (f \in H^2_{\mathcal{E}}(\mathbb{D}), z \in \mathbb{D}).$$

It is known that $\|T_\Phi\| = \|\Phi\|_\infty$, and T_Φ is an isometry if and only if Φ is inner [5, Proposition 2.2].

Let $A \in \mathcal{B}(H^2_{\mathcal{E}}(\mathbb{D}))$. Then $AT_z = T_z A$ if and only if A is an *analytic Toeplitz operator* that is, $A = T_\Phi$ for some $\Phi \in H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D})$.

Let J be defined on $L^2_{\mathcal{E}}(\mathbb{T})$ by

$$Jf(z) = \bar{z} f(\bar{z}), f \in L^2_{\mathcal{E}}(\mathbb{T}).$$

The J maps $\overline{zH^2_{\mathscr{E}}(\mathbb{D})}$ onto $H^2_{\mathscr{E}}(\mathbb{D})$, and J maps $H^2_{\mathscr{E}}(\mathbb{D})$ onto $\overline{zH^2_{\mathscr{E}}(\mathbb{D})}$. This J is a unitary operator with the following properties:

$$J^* = J, J^2 = I,\ JM_z^* = M_z J,\ JQ = PJ, \text{ and } JP = QJ,$$

where $Q := I - P$ the projection from $L^2_{\mathscr{E}}(\mathbb{T})$ to $\overline{zH^2_{\mathscr{E}}(\mathbb{D})} := L^2_{\mathscr{E}}(\mathbb{T}) \ominus H^2_{\mathscr{E}}(\mathbb{D})$.

The Hankel operator H_Φ from $H^2_{\mathscr{E}}(\mathbb{D})$ into $H^2_{\mathscr{E}_*}(\mathbb{D})$ is defined by

$$H_\Phi h = JQ(\Phi h) = PJ(\Phi h),\ \ h \in H^2_{\mathscr{E}}(\mathbb{D}).$$

The Hankel operator $H_\Phi = A$ is characterized by the operator equation

$$AT_z = T_z^* A. \tag{1.1}$$

It is easy to verify that $H_\Phi^* = H_{\Phi(\bar{z})^*}$. The Toeplitz and Hankel operators are related by the following equation

$$T_{\Phi\Psi} - T_\Phi T_\Psi = H^*_{\Phi^*} H_\Psi \quad \text{equivalently,} \quad T_{\check{\Phi}\tilde{\Psi}} - T_{\check{\Phi}} T_{\tilde{\Psi}} = H_\Phi H^*_{\Psi},$$

where $\check{\Phi}(z) = \Phi(\bar{z})$ and $\widetilde{\Phi}(z) = \Phi(\bar{z})^*$.

Theorem 1 ([5, p. 69]). *Let $T \in \mathscr{B}(\mathscr{H})$ and $T' \in \mathscr{B}(\mathscr{H}')$ be two contractions.*

(1) *Let $U \in \mathscr{B}(\mathscr{K})$ and $U' \in \mathscr{B}(\mathscr{K}')$ be the minimal isometric dilation of T and T', respectively. Then for any operator $W \in \mathscr{B}(\mathscr{H}, \mathscr{H}')$ satisfying $WT = T'W$, there exists $W_1 \in \mathscr{B}(\mathscr{K}, \mathscr{K}')$ such that $W_1 U = U' W_1$, $\|W_1\| = \|W\|$, $W = P_{\mathscr{H}'} W_1|_{\mathscr{H}}$ and $W(\mathscr{K} \ominus \mathscr{H}) \subset \mathscr{K}' \ominus \mathscr{K}'$.*

(2) *Let $U \in \mathscr{B}(\mathscr{K})$ and $U' \in \mathscr{B}(\mathscr{K}')$ be the co-isometric extension of T and T', respectively. Then for any operator $W \in \mathscr{B}(\mathscr{H}, \mathscr{H}')$ satisfying $WT = T'W$, there exists $W_1 \in \mathscr{B}(\mathscr{K}, \mathscr{K}')$ such that $W_1 U = U' W_1$, $\|W_1\| = \|W\|$, $W = W_1|_{\mathscr{H}}$.*

Theorem 2 (Douglas Theorem [2, Theorem 1]). *If $A, B \in \mathscr{B}(\mathscr{H})$, then the following statements are equivalent:*

(1) *$\mathscr{R}(A) \subseteq \mathscr{R}(B)$;*
(2) *$AA^* \leq \lambda^2 BB^*$ for some $\lambda \geq 0$;*
(3) *$A = BC$ for some $C \in \mathscr{B}(\mathscr{H})$.*

Moreover, if (1), (2), and (3) are valid, then there exists a unique operator C so that

(a) *$\|C\|^2 = \inf\{\mu | AA^* \leq \mu BB^*\}$;*
(b) *$\mathscr{N}(A) = \mathscr{N}(C)$;*
(c) *$\mathscr{R}(C) \subseteq \overline{\mathscr{R}(B^*)}$.*

Here $C = D^*$, where $D : \overline{\mathscr{R}(B^*)} \to \overline{\mathscr{R}(A^*)}$ defined by $D(B^* f) = A^* f$ and $D = 0$ on the orthogonal complement of $\overline{\mathscr{R}(B^*)}$.

Theorem 3 (Beurling-Lax-Halmos theorem [5, Chapter V, Theorem 3.3]). *Any T_z-invariant subspace of $\mathscr{M}$ of $H^2_{\mathscr{E}}(\mathbb{D})$ is of the form*

$$\mathscr{M} = \Theta H^2_{\mathscr{F}}(\mathbb{D}),$$

where $\mathscr{F}$ is a Hilbert space and $\Theta : \mathbb{D} \to \mathscr{B}(\mathscr{F}, \mathscr{E})$ is an inner function.

The main aim of this paper is to give the factorization of Hankel operators using Douglas theorem on range inclusion, factorization and majorization (cf. Theorem 2). We also studied the case when the range of the Toeplitz operator is included in the range of the Hankel operator (cf. Theorem 7). We have extended the results of [4] in the vector-valued Hardy space setup. We conclude by discussing the hyponormality of Hankel operators.

2. Main results

We begin this section with some observations.

Proposition 4. *Let* $\Psi \in H^\infty_{\mathscr{B}(\mathscr{E})}(\mathbb{D})$ *and* $\Phi \in L^\infty_{\mathscr{B}(\mathscr{E})}(\mathbb{T})$*. Then*

(1) $H_\Phi T_\Psi = H_{\Phi\Psi}$.
(2) $T^*_\Psi H_\Phi = H_\Phi T_{\tilde{\Psi}}$.

Proof. The proof of (1) follows directly from the definition. Let us prove (2).

$$T^*_\Psi H_\Phi = (H^*_\Phi T_\Psi)^* = (H_{\tilde{\Phi}} T_\Psi)^* = H^*_{\tilde{\Phi}\Psi} = H_{\Phi\tilde{\Psi}} = H_\Phi T_{\tilde{\Psi}}. \qquad \square$$

Remarks 5.

(1) For analytic function $\Phi \in H^\infty_{\mathscr{B}(\mathscr{E})}(\mathbb{D})$, the subspace $\mathscr{N}(T_\Phi)$ is T_z-invariant, equivalently, $\overline{\mathscr{R}(T^*_\Phi)}$ is T^*_z-invariant. By Theorem 3, there exists a Hilbert space $\mathscr{F}$ and an inner function $\Theta \in H^\infty_{\mathscr{B}(\mathscr{F},\mathscr{E})}(\mathbb{D})$ such that $\mathscr{N}(T_\Phi) = T_\Theta H^2_{\mathscr{F}}(\mathbb{D})$.

(2) For any $\Phi \in L^\infty_{\mathscr{B}(\mathscr{E})}(\mathbb{T})$, $\overline{\mathscr{R}(H^*_\Phi)}$ is T^*_z-invariant. By Theorem 3, there exists a Hilbert space $\mathscr{F}$ and an inner function $\Theta \in H^\infty_{\mathscr{B}(\mathscr{F},\mathscr{E})}(\mathbb{D})$ such that $\mathscr{N}(H_\Phi) = T_\Theta H^2_{\mathscr{F}}(\mathbb{D})$. If $\mathscr{R}(H_{\Phi_1}) \subseteq \mathscr{R}(H_{\Phi_2})$. Then, by Theorem 2, there exists $C \in \mathscr{B}(H^2_{\mathscr{E}}(\mathbb{D}))$ such that $H_{\Phi_1} = H_{\Phi_2} C$. Therefore, for any $\Delta \in H^\infty_{\mathscr{B}(\mathscr{E})}(\mathbb{D})$, using Proposition 4, we have

$$C^* T^*_\Delta H^*_{\Phi_2} = C^* H^*_{\Phi_2} T_{\tilde{\Delta}} = H^*_{\Phi_1} T_{\tilde{\Delta}} = T^*_\Delta H^*_{\Phi_1} = T^*_\Delta C^* H^*_{\Phi_2}$$
$$\left(C^* T^*_\Delta - T^*_\Delta C^*\right) H^*_{\Phi_2} = 0.$$

Therefore, $C^* T^*_\Delta - T^*_\Delta C^* = 0$ on $\overline{\mathscr{R}(H^*_{\Phi_2})}$ and $\mathscr{R}(CT_\Delta - T_\Delta C) \subseteq \mathscr{N}(H_{\Phi_2}) = T_\Theta H^2_{\mathscr{F}}(\mathbb{D})$.

Let $\Theta_1, \Theta_2 \in H^\infty_{\mathscr{B}(\mathscr{E})}(\mathbb{D})$ with Θ_2 be inner such that $\mathscr{R}(T_{\Theta_1}) \subseteq \mathscr{R}(T_{\Theta_2})$. Then, by Theorem 2, there exists $C \in \mathscr{B}(H^2_{\mathscr{E}}(\mathbb{D}))$ such that $T_{\Theta_1} = T_{\Theta_2} C$. Next, observe that

$$T_{\Theta_2} T_z C = T_z T_{\Theta_2} C = T_z T_{\Theta_1} = T_{\Theta_1} T_z = T_{\Theta_2} C T_z.$$

Since T_{Θ_2} is isometry, we have $CT_z = T_z C$ which implies $C = T_{\Theta_3}$ for some $\Theta_3 \in H^\infty_{\mathscr{B}(\mathscr{E})}(\mathbb{D})$. Thus $T_{\Theta_1} = T_{\Theta_2} T_{\Theta_3}$.

Again if $\Theta_1, \Theta_2 \in H^\infty_{\mathscr{B}(\mathscr{E})}(\mathbb{D})$ with Θ_1 is isometry and $\|\Theta_2\|_\infty \le 1$ such that $T_{\Theta_1} T^*_{\Theta_1} \le T_{\Theta_2} T^*_{\Theta_2}$, then $T_{\Theta_1} = T_{\Theta_2} T_{\Theta_3}$ for some $\Theta_3 \in H^\infty_{\mathscr{B}(\mathscr{E})}(\mathbb{D})$ which is inner. This fact is an easy consequence of [5, Proposition V.5.3].

By using the result in [3], we observe that if $\Theta_1, \Theta_2 \in H^\infty_{\mathscr{B}(\mathscr{E})}(\mathbb{D})$ with $\|\Theta_i\|_\infty \le 1$ for $i = 1, 2$ such that $T_{\Theta_1} T^*_{\Theta_1} \le T_{\Theta_2} T^*_{\Theta_2}$, then $T_{\Theta_1} = T_{\Theta_2} T_{\Theta_3}$ for some $\Theta_3 \in H^\infty_{\mathscr{B}(\mathscr{E})}(\mathbb{D})$ with $\|\Theta_3\|_\infty \le 1$.

In contrast to the aforementioned observations, we prove the following theorem, which also generalizes the factorization of contractive analytic functions [5, p. 205]. Although this result may be known to experts, we provide a different proof using Douglas theorem (cf. Theorem 2).

Theorem 6. *Let* $\Theta_1, \Theta_2 \in H^\infty_{\mathscr{B}(\mathscr{E})}(\mathbb{D})$*. Then* $\mathscr{R}(T_{\Theta_1}) \subseteq \mathscr{R}(T_{\Theta_2})$ *if and only if* $\Theta_1 = \Theta_2\Theta_3$ *for some* $\Theta_3 \in H^\infty_{\mathscr{B}(\mathscr{E})}(\mathbb{D})$.

Proof. Suppose that $\mathscr{R}(T_{\Theta_1}) \subseteq \mathscr{R}(T_{\Theta_2})$. Then, by Theorem 2, there exists an operator $W : \overline{\mathscr{R}(T^*_{\Theta_2})} \to \overline{\mathscr{R}(T^*_{\Theta_1})}$ such that $T^*_{\Theta_1} = W T^*_{\Theta_2}$. Since $T_{\Theta_i} T_z = T_z T_{\Theta_i}$, and $\overline{\mathscr{R}(T^*_{\Theta_i})}$ is T^*_z-invariant for $i = 1, 2$, we have

$$W T^*_z T^*_{\Theta_2} = W T^*_{\Theta_2} T^*_z = T^*_{\Theta_1} T^*_z = T^*_z T^*_{\Theta_1} = T^*_z W T^*_{\Theta_2}$$
$$W\left(T^*_z|_{\overline{\mathscr{R}(T^*_{\Theta_2})}}\right) = \left(T^*_z|_{\overline{\mathscr{R}(T^*_{\Theta_1})}}\right) W.$$

Therefore, by using (2) of Theorem 1, there exists W_1 such that $W_1 T^*_z = T^*_z W_1$ and $W = W_1|_{\overline{\mathscr{R}(T^*_{\Theta_2})}}$. Therefore, we get W^*_1 is an analytic Toeplitz operator T_{Θ_3}. Thus $T_{\Theta_1} = T_{\Theta_2} T_{\Theta_3}$ which implies $\Theta_1 = \Theta_2\Theta_3$. The converse trivially follows from the Theorem 2. $\square$

Theorem 7. *Let $\Phi, \Phi_1, \Phi_2, \Psi \in L^\infty_{\mathscr{B}(\mathscr{E})}(\mathbb{T})$. Then the following are true:*

(1) $\mathscr{R}(T_\Phi) \subseteq \mathscr{R}(H_\Psi)$ *if and only if* $\Phi = 0$.

(2) *Suppose that* Φ *is not identically zero, then* $\mathscr{R}(H_\Phi^*) \subseteq \mathscr{R}(T_\Phi^*)$ *if and only if* P *is bounded below on* $\overline{\Phi H^2_{\mathscr{E}}(\mathbb{D})}$.

(3) $\mathscr{R}(H_{\Phi_1}) \subseteq \mathscr{R}(H_{\Phi_2})$ *if and only if there exists* $\Omega \in H^\infty_{\mathscr{B}(\mathscr{E})}$ *such that* $H_{\Phi_1} = H_{\Phi_2} T_\Omega = H_{\Phi_2 \Omega}$.

Proof.

(1). Suppose that $\mathscr{R}(T_\Phi) \subseteq \mathscr{R}(H_\Psi)$. Then, for some $\lambda \geq 0$

$$T_\Phi T_\Phi^* \leq \lambda^2 H_\Psi H_\Psi^*$$
$$T_\Phi T_\Phi^* \leq \lambda^2 (T_{\breve{\Psi}\tilde{\Psi}} - T_{\breve{\Psi}} T_{\tilde{\Psi}}).$$

Thus for every $f \in H^2_{\mathscr{E}}(\mathbb{D})$, we have

$$\left\| T_{\Phi^*} f \right\|^2 + \lambda^2 \left\| T_{\tilde{\Psi}} f \right\|^2 \leq \lambda^2 \left\| \tilde{\Psi} f \right\|^2$$
$$\left\| P(\Phi^* f) \right\|^2 + \lambda^2 \left\| P(\tilde{\Psi} f) \right\|^2 \leq \lambda^2 \left\| \tilde{\Psi} f \right\|^2.$$

Since $\{z^m \eta : m \in \mathbb{N}, \eta \in \mathscr{E}\}$ is total in $H^2_{\mathscr{E}}(\mathbb{D})$, we have

$$\left\| P(\Phi^* z^m \eta) \right\|^2 + \lambda^2 \left\| P(\tilde{\Psi} z^m \eta) \right\|^2 \leq \lambda^2 \left\| \tilde{\Psi}(z^m \eta) \right\|^2.$$

As $m \to \infty$, from above expression, we conclude that

$$\left\| \Phi^* \right\|^2 + \lambda^2 \left\| \tilde{\Psi} \right\|^2 \leq \lambda^2 \left\| \tilde{\Psi} \right\|^2,$$

which implies $\Phi = 0$. The converse is evident.

(2). Suppose that $\mathscr{R}(H_\Phi^*) \subseteq \mathscr{R}(T_\Phi^*)$. Then, by Theorem 2, there exists $\lambda \geq 0$ such that $H_\Phi^* H_\Phi \leq \lambda^2 T_\Phi^* T_\Phi$, which implies

$$T_{\Phi^*\Phi} - T_{\Phi^*} T_\Phi \leq \lambda^2 T_\Phi^* T_\Phi.$$

Therefore, for every $f \in H^2_{\mathscr{E}}(\mathbb{D})$, we have

$$\left\| \Phi f \right\|^2 - \left\| T_\Phi f \right\|^2 \leq \lambda^2 \left\| T_\Phi f \right\|^2$$
$$\left\| P(\Phi f) \right\|^2 \geq (1 + \lambda^2)^{-1} \left\| \Phi f \right\|^2.$$

Which shows that the orthogonal projection operator P is bounded below on $\overline{\Phi H^2_{\mathscr{E}}(\mathbb{D})}$.

Conversely, if P is bounded below on $\overline{\Phi H^2_{\mathscr{E}}(\mathbb{D})}$, then there exists a constant $M \geq 0$ such that $\left\| P(\Phi f) \right\|^2 \geq M \left\| \Phi f \right\|^2$ for every $f \in H^2_{\mathscr{E}}(\mathbb{D})$. Therefore,

$$\left\| P(\Phi(\eta z^n)) \right\|^2 \geq M \left\| \Phi(\eta z^n) \right\|^2.$$

As $n \to \infty$, we have $M \leq 1$. Then choose $M = (1 + \mu^2)^{-1}$ for some $\mu \geq 0$, and apply the above argument in the necessity part to obtain the desired conclusion.

(3). First, we assume that $\mathscr{R}(H_{\Phi_1}) \subseteq \mathscr{R}(H_{\Phi_2})$. Then, by Theorem 2, we have

$$H_{\Phi_1} H_{\Phi_1}^* \leq \lambda^2 H_{\Phi_2} H_{\Phi_2}^*, \quad H_{\Phi_1} = H_{\Phi_2} C$$

such that $\mathscr{N}(C) = \mathscr{N}(H_{\Phi_1})$, $\|C\| \leq \lambda^2$, and $\mathscr{R}(C) \subseteq \overline{\mathscr{R}(H_{\Phi_2}^*)}$. Note that $C = W^*$, where $W : \overline{\mathscr{R}(H_{\Phi_2}^*)} \to \overline{\mathscr{R}(H_{\Phi_1}^*)}$ is defined by $W(H_{\Phi_2}^* f) = H_{\Phi_1}^* f$ and $W = 0$ on $\mathscr{R}(H_{\Phi_2}^*)^\perp$.

Since $T_z^* H_{\Phi_2} = H_{\Phi_2} T_z$, we have $\mathscr{N}(H_{\Phi_2})$ is T_z-invariant, equivalently $\overline{\mathscr{R}(H_{\Phi_2}^*)}$ is T_z^*-invariant. Therefore, by using (1.1), we get

$$W T_z^* H_{\Phi_2}^* = W H_{\Phi_2}^* T_z = H_{\Phi_1}^* T_z = T_z^* H_{\Phi_1}^* = T_z^* W H_{\Phi_2}^*$$

which implies

$$W \left(T_z^* |_{\overline{\mathscr{R}(H_{\Phi_2}^*)}} \right) = \left(T_z^* |_{\overline{\mathscr{R}(H_{\Phi_1}^*)}} \right) W.$$

Using (2) of Theorem 1, there exists $W_1 : H^2_{\mathscr{E}}(\mathbb{D}) \to H^2_{\mathscr{E}}(\mathbb{D})$ such that $W_1 T_z^* = T_z^* W_1$ and $W = W_1|_{\overline{\mathscr{R}(H^*_{\Phi_2})}}$. Thus W_1^* is an analytic Toeplitz operator T_Ω and $H_{\Phi_1} = H_{\Phi_2} W^* = H_{\Phi_2} T_\Omega$. The converse is obvious. □

Corollary 8. *Let $\Phi \in L^\infty_{\mathscr{B}(\mathscr{E})}(\mathbb{T})$. Then H_Φ is hyponormal if and only if $H_\Phi = H^*_\Phi T_\Omega$ for some $\Omega \in H^\infty_{\mathscr{B}(\mathscr{E})}(\mathbb{D})$ with $\|\Omega\|_\infty \le 1$.*

Proof. Suppose that H_Φ is hyponormal that is, $H_\Phi H^*_\Phi \le H^*_\Phi H_\Phi$ equivalently, $H_\Phi H^*_\Phi \le H_{\tilde{\Phi}} H^*_{\tilde{\Phi}}$. Now set $\Phi_1 = \Phi$ and $\Phi_2 = \widetilde{\Phi}$. By applying (3) of Theorem 7, we have $H_\Phi = H_{\tilde{\Phi}} T_\Omega = H^*_\Phi T_\Omega$ for some $\Omega \in H^\infty_{\mathscr{B}(\mathscr{E})}(\mathbb{D})$ with $\|\Omega\|_\infty \le 1$.

Conversely, suppose $H_\Phi = H^*_\Phi T_\Omega$ for some $\Omega \in H^\infty_{\mathscr{B}(\mathscr{E})}(\mathbb{D})$ with $\|\Omega\|_\infty \le 1$. Then $\|T_\Omega\| = \|\Omega\|_\infty \le 1$. Therefore,

$$H^*_\Phi H_\Phi - H_\Phi H^*_\Phi = H^*_\Phi (I - T_\Omega T^*_\Omega) H_\Phi \ge 0.$$

Hence H_Φ is hyponormal. □

Corollary 9. *Let $\Phi \in L^\infty_{\mathscr{B}(\mathscr{E})}(\mathbb{T})$. If H_Φ is hyponormal, then it is normal.*

Proof. Suppose H_Φ is hyponormal. By Corollary 8 and (2) of Proposition 4, we observe that

$$H_{\tilde{\Phi}} = H^*_\Phi = T^*_\Omega H_\Phi = H_\Phi T_{\tilde{\Omega}} = (H^*_\Phi)^* T_{\tilde{\Omega}} = H^*_{\tilde{\Phi}} T_{\tilde{\Omega}}.$$

Which shows that if H_Φ is hyponormal, then H^*_Φ is also hyponormal, and consequently, normal. □

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The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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