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Comptes Rendus
Mathématique

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Volume 362 (2024), p. 1793-1799

Online since: 25 November 2024

https://doi.org/10.5802/crmath.684

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www.centre-mersenne.org — e-ISSN : 1778-3569

Comptes Rendus. Mathématique
2024, Vol. 362, p. 1793-1799
https://doi.org/10.5802/crmath.684

Research article / *Article de recherche*
Geometry and Topology / *Géométrie et Topologie*

A simple generalization of a Willmore-type inequality

Une généralisation simple d'une inégalité de type Willmore

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Abstract. We give a simple generalization of the Willmore-type inequality in [1] without assuming that Ricci curvature is nonnegative everywhere.

Résumé. Nous donnons une généralisation simple de l'inégalité de type Willmore dans [1] sans supposer que la courbure de Ricci est partout non négative.

Keywords. Willmore-type inequality, Ricci curvature.

Mots-clés. Inégalité de type Willmore, courbure de Ricci.

2020 Mathematics Subject Classification. 53C20, 53C42.

Funding. This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT)(No. RS-2023-00242367).

Manuscript received 22 September 2022, revised 23 June 2023 and 30 January 2024, accepted 3 September 2024.

1. Introduction

The classical Willmore inequality for a bounded domain Ω of $\mathbb{R}^3$ with smooth boundary says that

$$\int_{\partial\Omega} H^2 \,\mathrm{d}\,\mathrm{vol}_{\partial\Omega} \geq 16\pi,$$

where H is the mean curvature of $\partial\Omega$ and $\mathrm{d}\,\mathrm{vol}_{\partial\Omega}$ is the volume form of $\partial\Omega$. In [1], Agostiniani, Fogagnolo, and Mazzieri obtained the following Willmore-type inequality for an n-dimensional complete noncompact Riemannian manifold (M, g) with the nonnegative Ricci curvature and Euclidean volume growth:

$$\int_{\partial\Omega} \left|\frac{H}{n-1}\right|^{n-1} \mathrm{d}\,\mathrm{vol}_{\partial\Omega} \geq \mathrm{AVR}(g)|S^{n-1}|, \tag{1}$$

where $|S^{n-1}|$ is the volume of the standard sphere and $\mathrm{AVR}(g)$ is the asymptotic volume ratio, i.e.

$$\mathrm{AVR}(g) = \lim_{r\to\infty} \frac{n\,\mathrm{vol}(B(p,r))}{r^n|S^{n-1}|},$$

where $B(p,r)$ is the r-ball centered at $p\in M$. They used the monotonicity-rigidity properties of the function $U_\beta(t)$ which is defined as follows:

$$U_\beta(t) = t^{-\beta(\frac{n-1}{n-2})}\int_{\{u=t\}}|\nabla u|^{\beta+1},$$

where u is the harmonic function which vanishes at infinity and $u=1$ on $\partial\Omega$.

In this paper, we will generalize the inequality without assuming $\mathrm{Ric}\geq 0$ everywhere. The condition $\mathrm{Ric}\geq 0$ is essential in [1] to apply results on harmonic functions.

Let $\rho(q)=\max\{(-\mathrm{Ric}(v,v))_+ \mid |v|=1,\ v\in T_qM\}$, where $f_+=\max\{0,f\}$. We define integral norms $\mathscr{R}_{n-1}$ as follows:

$$\mathscr{R}_{n-1}=\int_M \rho^{n-1}\,\mathrm{d}V,$$

where $\mathrm{d}V$ is the volume form of M.

Theorem 1. *Let (M,g) be an n-dimensional complete noncompact Riemannian manifold with Euclidean volume growth and* $\mathrm{Ric}\geq 0$ *outside $B(p,R_0)$. If $\Omega\subset M$ is a bounded and open subset with smooth boundary, then*

$$\left(\int_{\partial\Omega}\left|\frac{H}{n-1}\right|^{n-1}\mathrm{d}\mathrm{vol}_{\partial\Omega}\right)^{\frac{1}{n-1}} \geq \left(\mathrm{AVR}(g)|S^{n-1}|\right)^{\frac{1}{n-1}} - \frac{(2R_0)^{\frac{n-1}{n-2}}(\mathscr{R}_{n-1})^{\frac{1}{n-1}}}{n-1}. \tag{2}$$

If $\mathrm{Ric}\geq 0$, then $\mathscr{R}_{n-1}=0$, so we obtain (1).

2. Proof of Theorem 1

We will use the following notations. Let

$$\Omega_t=\{x\in M \mid d(x,\Omega)\leq t\}$$

$$\partial\Omega_t=\{x\in M \mid d(x,\Omega)=t\}.$$

Let γ_q be the outward normal geodesic such that $\gamma_q(0)=q$ and $\gamma_q'(0)$ is perpendicular to $\partial\Omega$ for $q\in\partial\Omega$. Let

$$t_q=\max\{t \mid d(\gamma_q(t),\partial\Omega)=t\}.$$

Then we have

$$M\setminus\Omega=\bigcup_{q\in\partial\Omega}\{\gamma_q(t)\mid t\leq t_q\}.$$

We denote by g_t the induced metric of $\partial\Omega_t$ from the metric g of M. Let $\mathrm{d}\mathrm{vol}_t$ be the volume form of $\partial\Omega_t$ induced from g_t. Then the volume form $\mathrm{d}\mathrm{vol}_{\partial\Omega}$ of $\partial\Omega$ is $\mathrm{d}\mathrm{vol}_0$ and the volume form of M satisfies $\mathrm{d}V=\mathrm{d}t\wedge\mathrm{d}\mathrm{vol}_t$. By identifying $\gamma_q(t)\in\partial\Omega_t$ with $q\in\partial\Omega$ for $t\leq t_q$, we define $\omega(t,q)$ and $h(t,q)$ as follows:

$$\mathrm{d}\mathrm{vol}_t=\omega(t,\cdot)\,\mathrm{d}\mathrm{vol}_{\partial\Omega} \tag{3}$$

$$\frac{\partial}{\partial t}\omega(t,q)=h(t,q)\omega(t,q), \tag{4}$$

where $h(t,\cdot)$ is the mean curvature of $\partial\Omega_t$. So $H=h(0,\cdot)$. We abbreviate $\omega(t,q)$ and $h(t,q)$ to $\omega(t)$ and $h(t)$, respectively. For $t<t_q$, h satisfies

$$h'+\frac{h^2}{n-1}\leq -\mathrm{Ric}\left(\frac{\partial}{\partial t},\frac{\partial}{\partial t}\right), \tag{5}$$

where $\frac{\partial}{\partial t}=\gamma_q'(t)$ and $|\frac{\partial}{\partial t}|=1$. For $t>t_q$, we let $\omega(t)=0$. Then $\omega(t)$ is defined for all $t>0$ and $\mathrm{vol}(\Omega_t\setminus\Omega)=\int_0^t\int_{\partial\Omega}\omega(s,q)\,\mathrm{d}\mathrm{vol}_{\partial\Omega}\,\mathrm{d}s$. Furthermore, $\omega'(t)=h(t)\omega(t)$ for $t<t_q$ and $\omega'(t)=0$ for $t>t_q$. We denote left and right limits $\lim_{\delta\to 0-}f(t+\delta)$ and $\lim_{\delta\to 0+}f(t+\delta)$ by $f(t-)$ and $f(t+)$, respectively.

Let

$$\psi(t)=\begin{cases} h_+(t) & \text{if } t<t_q \\ h_+(t-) & \text{if } t=t_q \\ 0 & \text{if } t>t_q, \end{cases} \tag{6}$$

where $h_+(t)=\max\{h(t),0\}$. Then we have

$$\omega'(t)\le\psi(t)\omega(t) \tag{7}$$

for $t<t_q$. Also $\psi'=h'$ on $\{h(t)>0\}$ and $\psi'=0$ for $t>t_q$.

Although both ω and ψ are continuous along $\gamma_q(t)$ for $t\ne t_q$, they are left continuous for all $t>0$, i.e. $\omega(t-)=\omega(t)$ and $\psi(t-)=\psi(t)$ for all $t>0$. Even if ω is not continuous at $t=t_q$, the left derivative $\lim_{\delta\to 0-}\frac{\omega(t+\delta)-\omega(t)}{\delta}$ is well defined for any $t>0$. On the other hand, the right derivative $\lim_{\delta\to 0+}\frac{\omega(t+\delta)-\omega(t)}{\delta}$ is not well defined, i.e. it can be $-\infty$. For a while, we will abbreviate the left derivative $\lim_{\delta\to 0-}\frac{f(t+\delta)-f(t)}{\delta}$ to $f'(t)$ for simplicity.

The left derivative $\omega'(t)$ is bounded on $[0,T]\times\partial\Omega$ for $T>0$ by the Jacobi equation $J''=R(\frac{\partial}{\partial t},J)\frac{\partial}{\partial t}$, where R is the curvature tensor and J is the Jacobi field. Also $(\psi^{n-1})'(t)$ is bounded on $[0,T]\times\partial\Omega$ since $h'(t)\ne-\infty$ (i.e. $\gamma_q(t)$ is not a focal point) at t when $h(t)\ge 0$.

Since $\rho=\max\{(-\operatorname{Ric}(v,v))_+ \mid |v|=1,\ v\in T_qM\}$, (5) becomes

$$\psi'+\frac{\psi^2}{n-1}\le\rho, \tag{8}$$

where we also let $\rho(\gamma_q(t))=0$ if $t>t_q$. We let

$$\mathscr{H}_{n-1}(t)=\int_{\partial\Omega_t}\psi^{n-1}\mathrm{d}\,\mathrm{vol}_t=\int_{\partial\Omega}\psi^{n-1}\omega.$$

Since $(\psi^{n-1}\omega)(t-)=(\psi^{n-1}\omega)(t)$ for every t, $\mathscr{H}_{n-1}(t)$ is left continuous, i.e.

$$\mathscr{H}_{n-1}(t-)=\lim_{\delta\to 0-}\int_{\partial\Omega}(\psi^{n-1}\omega)(t+\delta)=\int_{\partial\Omega}(\psi^{n-1}\omega)(t)=\mathscr{H}_{n-1}(t) \tag{9}$$

by the dominated convergence theorem.

Since $\frac{(\psi^{n-1}\omega)(t+\delta)-(\psi^{n-1}\omega)(t)}{\delta}\le 0$ for $t>t_q$ and $\delta<0$, we obtain that

$$\begin{aligned}\int_{\partial\Omega}&\frac{\psi^{n-1}(t+\delta)\omega(t+\delta)-\psi^{n-1}(t)\omega(t)}{\delta}\\ &\le\int_{\{q\in\partial\Omega\ |\ t\le t_q\}}\frac{(\psi^{n-1}(t+\delta)-\psi^{n-1}(t))\omega(t+\delta)+\psi^{n-1}(t)(\omega(t+\delta)-\omega(t))}{\delta}\\ &\longrightarrow\int_{\{q\in\partial\Omega\ |\ t\le t_q\}}(\psi^{n-1})'\omega+\psi^{n-1}\omega'=\int_{\partial\Omega}(\psi^{n-1})'\omega+\psi^{n-1}\omega'\end{aligned} \tag{10}$$

as $\delta\to 0-$ by the dominated convergence theorem for $t\le t_q$ and $\omega(t)=\psi(t)=0$ for $t>t_q$.

From now on, we redefine $f'(t)=\frac{\mathrm{d}}{\mathrm{d}t}f(t)$ as follows:

$$f'(t):=\limsup_{\delta\to 0-}\frac{f(t+\delta)-f(t)}{\delta}.$$

We call $f'(t)$ the left derivative of f for simplicity. Then the left derivative $\mathscr{H}'_{n-1}(t)$ satisfies

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\partial\Omega_t}\psi^{n-1}\mathrm{d}\mathrm{vol}_t &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\partial\Omega}\psi^{n-1}\omega \\ &= \limsup_{\delta\to 0-}\left(\int_{\partial\Omega}\frac{\psi^{n-1}(t+\delta)\omega(t+\delta)-\psi^{n-1}(t)\omega(t)}{\delta}\right) \\ &\le \int_{\partial\Omega}(\psi^{n-1})'\omega+\psi^{n-1}\omega' \\ &\le \int_{\partial\Omega}\psi^{n-2}((n-1)\psi'+\psi^2)\omega \\ &\le (n-1)\int_{\partial\Omega_t}\psi^{n-2}\rho\,\mathrm{d}\mathrm{vol}_t \\ &\le (n-1)\left(\int_{\partial\Omega_t}\psi^{n-1}\mathrm{d}\mathrm{vol}_t\right)^{\frac{n-2}{n-1}}\left(\int_{\partial\Omega_t}\rho^{n-1}\mathrm{d}\mathrm{vol}_t\right)^{\frac{1}{n-1}}\end{aligned} \tag{11}$$

by (8), (10).

Proposition 2. *For sufficiently large t, we have*

$$\left(\int_{\partial\Omega}|H|^{n-1}\,\mathrm{d}\mathrm{vol}_{\partial\Omega}\right)^{\frac{1}{n-1}} \ge \mathscr{H}_{n-1}(t)^{\frac{1}{n-1}}-(2R_0)^{\frac{n-1}{n-2}}(\mathscr{R}_{n-1})^{\frac{1}{n-1}}. \tag{12}$$

Proof. Since $\int_{\partial\Omega}|H|^{n-1}\ge\mathscr{H}_{n-1}(0)$, we only need to show that

$$\mathscr{H}_{n-1}(t)^{\frac{1}{n-1}}-\mathscr{H}_{n-1}(0)^{\frac{1}{n-1}}\le(2R_0)^{\frac{n-1}{n-2}}(\mathscr{R}_{n-1})^{\frac{1}{n-1}}.$$

By (11),

$$\frac{\mathscr{H}'_{n-1}(t)}{(n-1)\mathscr{H}_{n-1}(t)^{\frac{n-2}{n-1}}}\le\left(\int_{\partial\Omega_t}\rho^{n-1}\mathrm{d}\mathrm{vol}_t\right)^{\frac{1}{n-1}} \tag{13}$$

if $\mathscr{H}_{n-1}(t)\neq 0$.

If $g(x)=x^{\frac{1}{n-1}}$ and f is left continuous, then the left derivative $g(f)'$ satisfies that

$$\begin{aligned} g(f)'(a) &= \limsup_{\delta\to 0-}\left(\frac{g(f(a+\delta))-g(f(a))}{f(a+\delta)-f(a)}\right)\left(\frac{f(a+\delta)-f(a)}{\delta}\right) \\ &= g'(f(a))f'(a)\end{aligned} \tag{14}$$

if $f(a)\neq 0$. By (13) and (14), the left derivative $\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H}_{n-1}(t)^{\frac{1}{n-1}}$ satisfies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathscr{H}_{n-1}(t)^{\frac{1}{n-1}}\right)\le\left(\int_{\partial\Omega_t}\rho^{n-1}\,\mathrm{d}\mathrm{vol}_t\right)^{\frac{1}{n-1}} \tag{15}$$

if $\mathscr{H}_{n-1}(t)\neq 0$. If $\mathscr{H}_{n-1}(t)=0$, then the left derivative $(\mathscr{H}_{n-1}(t))^{\frac{1}{n-1}})'\le 0$ since $\mathscr{H}_{n-1}(s)^{\frac{1}{n-1}}\ge 0$ for any s. Thus (15) holds for any t.

Lemma 3. *Let f and $g\ge 0$ be left continuous on $[a,b]$ and $f(t)\ge f(t+)$ for all $t\in[a,b]$. If the left derivative f' satisfies that $f'(t)\le g(t)$ for any t and g is bounded, then*

$$f(b)-f(a)\le\int_a^b g.$$

Proof. For $\eta>0$, let $I=\{x\in[a,b]\mid f(b)-f(x)\le\int_x^b(g(t)+\eta)dt\}$ and $A=\inf I$. Then it is clear that $b\in I$. First, we show that $A\in I$. If not, there is a sequence $a_n\in I$ such that $a_n\to A$. Since $f(A)\ge f(A+)$, we have

$$f(b)-f(A)\le f(b)-f(A+)\le\lim_{n\to\infty}\int_{a_n}^b(g(t)+\eta)\,\mathrm{d}t=\int_A^b(g(t)+\eta)\,\mathrm{d}t.$$

Hence we obtain that $A\in I$.

Now we show that $A = a$. If not, for s satisfying $a \le s < A$,

$$f(b) - f(s) > \int_s^b (g(t)+\eta)\,\mathrm{d}t.$$

Since $f(b) - f(A) \le \int_A^b (g(t)+\eta)\,\mathrm{d}t$, we have

$$f(A) - f(s) > \int_s^A (g(t)+\eta)\,\mathrm{d}t.$$

Then

$$f'(A) = \limsup_{s\to A-} \frac{f(A)-f(s)}{A-s} \ge \limsup_{s\to A-} \frac{1}{A-s}\int_s^A (g(t)+\eta)\,\mathrm{d}t = g(A)+\eta$$

since g is left continuous. But this is a contradiction to our assumption. So we obtain that $A = a$, i.e. $f(b) - f(a) \le \int_a^b (g(t)+\eta)\mathrm{d}t$. Since η is arbitrarily chosen, letting $\eta \to 0$, we obtain Lemma 3. □

By the same reason that $\mathscr{H}_{n-1}$ is left continuous, $\left(\int_{\partial\Omega_t} \rho^{n-1}\,\mathrm{d}\mathrm{vol}_t\right)^{\frac{1}{n-1}}$ is left continuous. Hence, by (15) and Lemma 3 with $f = \mathscr{H}_{n-1}$ and $g = \left(\int_{\partial\Omega_t} \rho^{n-1}\,\mathrm{d}\mathrm{vol}_t\right)^{\frac{1}{n-1}}$,

$$\mathscr{H}_{n-1}(t)^{\frac{1}{n-1}} - \mathscr{H}_{n-1}(0)^{\frac{1}{n-1}} \le \int_0^t \left(\int_{\partial\Omega_s} \rho^{n-1}\mathrm{d}\mathrm{vol}_s\right)^{\frac{1}{n-1}} \mathrm{d}s. \tag{16}$$

Since the diameter of $B(p,R_0)$ is not larger than $2R_0$, $B(p,R_0)\setminus\Omega \subset \Omega_{t_0+2R_0}\setminus\Omega_{t_0}$ for some $t_0 \ge 0$. For $t > t_0 + 2R_0$,

$$\begin{aligned}\mathscr{H}_{n-1}(t)^{\frac{1}{n-1}} - \mathscr{H}_{n-1}(0)^{\frac{1}{n-1}} &\le \int_{t_0}^{t_0+2R_0} \left(\int_{\partial\Omega_s} \rho^{n-1}\,\mathrm{d}\mathrm{vol}_s\right)^{\frac{1}{n-1}} \mathrm{d}s \\ &\le \left(\int_M \rho^{n-1}\,\mathrm{d}V\right)^{\frac{1}{n-1}} (2R_0)^{\frac{n-2}{n-1}},\end{aligned} \tag{17}$$

so we have Proposition 2. □

Now we calculate $\mathscr{H}_{n-1}(t)$ with $\mathrm{vol}(\partial\Omega_t)$. Let $A(t) = \mathrm{vol}_t(\partial\Omega_t) = \int_{\partial\Omega} \omega(t,\cdot)$. Then $A(t)$ is left continuous similarly as in (9). Similarly as in (11),

$$\begin{aligned}A'(t) &\le \int_{\partial\Omega} \omega' \\ &\le \int_{\partial\Omega_t} \psi\,\mathrm{d}\mathrm{vol}_t \\ &\le \left(\int_{\partial\Omega_t} \psi^{n-1}\,\mathrm{d}\mathrm{vol}_t\right)^{\frac{1}{n-1}} \left(\int_{\partial\Omega_t} \mathrm{d}\mathrm{vol}_t\right)^{\frac{n-2}{n-1}} \\ &\le \mathscr{H}_{n-1}(t)^{\frac{1}{n-1}} A(t)^{\frac{n-2}{n-1}}.\end{aligned} \tag{18}$$

by (7). By (14), (18), and the left continuity of $A(t)$,

$$(n-1)\left(A^{\frac{1}{n-1}}(t)\right)' \le \mathscr{H}_{n-1}(t)^{\frac{1}{n-1}}. \tag{19}$$

From Proposition 2, we obtain that

$$\left(\int_{\partial\Omega} |H|^{n-1}\,\mathrm{d}\mathrm{vol}_{\partial\Omega}\right)^{\frac{1}{n-1}} \ge (n-1)\left(A^{\frac{1}{n-1}}(t)\right)' - (2R_0)^{\frac{n-1}{n-2}} (\mathscr{R}_{n-1})^{\frac{1}{n-1}}. \tag{20}$$

Now we calculate the left derivative $\left(A^{\frac{1}{n-1}}(t)\right)'$ with $\mathrm{AVR}(g)$.

Lemma 4. *Let $f : [0,\infty) \to \mathbb{R}$ have left derivative and satisfy $f(t+) \le f(t)$ for all t. If $\limsup_{t\to\infty} \frac{f(t)}{t} = a > 0$, then the left derivative $f'(t)$ satisfies*

$$\limsup_{t\to\infty} f'(t) \ge a. \tag{21}$$

Proof. Assume $\limsup_{t\to\infty} f'(t) < a-2\epsilon < a$. Then there exists $r_0 > 0$ such that $f'(t) < a-\epsilon$ for $t \ge r_0$. Take a large $R > r_0$. By Lemma 3,

$$f(R) - f(r_0) \le \int_{r_0}^{R} (a-\epsilon) = (a-\epsilon)(R-r_0), \tag{22}$$

which implies that

$$\limsup_{t\to\infty} \frac{f(t)}{t} \le a-\epsilon.$$

It is a contradiction to our condition. Hence,

$$\limsup_{t\to\infty} f'(t) \ge a, \tag{23}$$

which completes the proof of Lemma 4. □

We let $f(t) = \mathrm{vol}(\Omega_t \setminus \Omega)^{\frac{1}{n}}$, where $\mathrm{vol}(\Omega_t \setminus \Omega) = \int_0^t \int_{\partial\Omega} \omega(s,q)\,\mathrm{d}\mathrm{vol}_{\partial\Omega}\,\mathrm{d}s$. Although $\mathrm{vol}(\Omega_t \setminus \Omega)$ may not be differentiable since $\omega(t)$ is not continuous at t_q, f is continuous. We can obtain the left derivative as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{vol}(\Omega_t \setminus \Omega) = \int_{\partial\Omega} \omega(t,q)\,\mathrm{d}\mathrm{vol}_{\partial\Omega} = A(t). \tag{24}$$

Since $\mathrm{vol}(\Omega_t \setminus \Omega)$ is continuous, the left derivative satisfies

$$f'(t) = \frac{1}{n}\frac{A(t)}{\mathrm{vol}(\Omega_t \setminus \Omega)^{\frac{n-1}{n}}}$$

by (14).

Since $\frac{\mathrm{vol}(\Omega_R\setminus\Omega)}{\mathrm{vol}(B(p,R))} \to 1$ as $R \to \infty$, we have

$$\lim_{t\to\infty} \frac{f(t)}{t} = \left(\frac{\mathrm{AVR}(g)|S^{n-1}|}{n}\right)^{\frac{1}{n}}. \tag{25}$$

From $f'(t) = \frac{1}{n}\frac{A(t)}{\mathrm{vol}(\Omega_t\setminus\Omega)^{\frac{n-1}{n}}}$ and Lemma 4, we obtain that

$$\begin{aligned}\left(\frac{\mathrm{AVR}(g)|S^{n-1}|}{n}\right)^{\frac{1}{n}} &\le \limsup_{t\to\infty} f'(t) = \limsup_{t\to\infty} \frac{1}{n}\frac{A(t)}{\mathrm{vol}(\Omega_t \setminus \Omega)^{\frac{n-1}{n}}} \\ &= \limsup_{t\to\infty} \frac{1}{n}\frac{A(t)}{\mathrm{vol}(B(p,t))^{\frac{n-1}{n}}}.\end{aligned} \tag{26}$$

Although $A^{\frac{1}{n-1}}(t)$ may not be continuous, we have $A^{\frac{1}{n-1}}(t+) \le A^{\frac{1}{n-1}}(t)$ since $\omega(t) \ge \omega(t+)$. By Lemma 4 and (25), (26),

$$\begin{aligned}\limsup_{t\to\infty} (A^{\frac{1}{n-1}}(t))' &\ge \limsup_{t\to\infty} \frac{A^{\frac{1}{n-1}}(t)}{t} \\ &\ge n^{\frac{1}{n-1}} \lim_{t\to\infty} \left(\frac{\mathrm{AVR}(g)|S^{n-1}|}{n}\right)^{\frac{1}{n(n-1)}} \frac{\mathrm{vol}(B(p,t))^{\frac{1}{n}}}{t} \\ &= \left(\mathrm{AVR}(g)|S^{n-1}|\right)^{\frac{1}{n-1}}.\end{aligned}$$

Consequently, from (20),

$$\left(\int_{\partial\Omega}\left(\frac{|H|}{n-1}\right)^{n-1}\right)^{\frac{1}{n-1}} \ge \left(\mathrm{AVR}(g)|S^{n-1}|\right)^{\frac{1}{n-1}} - \frac{1}{n-1}(2R_0)^{\frac{n-1}{n-2}}(\mathscr{R}_{n-1})^{\frac{1}{n-1}},$$

which completes the proof of Theorem 1.

Acknowledgements

The author would like to express his gratitude to the referee for their helpful comments, which improve this paper very much.

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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