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Constructing many-twist Möbius bands with small aspect ratios

Construction de bandes de Möbius à torsion multiple avec petits rapports d'aspect

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Abstract. This paper presents a construction of a folded paper ribbon knot that provides a constant upper bound on the infimal aspect ratio for paper Möbius bands and annuli with arbitrarily many half-twists. In particular, the construction shows that paper Möbius bands and annuli with any number of half-twists can be embedded with aspect ratio less than 6.

Résumé. Cet article présente une construction d'un nœud de ruban de papier plié qui fournit une limite supérieure constante sur le rapport d'aspect infinitésimal pour les bandes de Möbius en papier et les anneaux avec un nombre arbitraire de demi-torsions. En particulier, la construction montre que les bandes de Möbius en papier et les anneaux avec un nombre arbitraire de demi-torsions peuvent être plongés avec un rapport d'aspect inférieur à 6.

Keywords. Möbius Band, Halpern–Weaver Conjecture, Folded Ribbon Knots Isometric Embedding, Optimization.

Mots-clés. Bande de Möbius, conjecture de Halpern–Weaver, nœuds de rubans pliés Emboîtement isométrique, optimisation.

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1. Introduction

In 1977, Halpern and Weaver conjectured that the infimal aspect ratio of an embedded paper Möbius band is $\sqrt{3}$ [4]. This conjecture was recently proven by Schwartz in [7]. Shortly after, Schwartz proved that the infimal aspect ratio for the 2 half-twist annulus is 2 [6]. Noah Montgomery independently showed this result using alternative methods (unpublished). Moreover, Brown and Schwartz conjecture the infimal aspect ratio for the 3-half-twist embedded paper Möbius band is 3 [1]. Brown (unpublished) and Ana and Mia Jain [5] have independently found 5-twist bands with aspect ratio $5+\epsilon$, indicating that the infimal aspect ratio for embedded paper Möbius bands with 5 twists is at most 5. The Jain twins refer to their construction as the "Möbius star".

The seeming pattern of more twists requiring a longer band raises the question: What is the asymptotic growth of the minimal aspect ratio λ_n as a function of the number of twists n? Noah Montgomery found a construction with length complexity $O(\sqrt{n})$ (unpublished), but did not produce any lower bound. This paper puts the big-O question to rest by constructing an $O(1)$ solution. The paper does not show that the construction's constant bound is tight. That is, determining the value of $\limsup\{\lambda_n\}$ is still an open problem. The construction is actually a folded ribbon (un)knot which can be arbitrarily well-approximated by paper bands. Its folded ribbon knot form negatively answers Conjecture 39 in [3].

Theorem 1 (Main Theorem). *There exists a constant λ such that for any n, there exists a paper band (defined below) with n half-twists and aspect ratio less than λ. In fact, it suffices to let $\lambda = 6$.*

Theorem 1 is an immediate corollary of two lemmas. The first lemma is a statement about a family of objects known as folded ribbon knots. Roughly speaking, a folded ribbon knot is a folded strip of paper which lies in the plane (See Remark 6 for more). The *ribbon linking number* of a folded ribbon knot $\mathcal{K}$ is the linking number between its centerline and one boundary component. The folded ribbonlength $\mathrm{Rib}(\mathcal{K})$ of $\mathcal{K}$ is the aspect ratio of the strip of paper, before folding. See Definition 8 for more detail.

Lemma 2 (High-Link Paper Ribbon Knots). *There is a family $\{\mathcal{K}_n\}$ of folded ribbon knots such that*

- *If n is odd, $\mathcal{K}_n$ is a topological Möbius band with ribbon linking number $\pm n$.*
- *If n is even, $\mathcal{K}_n$ is a topological annulus with ribbon linking number $\pm n/2$.*
- *There exists a constant λ such that for all $n \in \mathbb{N}$, $\mathcal{K}_n$ has folded ribbonlength $Rib(\mathcal{K}_n) < \lambda$. In fact, $\lambda = 6$ suffices.*

Lemma 3 (Approximability). *For each n, there is a sequence of n-twist paper bands in $\mathbb{R}^3$ which converge pointwise to the folded ribbon knot $\mathcal{K}_n$, and whose aspect ratios converge to $Rib(\mathcal{K}_n)$.*

Section 2 introduces necessary definitions and terminology. Section 3.1 defines a particular family $\{\mathcal{K}_n\}_{n\in\mathbb{N}}$, and shows that it satisfies bullet 3 of Lemma 2. Section 3.2 shows that this family satisfies bullets 1 and 2, completing the proof of Lemma 2. Section 3.3 proves Lemma 3. Section 4 gives an explicit value for the bounding aspect ratio λ of Theorem 1.

2. Background

Definition 4 (Paper Band, Aspect Ratio). *Formally, a paper Möbius band is a smooth locally isometric embedding of the Möbius strip*

$$([0,1]\times[0,\lambda])/\sim; \quad (t,0)\sim(1-t,\lambda)$$

into $\mathbb{R}^3$. Similarly, a paper annulus is a smooth locally isometric embedding of the cylinder $([0,1]\times[0,\lambda])/((t,0)\sim(t,\lambda))$ into $\mathbb{R}^3$. Refer to these maps collectively as paper bands. *λ is called the* aspect ratio *of the band.*

Definition 5 (Center line, half twist). *The center line of a band is the image of $\{0.5\}\times[0,\lambda]$ under the embedding. Define an n* half-twist *paper band to be*

- *A paper Möbius band for which the boundary and center line have linking number $\pm n$ (for n odd).*
- *A paper annulus for which one of the boundaries and the center line have linking number $\pm n/2$ (for n even).*

Remark 6. Many[1] paper bands can be gently pressed down to lie in the plane, at which point the image is a union of rectangles, parallelograms, and trapezoids, joined to one another at creases in sequence. Such an object is known as a *folded ribbon knot*.

Definition 7 (Folded Ribbon Knot). *(Taken from [3].) Formally, a* folded ribbon knot *is a piecewise linear immersion of an annulus or Mobius band into the plane, where the fold lines are the only singularities, and where the crossing information is consistent.*

Definition 8 (Folded Ribbonlength). *The centerline of a folded ribbon knot $\mathcal{K}$ is a closed polygonal curve. The ratio of the total length of this curve to the ribbon knot's width is the* folded ribbonlength *of $\mathcal{K}$, denoted $Rib(\mathcal{K})$. In this paper, all widths are taken to be 1, so this is just the length of the centerline.*

Definition 9 (Prefold Diagram). *To a folded ribbon knot we can associate a* prefold diagram, *which is a rectangle with non-intersecting solid and dotted line segments (prefolds) drawn on it. Each line segment represents a fold, and the texture of the segment dictates which way the fold goes. One can imagine the rectangle as a strip of paper, with the side facing the viewer colored red, and the other side colored blue. Then, a solid line indicates folding so that the red side is on the inside, and a dotted line indicates a fold which has a blue inside. Figures 1 and 2 provide clarifying examples.*

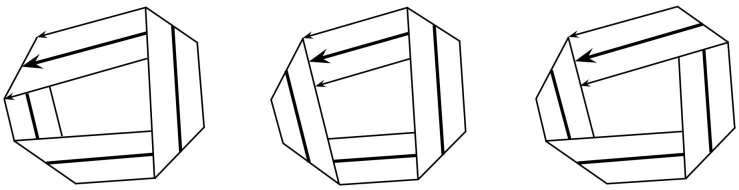

Figure 1. Three different ribbon knots. Their prefold diagrams all have line segments in the same places, but they differ in which segments are solid or dashed. Reused with permission from [2].

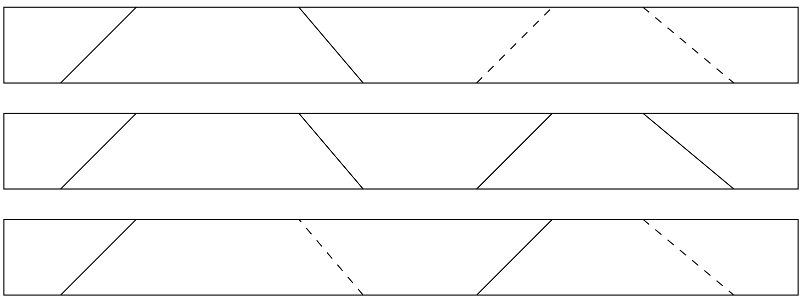

Figure 2. The prefold diagrams for the above three ribbon knots. The top prefold diagram corresponds to the left ribbon knot, the middle with middle, and bottom with right.

[1] It is likely not all paper bands have this property. A particular likely counterexample is the cap, an efficient 3-twist band featured in [1]. This counterexample was pointed out to me by Richard Schwartz.

3. Construction

3.1. *Folded Ribbon Knot*

Most constructions aimed at this problem are centered around the "belt trick": Coil a belt, and then pull the ends apart without allowing them to rotate. The coils turn into twists. This is useful because it means one can construct a many-twist band by tightly coiling the band, yielding very many twists while using a small length of band. The issue with this is that one end of the band ends up confined in a very small space, which prevents reconnection of the two ends without using a very large amount of band to "escape."

Here's the key idea for this paper: If we wrap very tightly at a large angle, then we can escape using a constant length of band. Construct an "escape accordion" (prefold diagram pictured in Figure 3) by folding along parallel lines, 45 degrees rotated from the sides of the band.

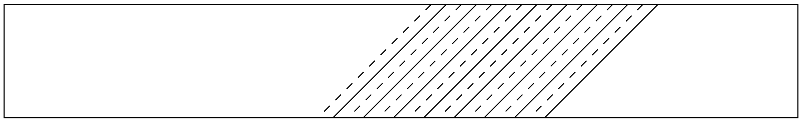

Figure 3. The prefold diagram for the escape accordion

Color the front side of the band light red and the back side dark blue. Then, after folding the accordion, the band will look like Figure 4.

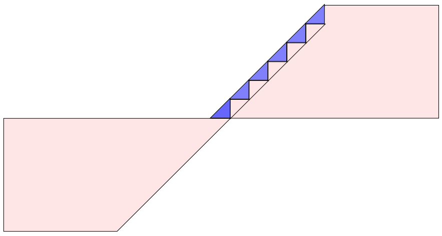

Figure 4. The escape accordion made from colored paper

The key insight about the accordion is that its construction uses a parallelogram with base 2,[2] regardless of the distance between folds. Thus, if we want to achieve n half-twists using ϵ additional length of band, we can let there be $\lceil n/2\epsilon \rceil$ folds in the accordion. A base of 2 is required so that the two ends of the band do not crash into each other during the wrapping step (see Figure 6). Adding in the prefolds for the wrapping step yields a new prefold diagram, pictured in Figure 5.

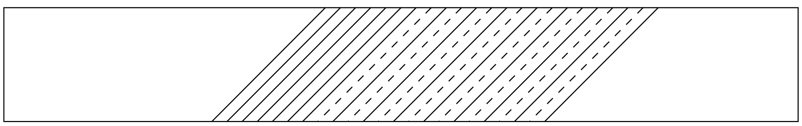

Figure 5. The prefold diagram for the accordion and the wrapping

Fold up figure 5, retaining the same red-blue coloring used in Figure 4, to obtain Figure 6.

All that's left in the construction is to reattach the ends, which requires a length of band l independent of the number of twists. Fixing an $\epsilon > 0$ of our choosing, we recover a family $\{\mathcal{K}_n\}_{n\in\mathbb{N}}$ of folded ribbon knots with $\text{Rib}(\mathcal{K}_n) < 2 + l + \epsilon$.

[2] Recall that here and throughout the paper, paper bands and ribbon knots are assumed to have width 1.

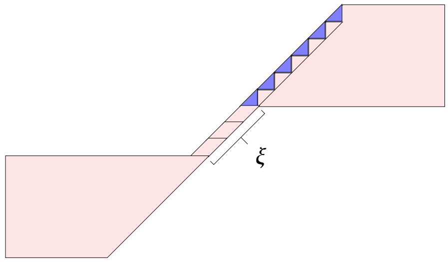

Figure 6. The full construction, up to reattaching the ends. Notice that for any fixed number of twists, the distance labeled ξ can be made arbitrarily small with an adequately skinny accordion.

3.2. *Linking Number*

To prove the remainder of Lemma 2, make use of the following result:

Lemma 10 ([3, Lemma 11]). *The ribbon linking number of a folded ribbon knot is determined by the combinatorial information of its folds and crossings. Each fold and crossing has a certain "local contribution" (see Figure 7), and the sum of these local contributions is the ribbon linking number.*

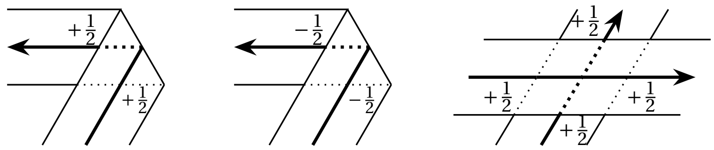

Figure 7. Local contributions of different "events" for Möbius bands (see Remark 11 for the annulus case). Folds have local contribution ± 1, while crossings have local contribution ± 2.

The term "crossing" here refers to a place where the centerline crosses itself. General overlap of the folded ribbon knot with itself does not count as a crossing. The spirit of Lemma 10 is that aside from simple folds and crossings, every overlap which would contribute to the linking number can be removed by decreasing the ribbon width. Since the linking number is invariant under such a change, all overlaps aside from folds and crossings may be safely ignored.

Here's the upshot: the sections depicted in Figure 6 of the knots $\mathcal{K}_n$ do not have any crossings, so the only information relevant to calculating the linking number is the folding information. There are four types of folds to consider, each with their own local contribution and realization in the prefold diagram:

(1) Right underfolds contribute $+1$ to the ribbon linking number and appear as downward sloping dashed lines in prefold diagrams
(2) Right overfolds, -1 - downward sloping solid lines
(3) Left underfolds, -1 - upward sloping dashed lines
(4) Left overfolds, $+1$ - upward sloping solid lines

Remark 11. Note that the above contributions are *only for Möbius strips.* In a Möbius strip, either side of the center line is part of the same single boundary. Compared to an annulus, then, each fold creates twice as many intersections between the centerline and a boundary component, and thus contributes twice as much to the linking number in the Möbius band case versus the annulus case. Hence, the contribution of a fold in the annulus case is $\pm\frac{1}{2}$, not ± 1. The distinct cases in the definition of an n half-twist paper band exist to counterbalance this artifact.

Exercise. *Using the described method, calculate the ribbon linking number of the ribbons corresponding to the prefold diagrams in Figure 2. Confirm that your answer matches what you would visually infer from Figure 1.*

Using this counting method, we can see in Figure 5 that the folds of the accordion cancel out in pairs, while the folds of the wrapping step compound, causing the linking number to accumulate. It takes n consecutive solid lines in the prefold diagram to create a band with n half-twists. Note that Figure 5 does not include the prefolds corresponding to how the ends are reconnected. The folds which are added to ensure the ends of the band connect will contribute some additional linking or unlinking, but any reasonable method only contributes a constant amount, so this does not matter. Lemma 2 is thus proven.

3.3. *Smooth Approximation*

We now prove Lemma 3, which states that the folded ribbon knots $\mathscr{K}_n$ can be well-approximated by smooth Möbius bands.

Proof. The main idea is to model each fold with a very tight turnaround, or pseudofold. For other examples of a similar procedure, see [1, 4, 6].

Definition 12 (Pseudofold). *As defined in [4], proof of Lemma 9.1, a pseudofold is based on a plane curve $\gamma(\delta, t)$ (parameterized by arc length t) with curvature $\kappa(t)$ satisfying:*

- *$\kappa(t)$ is smooth and has compact support (bump function)*
- $\kappa(t) \geq 0$
- $\int \kappa(t) = \pi$

$\gamma(\delta, t)$ follows the x-axis for some time, turns around smoothly, and then follows the line $y = \delta$ is the other direction. The length of the curved part is $c\delta$ for some constant c depending on the particular bump function chosen. Let the curved part correspond to $t \in [0, c\delta]$

Given such a curve γ, one can construct the chart

$$(t, s) \longmapsto (dt + s, \gamma_x(\delta, t), \gamma_y(\delta, t))$$

where γ_x and γ_y are the components of γ. A pseudofold *is a subsurface given by such a chart. Note that the parameter d determines on the pseudofold angle. See Figure 8 for a clarifying illustration.*

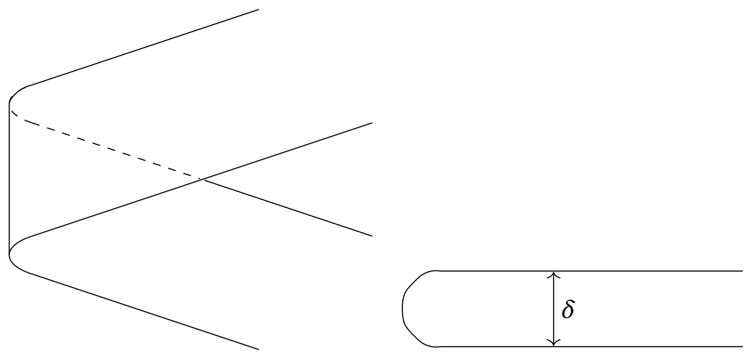

Figure 8. The top and side views of a pseudofold. The side view is simply the graph of $\gamma(\delta, t)$ in the plane.

Separate each layer of the folded ribbon knot vertically by some small distance δ and then connect the layers with pseudofolds. Let the *size* of a pseudofold be the height disparity between the layers it connects. The pseudofolds of the accordion all have size δ. Those corresponding to the wrapping have sizes $(m+1)\delta$, $(m+2)\delta$, ..., $(m+n)\delta$, assuming there are m accordion folds and n wrapping folds. We can represent this in a prefold diagram. In this diagram, let light green parallelograms represent pseuofolds which replace solid lines, and let darker purple parallelograms represent pseudofolds which replace dashed lines. This is pictured in Figure 9. Figure 10 depicts the result of folding the prebend diagram.

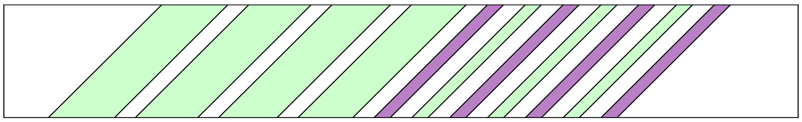

Figure 9. The prefold diagram for the smooth approximation. The base of each parallelogram is $c\sqrt{2}$ times the size of the corresponding pseudofold. c is the same constant used in Definition 12. It depends on the particular curve γ used to construct the pseudofolds.

Every error in the approximation is proportional to δ. Thus, as δ goes to 0, the additional band length and the distance between any particular point on the ribbon and its approximating point on the band go to 0. Furthermore, these bands respect the folding information of $\mathcal{K}_n$, so they are n-twist bands. □

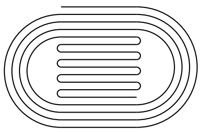

Figure 10. A side view of the complicated part of a smooth approximation

4. Parity

The construction so far applies to both Möbius bands and annuli. Which one is constructed comes down to how the ends of the band are connected to one another. Letting one side be colored red and the other blue, a Möbius band is obtained from taping red to blue, while an annulus is obtained by taping red to red. Figures 11 and 12 depict two reasonably efficient ways to reconnect the ends for each type of band.

This section's constructions give many-twist Möbius bands with aspect ratio $3+2\sqrt{2}+\epsilon \approx 5.83$ and many-twist annuli with aspect ratio $\sqrt{2\alpha^2+2\sqrt{2}\alpha+2}+\sqrt{2}\alpha+2+\epsilon \approx 5.38$, where α is the larger root of $(1+7\sqrt{2})x^2-12x+2\sqrt{2}$. Thus, we can take $\lambda = 6$ in the Main Theorem. The disparity between cases comes from the fact that the reconnection in the Möbius band case is less efficient.[3]

[3]The need for two distinct reconnection methods was pointed out to me by Luke Briody. A miscalculation by the author originally gave a Möbius band aspect ratio of $2+3\sqrt{2}$. I thank an anonymous referee for pointing out the error.

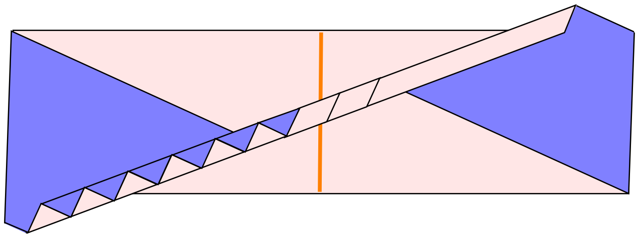

Figure 11. A fully constructed paper annulus. The orange line in the middle indicates where the strip of paper is taped/glued to itself. The fact that there is the same color (red) on each side of the line corresponds to the fact that this is an annulus, not a Möbius Band.

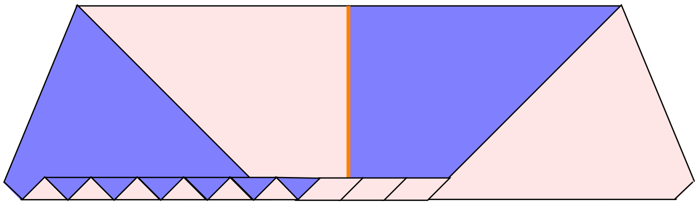

Figure 12. A fully connected many-twist paper Möbius band. The orange gluing line has opposite colors on either side of it, indicating that the band has an odd number of half-twists.

Note that each reconnection method introduces a handful of folds, but the amount is constant in the number of half-twists in the band. Additionally, reconnection in the annulus case introduces many center-line crossings, but these can be removed by applying an ambient isotopy, indicating that they do not change the linking number. In total, each reconnection method only contributes a constant amount to the ribbon knot's linking number.

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Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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