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Expansion properties of Whitehead moves on cubic graphs

Sur les propriétés d'expansion de transformations de Whitehead sur les graphes trivalents

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Abstract. The present note concerns the “graph of graphs” that has cubic graphs as vertices connected by edges represented by the so-called Whitehead moves. Here, we prove that the outer-conductance of the graph of graphs tends to zero as the number of vertices tends to infinity. This answers a question of K. Rafi in the negative.

Résumé. Cette note porte sur le « graphe des graphes », dont les sommets sont des graphes trivalents reliés par des arêtes correspondant aux transformations de Whitehead. Nous montrons ici que la conductance externe de ce graphe tend vers zéro lorsque le nombre de sommets tend vers l'infini. Cela donne une réponse négative à la question posée par K. Rafi.

Keywords. Trivalent graph, cubic graph, expander graph, Whitehead move, conductance.

Mots-clés. Graphes trivalents, graphes expanseurs, transformation de Whitehead, conductance.

2020 Mathematics Subject Classification. 05C48, 05C75, 05C90.

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1. Introduction

The present note is concerned with the graph of connected 3-regular, or cubic, graphs. Such a “graph of graphs” Γ_n , represents what can be informally described as the “deformations space” of cubic graphs on $2n$ vertices under the Whitehead moves. In particular, we shall investigate its expansion properties.

Let g be a cubic graph (we allow graphs with multiple edges and loops), with set of vertices $V(g)$, and set of edges $E(g)$. Then, for an edge $e \in E(g)$ which is not a loop, there are two possible Whitehead moves $w_e^{(1)}$ and $w_e^{(2)}$ on e , depicted in Figure 1, which are local transformations of g .

If $e \in E(g)$ is a loop, we define the respective Whitehead moves as having no effect on g . Let us note that this convention is not arbitrary. Indeed, we may represent g as a multigraph: a set

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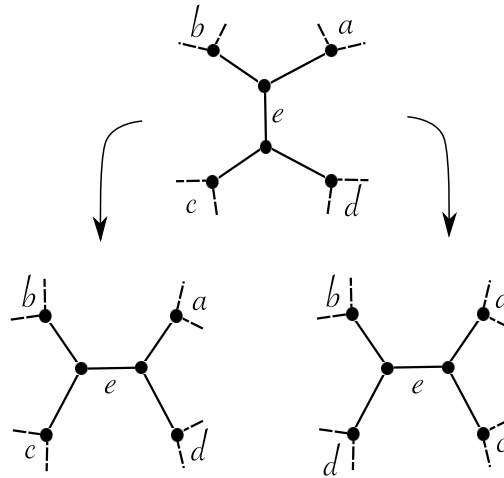


Figure 1. Two possible Whitehead moves performed on an edge ε .

of half-edges incident to vertices and paired by a fixed-point-free involution. Then a Whitehead move does not affect the pairing of half-edges and only changes which vertices the half-edges are adjacent to. If e is a loop, the two vertices u, v adjacent to its half-edges coincide: thus changing the vertex assignment of one of the other two half-edges at u and of one of the other two half-edges at v does nothing.

Let the vertices of Γ_n be all possible cubic graphs on $2n$ vertices ($n \geq 1$) up to isomorphism, where two vertices $g_1, g_2 \in V(\Gamma)$ are connected by a directed edge for each Whitehead move w_ε , for $\varepsilon \in E(g_1)$, such that $w_\varepsilon(g_1)$ is isomorphic to g_2 . It is easy to see that being related by a Whitehead move is an equivalence relation on the set of isomorphism classes of cubic graphs on $2n$ vertices ($n \geq 1$).

In this note, a directed version of this graph of graphs is considered, as it is common to do so in the context of dynamical systems. Moreover, as Whitehead moves are clearly reversible, this choice does not affect the expansion property of Γ_n .

We introduce the notion of outer-conductance ϕ_{out} because there is no standard definition of expansion for directed non-regular multigraphs, despite them being a natural class of objects to consider.

Outer-conductance captures how easy it is to escape a given subset of the graph. Therefore, having the outer-conductance tend to 0 is a way of saying that the graphs have poor mixing and therefore cannot be “expanders” in this sense.

Namely, we show that the following theorem holds which answers the question about “edge expansion” of Γ_n posed by K. Rafi [11].

Theorem A (Theorem 2). *Let Γ_n be the graph of cubic graphs on $2n$ ($n \geq 1$) vertices connected by Whitehead moves. Then Γ_n is connected and $\phi_{\text{out}}(\Gamma_n) \rightarrow 0$, as $n \rightarrow \infty$.*

Since Γ_n contains loops, it is aperiodic. As a byproduct of this and Theorem Theorem 2, we obtain one more statement about the combinatorics of Γ_n .

Corollary B (Corollary 4). *For each $n \geq 1$, the graph Γ_n is strongly connected and aperiodic. There exists a Perron number ρ_n such that the number of length ℓ paths in Γ_n is asymptotic to $\text{const} \cdot \rho_n^\ell$, as $\ell \rightarrow \infty$.*

Finally, in Section 5, we mention some objects connected to Γ_n .

First and foremost, the graph of graphs is related to moduli spaces of surface triangulations [5, 6] and pants decomposition of surfaces [12], although not entirely equivalent to said objects. The definition considered here is a directed version of the graph of graphs considered in [12], and is quasi-isometric to the latter.

Also, the graph of graphs considered in [12] is quasi-isometric to the “wide” component of the thick part of the moduli space of a genus $g \geq 2$ surface. This means that Γ_n describes pants decompositions of the genus $g = n + 1$ surface in which the set of separating curves contains only sufficiently short geodesics.

Similar local transformations are considered in [8] and two such pants decompositions represent adjacent vertices in Γ_n if they are connected by a (I) -move¹, while a (IV) -move² is always performed on a handle that corresponds to a loop in the pants decomposition graph, and thus is not taken into account when passing to Γ_n . In this way, there exists a graph morphism from the directed pants graph to Γ_n which sends pants decomposition to their dual graphs, and edges to edges.

As shown in [14, 15], the vertex set of Γ_n coincides with the vertex set of the pants graph quotient by the action of the mapping class group. Thus, Γ_n is quasi-isometric to the latter.

Thus, knowing the combinatorial properties of Γ_n could be useful in the study of the geometric and combinatorial properties of the pants graph.

2. Conductance and expansion

Let G be a connected directed graph with vertex set $V(G)$ and edge set $E(G)$. For a subset $S \subset V(G)$, let $d(S)$ denote the sum of vertex out-degrees in S , i.e. $d(S) = \sum_{v \in S} d_{\text{out}}(v)$, where $d_{\text{out}}(v)$ is the number of edges of the form (v, u) where $u \in V(G)$. The *boundary* $\partial(S)$ of a vertex subset $S \subset V(G)$ is defined as the set of directed edges in $E(G)$ joining a vertex in S with a vertex in $V(G) \setminus S$, that is $\partial(S) = \{e = (u, v) \in E(G) \mid u \in S, v \in S^c\}$.

The *outer-conductance* of $S \subset V(G)$ is defined as follows:

$$\phi_{\text{out}}(S) = \frac{|\partial(S)|}{\min\{d(S), d(V(G) - S)\}}$$

and the outer-conductance of G is

$$\phi_{\text{out}}(G) = \min_{S \subset V(G)} \phi_{\text{out}}(S).$$

This is a generalization of the notion of conductance [2, Section 6.2]. Since we work with directed graphs, the definition is adapted so that the volume is measured with respect to the number of out-going edges. By doing so the outer-conductance measures how hard it is to escape a subset of the graph.

As a generalization of the notion of expander families to the case of directed graphs with vertex degrees (both out-degree and in-degree) growing with the number of vertices, we introduce Definition 1.

Definition 1. *We say that a sequence of directed graphs G_n with out-degree and in-degree growing with the number of vertices is an outer-expander family if the conductance $\phi_{\text{out}}(G_n)$ is uniformly bounded away from 0 as n tends to infinity.*

Now we state our main result.

Theorem 2. *Let Γ_n be the directed graph of cubic graphs on $2n$ ($n \geq 1$) vertices connected by Whitehead moves. Then Γ_n is connected and $\phi_{\text{out}}(\Gamma_n) \rightarrow 0$, as $n \rightarrow \infty$. Therefore $\{\Gamma_n\}_{n \in \mathbb{N}}$ is not an outer-expander family.*

¹Called an A -move in [7]

²Called an S -move in [7]

3. Proof of Theorem 2

Let Γ_n be the directed graph of (isomorphism classes of) cubic graphs on $2n$ vertices ($n \geq 1$) with edges corresponding to Whitehead moves as defined in the introduction. The following two claims will imply Theorem 2.

3.1. Γ_n is connected

This result is mentioned in [11] as attributed to Coco Zhang. We give a proof below by using the classical facts about tree rotations and rebalancing known in computer science.

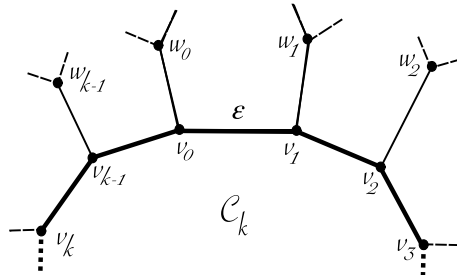


Figure 2. The edge $\varepsilon = (v_0, v_1)$ that belongs to a k -cycle C_k in $g \in V(\Gamma_n)$ on which a Whitehead move may be performed.

Let g be a cubic graph, and let C_k be an indecomposable k -cycle (with $k \geq 2$) in g , i.e. an induced cycle subgraph of g . If we perform a Whitehead move on the edge ε that belongs to a k -cycle C_k , a part of which is depicted in Figure 2, we obtain a modified graph (partly) depicted in Figure 3.

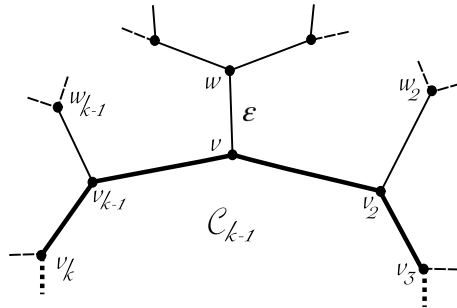


Figure 3. The resulting $(k - 1)$ -cycle C_{k-1} in $\tilde{g} = w_\varepsilon(g)$: the edge $\varepsilon = (v, w)$ does not belong to C_{k-1} .

The overall changes in the structure of $\tilde{g} = w_\varepsilon(g)$ as compared to g are local and amount to the following:

- The k -cycle C_k has been transformed into a $(k - 1)$ -cycle C_{k-1} ;
- If v_1, w_1 were part of an l -cycle C_l then C_l was transformed into an $(l + 1)$ -cycle C_{l+1} ;
- Same applies to any cycle that previously contained w_0 and/or v_0 .

By performing a total of $k - 1$ Whitehead moves on the edges in C_k , we reduce it to a loop, as depicted in Figure 4. Here we reduced the total amount of k -cycles with $k \geq 2$ by one, although the lengths of some other cycles could have been augmented.

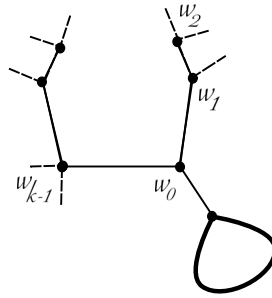


Figure 4. The k -cycle C_k ($k \geq 2$) is finally reduced to a loop by $k - 1$ consecutive Whitehead moves.

By repeating the above procedure on the remaining k -cycles (with $k \geq 2$) of the resulting graph, we shall reduce it to a (n, b) -lollipop tree, as defined below, an example of which is depicted in Figure 5. Such a tree has $2n$ vertices and $b = n + 1$ loops: note that b is the 1st Betty number of the initial graph g , since Whitehead moves preserve the number of vertices and edges of g , as well as the rank of $\pi_1(g)$.

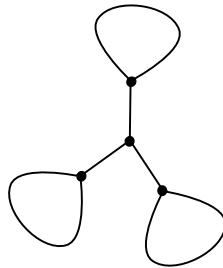


Figure 5. A $(2, 3)$ -lollipop tree.

There exists a natural isomorphism between our (n, b) -lollipop trees with a chosen loop (i.e. rooted (n, b) -lollipop trees with a chosen loop ε^* as a root), and binary rooted trees. Namely, let us take a rooted (n, b) -lollipop tree (T, ε^*) and remove all the loops as well as the edge (v^*, v) adjacent to ε^* (with ε^* being the loop at vertex v). The result, (\tilde{T}, v^*) is a binary rooted tree on $2n - 1$ vertices with n leaves.

The general theory of binary trees and tree rebalancing [3, Section 12–Section 13] shows that we can bring (\tilde{T}, v^*) to a unique complete rooted binary tree T_n on n vertices by a series of rotations. As Figure 6 shows, a tree rotation can be achieved by a Whitehead move. This proves that Γ_n is connected.

3.2. The outer-conductance of Γ_n tends to 0

A *bridge* in a connected graph is an edge such that its removal produces several connected components. A graph with a bridge is called *bridged*, while a graph that has no bridges at all is called *bridgeless*.

Let $B_n \subset V(\Gamma_n)$ be the set of (isomorphism classes of) connected cubic graphs on $2n$ vertices that have at least one bridge. We shall estimate the probability $\mathbb{P}(g \in B_n)$ asymptotically. For this matter, notice that if $U_n = V(\Gamma_n)$ is the set of unlabelled cubic graphs (i.e. their isomorphism classes), and L_n is the set of labelled ones, then $\mathbb{P}(g \in U_n \text{ has a bridge}) = \mathbb{P}(g \in L_n \text{ has a bridge}) + o(1)$, since according to [1] we have that $|L_n| \sim |U_n| \cdot (2n)!$, as $n \rightarrow \infty$.

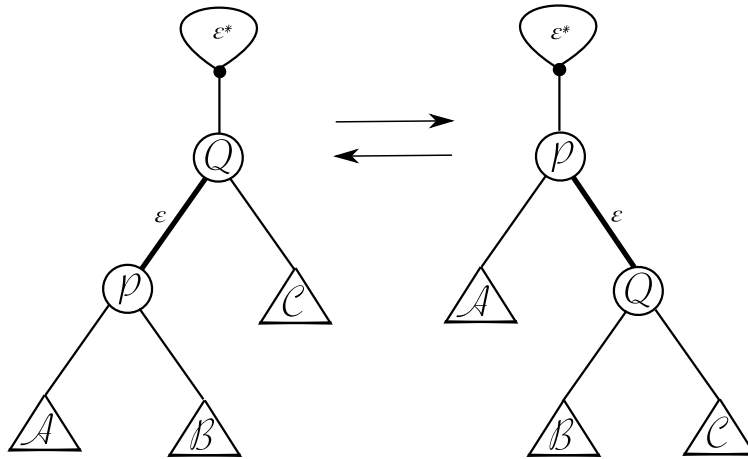


Figure 6. A left and right rotation of a rooted binary tree (corresponding to an (n, b) -lollipop tree with root ε^*) realised by a Whitehead move on edge e . Here P, Q are the vertices of ε , and A, B , and C are the respective sub-trees

A *Hamiltonian circuit* in a (connected) graph is a closed edge path that visits each vertex once (i.e. without self-intersection). A graph is called *Hamiltonian* if it admits a Hamiltonian circuit.

It is clear that a graph that has a bridge cannot be Hamiltonian, and hence every Hamiltonian cubic graph is bridgeless.

Let $\text{Bridged} = \{g \in L_n \mid g \text{ has a bridge}\}$ and $\text{Looped} = \{g \in L_n \mid g \text{ has a loop}\}$, and let nH be the set of graphs in L_n that do not have a Hamiltonian circuit. Then

$$\mathbb{P}(g \in \text{Looped}) \leq \mathbb{P}(g \in \text{Bridged}) \leq \mathbb{P}(g \in \text{Looped}) + \mathbb{P}(g \notin \text{Looped} \text{ and } g \in nH),$$

since $\text{Looped} \subseteq \text{Bridged}$, as well as $\text{Bridged} \subseteq nH$.

By applying [9, Theorem 9.5], we have that $\mathbb{P}(g \in \text{Looped}) = 1 - e^{-1} + o(1)$, while [9, Theorem 9.23] implies that $\mathbb{P}(g \notin \text{Looped} \text{ and } g \in nH) = o(1)$, as $n \rightarrow \infty$. Thus, $\mathbb{P}(g \in \text{Bridged}) = 1 - e^{-1} + o(1)$ asymptotically, as $n \rightarrow \infty$.

Together with the remarks above about the probabilities for labelled and unlabelled graphs, we obtain that $\mathbb{P}(g \in B_n)$, for $g \in V(\Gamma_n)$, satisfies $\mathbb{P}(g \in B_n) = 1 - e^{-1} + o(1)$, as $n \rightarrow \infty$. Thus, $|B_n| \sim (1 - e^{-1})|V(\Gamma_n)|$, and $|V(\Gamma_n) \setminus B_n| \sim e^{-1}|V(\Gamma_n)|$.

Note that the only vertices in B_n connected with vertices in $V(\Gamma_n) \setminus B_n$ are those $g \in B_n$ having a single bridge, and the corresponding Whitehead moves have to be performed exactly on the bridge edge of g .

Indeed, let $g \in V(\Gamma_n)$ have at least one bridge and let $\varepsilon^* = (u, v)$ denote one of its bridges. Then we can check case-by-case what the result of a Whitehead move on ε^* could be.

For this purpose, let us introduce the following equivalence relation on the vertices of g : for $v, w \in V(g)$, we write $v \sim_g w$ if and only if there are two edge-disjoint paths connecting v to w . Then we write $[v]_g = \{w \in V(g) \text{ s.t. } v \sim_g w\}$ for the equivalence class of v .

Let $\{a, b, c, d\}$ be the vertices adjacent to the bridge $\varepsilon^* \in E(g)$, and let $\tilde{g} = w_{\varepsilon^*}(g)$ be the graph resulting from a Whitehead move performed on ε^* . Then we consider the following five possible cases.

- (1) $[a]_g = [b]_g = [c]_g = [d]_g$: in this case ε^* is not a bridge neither in g , nor in \tilde{g} , as shown in Figure 7–Figure 8.

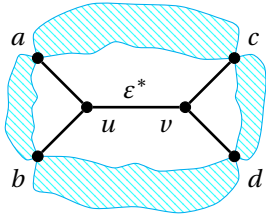


Figure 7. Local picture inside g

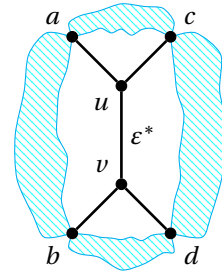


Figure 8. Local picture inside \tilde{g}

(2) $[a]_g = [b]_g = [c]_g \neq [d]_g$: same as above, ϵ^* is not a bridge neither in g , nor in \tilde{g} , see Figure 9–Figure 10.

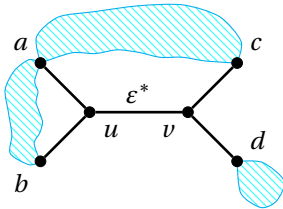


Figure 9. Local picture inside g

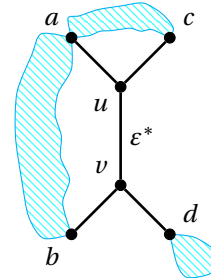


Figure 10. Local picture inside \tilde{g}

(3) $[a]_g = [b]_g \neq [c]_g = [d]_g$: in this case, ϵ^* is a bridge in g , but not in \tilde{g} , see Figure 11–Figure 12.

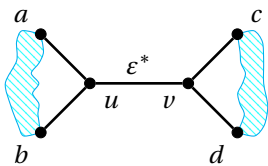


Figure 11. Local picture inside g

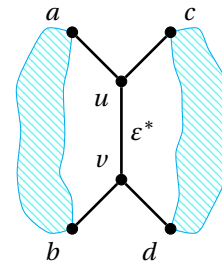


Figure 12. Local picture inside \tilde{g}

- (4) $[a]_g = [b]_g \neq [c]_g$, $[a]_g = [b]_g \neq [d]_g$, and $[c]_g \neq [d]_g$: again, ε^* is a bridge in g , but not in \tilde{g} , see Figure 13–Figure 14.

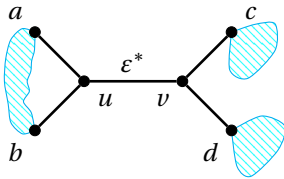


Figure 13. Local picture inside g

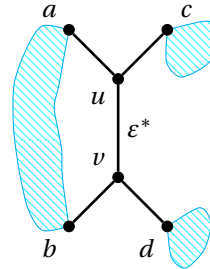


Figure 14. Local picture inside \tilde{g}

- (5) $[a]_g, [b]_g, [c]_g, [d]_g$ are all distinct: obviously, ε^* is a bridge in both g and \tilde{g} , see Figure 15–Figure 16.

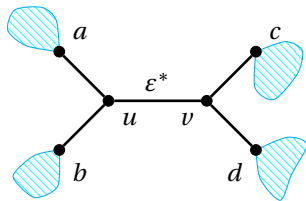


Figure 15. Local picture inside g

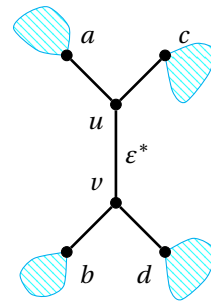


Figure 16. Local picture inside \tilde{g}

According to the observations above, if a bridged graph $g \in B_n$ can be transformed into a bridgeless one $\tilde{g} = w_\varepsilon(g)$, then ε is the only bridge of g . Every time, both Whitehead moves either succeed or fail to bring \tilde{g} outside B_n . This implies that $|\partial(B_n)| \leq 2|B_n|$. Thus the outer-conductance of the set B_n satisfies

$$\phi_{\text{out}}(B_n) = \frac{|\partial(B_n)|}{\min\{d(B_n), d(V(\Gamma_n) \setminus B_n)\}} \leq \frac{2|B_n|}{6n|V(\Gamma_n) \setminus B_n|} \sim \frac{2|V(\Gamma_n)|(1 - e^{-1})}{6n|V(\Gamma_n)|e^{-1}} = \frac{e - 1}{3n} \rightarrow 0,$$

as $n \rightarrow \infty$.

□

Remark 3. It appears that Γ_n is itself a relatively “thin” neighbourhood of B_n . Indeed, each length k cycle in a graph $g \in V(\Gamma_n)$ can be reduced by $k - 2$ Whitehead moves to a 2-cycle, and one more Whitehead move will create an edge incident to a loop. The standard asymptotic bound on girth is $\text{girth}(g) \lesssim 2 \log_2 n$. If $v = |V(\Gamma_n)|$, then $v \sim n^n$ [1], and thus each vertex of Γ_n happens to be within $\sim \log_2 \log_2 v$ distance from B_n . As shown in [12], the diameter of Γ_n satisfies $d = \text{diam}(\Gamma_n) \sim n \log_2 n$, and thus each vertex of Γ_n is within $\log_2 d$ of B_n .

4. Combinatorics of paths

Let A be a non-zero square $k \times k$ ($k \geq 2$) matrix with non-negative integer entries. Such a matrix is called *non-negative*, for brevity. If all the entries of A are actually positive, then A is called *positive*.

If there exists a permutation matrix P such that PAP^{-1} has upper-triangular block form, A is called *reducible*. Otherwise, A is *irreducible*.

For $i \in \{1, \dots, k\}$, the i -th *period* of A is the greatest common divisor of all integers m such that the i -th diagonal entry of A^m is positive. If A is irreducible, then all its periods coincide, and each of them equals *the period* of A . If the period of A equals 1, then A is called *aperiodic*.

An aperiodic and irreducible non-negative matrix A is called *primitive*. If A is a primitive matrix, then its spectral radius is a Perron number, as a consequence of the Perron–Frobenius theorem [10, Theorem 4.5.11]. Recall that a *Perron number* is an algebraic integer $\rho > 1$ such that all of its other Galois conjugates have modulus strictly less than ρ . Perron numbers often appear in dynamical context, cf. [10]. Another property of the spectral radius of a primitive matrix A is that this eigenvalue has multiplicity one.

Let Γ be a directed graph, and let A be its adjacency matrix defined as follows:

$$A_{uv} = \text{the number of directed edges joining } u \text{ to } v, \text{ if } u \neq v,$$

and

$$A_{vv} = \text{the number of loops incident to } v.$$

Then A is irreducible if and only if Γ is *strongly connected*, i.e. there exists a path of directed edges in Γ between any pair of distinct vertices. If Γ is strongly connected and the greatest common divisor of its directed cycle lengths equals 1, then A is aperiodic, and thus primitive. Thus, in this case we can deduce that the spectral radius of A is a Perron number by considering the combinatorics of Γ .

Corollary 4. *The graph Γ_n is strongly connected and aperiodic. The number of length ℓ paths in Γ_n is asymptotic to $\text{const} \cdot \rho_n^\ell$, with ρ_n a Perron number, if $\ell \rightarrow \infty$.*

Proof. For every $n \geq 1$, there exists an (n, b) -lollipop tree $t \in V(\Gamma_n)$, with $b = n + 1 \geq 2$ loops $\varepsilon_1, \dots, \varepsilon_b$. Then, $w_{\varepsilon_i}(t) = t$, for any $i = 1, \dots, b$, and thus Γ_n has loops. This implies that Γ_n is aperiodic.

Remark 5. There exist a 3-cycle and a 4-cycle in each Γ_n , for $n \geq 3$. Namely, we can choose the cycle in Figure 17. The subgraph of Γ_3 depicted in Figure 17 embeds in Γ_n by replacing the left-most loop in each cubic graph with a sequence of “lollipop” graphs.

Moreover Γ_n is strongly connected, as follows from the proof of Theorem 2. Thus, we obtain that A_n is primitive, and can now apply the Perron–Frobenius theorem.

Hence the spectral radius $\rho_n = \rho(A_n)$ is a multiplicity 1 eigenvalue with a positive eigenvector v , and ρ_n is a Perron number. Then the asymptotic number of paths follows from [10, Theorem 4.5.12] by a standard computation. □

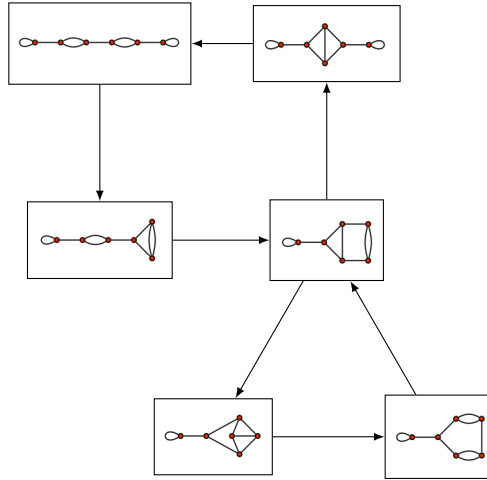


Figure 17. Two cycles in Γ_3 of coprime lengths having a common vertex. Each of them is a cubic graph containing a “lollipop” (a loop incident to an edge) subgraph on the left. By replacing this lollipop subgraph with a lollipop tree on $2n - 4$ vertices minus a loop, we obtain a similar picture inside Γ_n for every $n \geq 4$ as well.

5. Related objects

Above we consider the graph of graphs Γ_n of all possible cubic graphs on $2n$ vertices, where two vertices, $g_1, g_2 \in V(\Gamma_n)$, are connected by a directed edge for each Whitehead move w_e , for $e \in E(g_1)$, such that $w_e(g_1)$ is isomorphic to g_2 .

However, there are several ways to modify this definition. Indeed, we could modify the edges of Γ_n in order to turn it into an undirected graph, a simple graph or other types of graphs. These modifications can provide relations between the graph of graphs and other mathematical problems. In this section we will explore some of these connections.

5.1. Matchings on graphs

Let us consider two edges e, f in $g \in \Gamma_n$ that are not incident to each other. Note that the Whitehead move $w_e^{(k)}(g)$, $k = 1, 2$, only affects the 4 half-edges incident to the vertices of e (and not belonging to e). The same holds for $w_f^{(k)}(g)$, $k = 1, 2$. Thus, the respective sets of half-edges are disjoint, which implies that $w_e(g)$ and $w_f(g)$ commute. We will call this transformation a *simultaneous Whitehead move on the set $\{e, f\}$* , cf. [12].

Thus, we can consider a modified graph Γ'_n with $V(\Gamma'_n) = V(\Gamma_n)$ where $g_1, g_2 \in V(\Gamma'_n)$ are connected by a directed edge if there exists a Whitehead move on some set of non-incident edges of g_1 mapping g_1 to g_2 .

Each vertex matching of cardinality $m \geq 1$ in g_1 gives rise to 2^m simultaneous Whitehead moves. Thus, knowing the number of cardinality $m \geq 1$ matchings in g_1 allows us to compute the degree of g_1 in Γ'_n .

5.2. Outer space and its spine

Let F_n be the free group on $n \geq 2$ letters. In [4], Culler and Vogtmann introduced a space X_n on which the group $\text{Out}(F_n)$ acts. This space can be thought of as analogous to the Teichmüller space of a surface with the action of the mapping class group of the surface.

The description of X_n is complicated, and we will not detail it here. We will only briefly mention the relevant aspects of this object and how it could relate to our graph of graphs.

The space X_n is an infinite, finite-dimensional simplicial complex. A point in X_n is defined as a metric graph (where edges have assigned lengths) of total length 1 with fundamental group F_n equipped with a homotopy equivalence to the bouquet of n circles, called a *marking*. Each open simplex in X_n is obtained by varying the lengths of the edges in a fixed marked topological graph in such a way that their sum remains equal to 1. The group $\text{Out}(F_n)$ acts on X_n by change of markings.

Two given k -simplices x_1 and x_2 share a face of codimension 1 if one can pass from a point in x_1 to a point in x_2 by collapsing an edge and re-opening it, i.e., by a continuous version of a Whitehead move. Observe that some simplices have missing faces corresponding to loops in the graph, since collapsing a loop would change the fundamental group. Note that simplices of maximal dimension correspond to marked trivalent metric graphs.

The space X_n admits a *spine* K_n which is a deformation retract of X_n : the vertices of K_n are marked graphs considered without edge lengths. This spine K_n has the structure of a simplicial complex, in fact it can be identified with the geometric realization of the partially ordered set of open simplices of X_n .³ The group $\text{Out}(F_n)$ acts on K_n with compact quotient.

Our graph of graphs Γ_n is related to $K_{n+1}/\text{Out}(F_{n+1})$ in the following way. Let us consider the dual graph to the collection of top-dimensional simplices in X_{n+1} , then take its quotient by $\text{Out}(F_{n+1})$. This will yield a graph that Γ_n admits a morphism onto.

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³This is, word for word, the description given in [13].

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