ACADÉMIE
DES SCIENCES
INSTITUT DE FRANCE

Comptes Rendus
Mathématique

Olivier Bisson and Xavier Pennec

Differential of the Stretch Tensor for Any Dimension with Applications to Quotient Geodesics

Volume 362 (2024), p. 1847-1856

Online since: 25 November 2024

https://doi.org/10.5802/crmath.692

CENTRE MERSENNE

The Comptes Rendus. Mathématique *are a member of the Mersenne Center for open scientific publishing*
www.centre-mersenne.org — e-ISSN : 1778-3569

ACADÉMIE
DES SCIENCES
INSTITUT DE FRANCE

Comptes Rendus. Mathématique
2024, Vol. 362, p. 1847-1856
https://doi.org/10.5802/crmath.692

Research article / *Article de recherche*
Geometry and Topology, Mechanics / *Géométrie et Topologie, Mécanique*

Differential of the Stretch Tensor for Any Dimension with Applications to Quotient Geodesics

Différentielle du tenseur de déformation en toute dimension et applications aux géodésiques quotient

Olivier Bisson [*,a] **and Xavier Pennec** [a]

[a] Université Côte d'Azur, INRIA, France
E-mails: olivier.bisson@inria.fr, xavier.pennec@inria.fr

Abstract. The polar decomposition $X = WR$, with $X \in \mathrm{GL}(n,\mathbb{R})$, $W \in \mathscr{S}_+(n)$, and $R \in \mathscr{O}_n$, suggests a right action of the orthogonal group $\mathscr{O}_n$ on the general linear group $\mathrm{GL}(n,\mathbb{R})$. Equipped with the Frobenius metric, the $\mathscr{O}_n$-principal bundle $\pi : X \in \mathrm{GL}(n,\mathbb{R}) \mapsto X\mathscr{O}_n \in \mathrm{GL}(n,\mathbb{R})/\mathscr{O}_n$ becomes a Riemannian submersion. In this note, we derive an expression for the derivative of its unique symmetric section $s\circ\pi$ in any dimension, in terms of a solution to a Sylvester equation. We discuss how to solve this type of equation and verify that our formula coincides with those derived in the literature for low dimensions. We apply our result to the characterization of geodesics of the Frobenius metric in the quotient space $\mathrm{GL}(n,\mathbb{R})/\mathscr{O}_n$.

Résumé. La décomposition polaire $X = WR$, avec $X \in \mathrm{GL}(n,\mathbb{R})$, $W \in \mathscr{S}_+(n)$ et $R \in \mathscr{O}_n$, suggère une action à droite du groupe orthogonal $\mathscr{O}_n$ sur le groupe général linéaire $\mathrm{GL}(n,\mathbb{R})$. Équipé de la métrique de Frobenius, le fibré $\mathscr{O}_n$-principal $\pi : X \in \mathrm{GL}(n,\mathbb{R}) \mapsto X\mathscr{O}_n \in \mathrm{GL}(n,\mathbb{R})/\mathscr{O}_n$ devient une submersion Riemannienne. Dans cet article, nous obtenons une expression pour la dérivée en toute dimension de son unique section symétrique $s\circ\pi$, en termes d'une solution d'une équation de Sylvester. Nous discutons comment résoudre ce type d'équation et vérifions que notre formule coïncide avec celles dérivées dans la littérature en basses dimensions. Nous appliquons notre résultat à la caractérisation des géodésiques de la métrique de Frobenius dans l'espace quotient $\mathrm{GL}(n,\mathbb{R})/\mathscr{O}_n$.

Keywords. Polar Decomposition, stretch Tensor, quotient Geodesics.

Mots-clés. Décomposition polaire, tenseur de déformation, géodésiques quotient.

2020 Mathematics Subject Classification. 53A17, 53C22, 53C80, 15A24.

Funding. This work was supported by ERC grant #786854 G-Statistics from the European Research Council under the European Union's Horizon 2020 research and innovation program, and by the French government through the 3IA Côte d'Azur Investments ANR-19-P3IA-0002 managed by the National Research Agency.

Manuscript received 12 July 2024, revised 13 September 2024, accepted 22 September 2024.

[*] Corresponding author

1. Introduction

The *polar decomposition theorem* states that every invertible matrix $X \in \mathrm{GL}(n,\mathbb{C})$ can be written uniquely in the form

$$X = WR$$

where W is self-adjoint and positive definite, and R is unitary [9]. If X is a real invertible matrix, the polar decomposition separates it into a real symmetric positive-definite matrix $W := \left(XX^\top\right)^{1/2} \in \mathscr{S}_+(n)$ and an orthogonal matrix $R \in \mathscr{O}_n$, where $X^\top$ denotes the transpose of X and $Q^{1/2}$ the unique symmetric square root of a positive-definite symmetric matrix Q. Let us define the map $s : X\mathscr{O}_n \in \mathrm{GL}(n,\mathbb{R})/\mathscr{O}_n \mapsto W := \left(XX^\top\right)^{1/2} \in \mathscr{S}_+(n)$, that is, s is a section of the fiber bundle $\pi : \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})/\mathscr{O}_n$ with values in $\mathscr{S}_+(n) \subset \mathrm{GL}(n,\mathbb{R})$. In this note, we derive a very simple expression for the derivative of

$$s \circ \pi : X \in \mathrm{GL}(n,\mathbb{R}) \longmapsto W \in \mathscr{S}_+(n)$$

in terms of a solution to a Sylvester equation. We emphasize that this result holds in any dimension, and although explicit expressions for low dimensions exist in the literature, we provide an implicit formula that mainly relies on solving a well-known equation. Moreover, we discuss how we can numerically derive a solution to this equation with the help of the familiar Jacobi algorithm [7]. We present this formula in Theorem 1.

Theorem 1. *Let $X = WR \in \mathrm{GL}(n,\mathbb{R})$ with $W \in \mathscr{S}_+(n)$ and $R \in \mathscr{O}_n$. For all $\delta_X \in \mathscr{T}_X\mathrm{GL}(n,\mathbb{R})$, the Fréchet derivative of $s \circ \pi : X \in \mathrm{GL}(n,\mathbb{R}) \mapsto W := \left(XX^\top\right)^{1/2} \in \mathscr{S}_+(n)$ at X in the direction of the tangent vector δ_X is*

$$d_X(s\circ\pi)(\delta_X) = \mathbf{T}_W^{-1}\left(\delta_X X^\top + X\delta_X^\top\right), \tag{1}$$

where $\mathbf{T}_E^{-1}(Z)$ denotes the unique solution A to the Sylvester equation $EA + AE = Z$ for symmetric matrices E and Z.

Our main motivation comes from the computation of geodesics of the Frobenius metric on the quotient space $\mathrm{GL}(n,\mathbb{R})/\mathscr{O}_n$ for which $\pi : \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})/\mathscr{O}_n$ becomes a Riemannian submersion [12], see Section 3. We use Formula 1 to derive the horizontal lift of any tangent vector in the tangent bundle $\mathscr{T}(\mathrm{GL}(n,\mathbb{R})/\mathscr{O}_n)$ parameterized by the tangent bundle of the symmetric section s, presented in Theorem 6. Given an explicit expression of the Riemannian exponential map over $\mathrm{GL}(n,\mathbb{R})$, one can obtain the explicit expression of the geodesics over the quotient space $\mathrm{GL}(n,\mathbb{R})/\mathscr{O}_n$ simply by projecting the *horizontal* geodesics of $\mathrm{GL}(n,\mathbb{R})$, see [5]. Let $X \in \mathrm{GL}(n,\mathbb{R})$, $\pi(X) \in \mathrm{GL}(n,\mathbb{R})/\mathscr{O}_n$, we make sense of a *quotient tangent vector* $\delta_{\pi(X)} \in \mathscr{T}_{\pi(X)}(\mathrm{GL}(n,\mathbb{R})/\mathscr{O}_n)$ as an $\mathscr{O}_n$-equivariant and horizontal vector field along the fiber $\pi(X) = X\mathscr{O}_n$. Thanks to [12], we obtain that geodesics in $\mathrm{GL}(n,\mathbb{R})/\mathscr{O}_n$ are images of straight lines

$$t \longmapsto X + t\mathbf{T}_{WW^\top}^{-1}\Big(W\delta_W + \delta_W W\Big)X$$

in $\mathrm{GL}(n,\mathbb{R})$ through the quotient map π, restricted to the time interval around $t = 0$ where they remain full rank and where $\delta_W \in T_W\mathscr{S}_+(n)$ is obtained by pushing forward $\delta_{\pi(X)}$ along the section s, that is, $\delta_W := d_{\pi(X)}s(\delta_{\pi(X)})$ or, equivalently, $\delta_{\pi(X)} = d_W s^{-1}(\delta_W) \in \mathscr{T}_{\pi(X)}(\mathrm{GL}(n,\mathbb{R})/\mathscr{O}_n)$.

In continuum mechanics, one often considers X to be the gradient of a transformation, and one considers the orthogonal matrix R and the symmetric positive definite matrix W to be the rotation tensor and the left stretch tensor of this transformation, respectively. Naturally, the problem of evaluating the derivatives of the rotation and stretch tensors in the polar decomposition arises in various fields of applied mathematics and kinematics, see for instance [13]. The computation of such derivatives is often difficult because it usually involves matrix square roots and the matrix sign function [6]. Explicit 2-dimensional and 3-dimensional expressions for the rates and derivatives of the stretch tensor have been derived, for example, in [3, 8], and [14]. Most of these results differentiate the symmetric square root and solve tensor equations, as in [10].

While many low-dimensional equivalent formulae are now available, most of them are scattered throughout the litterature. Although purely *algebraic*, the proofs often rely on elaborated tensor calculus and low-dimensional relations with matrix invariants; alternatively, an *analytical* expression can be found in [4]. In comparison, Formula 1 only requires solving a simple Sylvester equation and holds in any dimension. For completeness, we also verify that the derivative of $s\circ\pi$ in Theorem 1 coincides with the already known formulas ([3, 14]) for the derivative of the right stretch tensor in dimensions 2 and 3.

2. Results and Proofs

2.1. *Proof of Theorem 1*

The polar decomposition $X = WR$, with $X \in \mathrm{GL}(n,\mathbb{R})$, $W := \left(XX^\top\right)^{1/2} \in \mathscr{S}_+(n)$ and $R \in \mathscr{O}_n$ suggests a right action of the orthogonal group $\mathscr{O}_n$ on the general linear group $\mathrm{GL}(n,\mathbb{R})$. Since the group action is smooth, free and proper [12], the quotient map

$$\pi : X \in \mathrm{GL}(n,\mathbb{R}) \longmapsto X\mathscr{O}_n \in \mathrm{GL}(n,\mathbb{R})/\mathscr{O}_n$$

endows the orbit space $\mathrm{GL}(n,\mathbb{R})/\mathscr{O}_n$ with a quotient manifold structure. Furthermore, $\mathscr{S}_+(n)$, seen as the image of the section $s : X\mathscr{O}_n \in \mathrm{GL}(n,\mathbb{R})/\mathscr{O}_n \mapsto W := \left(XX^\top\right)^{1/2} \in \mathscr{S}_+(n)$, is diffeomorphic to itself through the map

$$\Phi : Q \in \mathscr{S}_+(n) \longmapsto QQ^\top \in \mathscr{S}_+(n),$$

whose inverse is $\Phi^{-1} : Q \in \mathscr{S}_+(n) \mapsto Q^{1/2} \in \mathscr{S}_+(n)$. Define the submersion

$$\phi : X \in \mathrm{GL}(n,\mathbb{R}) \longrightarrow XX^\top \in \mathscr{S}_+(n)$$

and note that $XX^\top = WRR^\top W^\top = WW^\top$, so we write $\phi = \Phi\circ(s\circ\pi)$, see Figure 1.

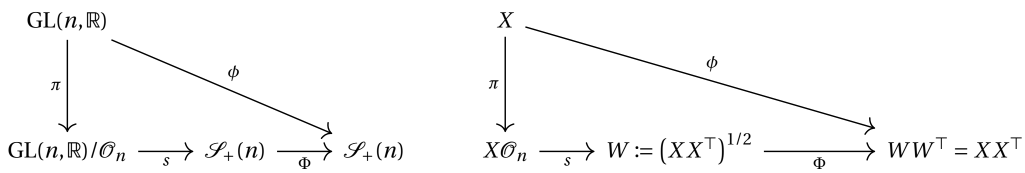

Figure 1. Commutative diagram for ϕ, $s\circ\pi$ and Φ, from [12].

Proof of Theorem 1. Let $X = WR \in \mathrm{GL}(n,\mathbb{R})$ with $W \in \mathscr{S}_+(n)$ and $R \in \mathscr{O}_n$. Let us write

$$s\circ\pi = \Phi^{-1}\circ\phi \text{ with } \Phi^{-1}\circ\phi : X \longmapsto (XX^\top)^{1/2} =: W$$

and where $(\cdot)^{1/2} : Q \in \mathscr{S}_+(n) \mapsto Q^{1/2} \in \mathscr{S}_+(n)$ denotes the symmetric square root of a matrix, which is uniquely defined for any self-adjoint, positive-definite matrix Q, [9]. It is rather clear that the differential of ϕ at X writes for all $\delta_X \in \mathscr{T}_X\,\mathrm{GL}(n,\mathbb{R})$

$$d_X\phi(\delta_X) = \delta_X X^\top + X\delta_X^\top. \tag{2}$$

Let us now compute the derivative of $\Phi^{-1} : Q \in \mathscr{S}_+(n) \mapsto Q^{1/2} \in \mathscr{S}_+(n)$. Let $\delta_Q \in \mathscr{T}_Q\mathscr{S}_+(n)$. By differentiating both sides of the equation $Q^{1/2}Q^{1/2} = Q$ and by applying the product rule we obtain $Q^{1/2}K + KQ^{1/2} = \delta_Q$, where $K = d_Q\Phi^{-1}(\delta_Q)$. We can therefore express K in terms of a solution of a Sylvester equation:

$$d_Q\Phi^{-1}(\delta_Q) = \mathbf{T}^{-1}_{Q^{1/2}}\left(\delta_Q\right). \tag{3}$$

Now we can compute $d_X(s\circ\pi)$ by applying the chain rule to its composite functions, whose derivatives are written in (2) and (3) respectively. We have

$$\begin{aligned} d_X(s\circ\pi)(\delta_X) &= d_X(\Phi^{-1}\circ\phi)(\delta_X) \\ &= d_{\phi(X)}\Phi^{-1}\circ d_X\phi(\delta_X) \\ &= d_{\phi(X)}\Phi^{-1}(\delta_X X^\top + X\delta_X^\top) \\ &= \mathbf{T}_W^{-1}(\delta_X X^\top + X\delta_X^\top). \end{aligned}$$

□

Remark 2. If one considers the left action of $\mathscr{O}_n$ on $\mathrm{GL}(n,\mathbb{R})$, the Fréchet derivative of $\widetilde{s}\circ\widetilde{\pi}$: $X = RW \in \mathrm{GL}(n,\mathbb{R}) \mapsto \left(X^\top X\right)^{1/2} =: W \in \mathscr{S}_+(n)$ at X in the direction of the tangent vector $\delta_X \in \mathscr{T}_X\mathrm{GL}(n,\mathbb{R})$ writes

$$d_X(\widetilde{s}\circ\widetilde{\pi})(\delta_X) = \mathbf{T}_W^{-1}\left(X^\top\delta_X + \delta_X^\top X\right),$$

and this result can be directly obtained from the proof of Theorem 1.

Additionally, we give an equivalent way to characterize the differential of the stretch-tensor.

Proposition 3. *Let $X = WR \in \mathrm{GL}(n,\mathbb{R})$ with $W \in \mathscr{S}_+(n)$ and $R \in \mathscr{O}_n$. For all $\delta_X \in \mathscr{T}_X\mathrm{GL}(n,\mathbb{R})$, $d_X(s\circ\pi)(\delta_X)$ is the unique symmetric matrix such that $Wd_X(s\circ\pi)(\delta_X) - X\delta_X^\top$ is skew-symmetric.*

Proof of Proposition 3. Let us denote $A = d_X(s\circ\pi)(\delta_X)$. By Theorem 1, A is the unique symmetric matrix solution to the Sylvester equation

$$WA + AW = \delta_X X^\top + X\delta_X^\top. \tag{4}$$

We can rewrite (4) to obtain

$$WA - X\delta_X^\top = \delta_X X^\top - AW = -\left(AW - \delta_X X^\top\right) = -\left(WA - X\delta_X^\top\right)^\top$$

In other words, $Wd_X(s\circ\pi)(\delta_X) - X\delta_X^\top$ is skew-symmetric. Conversely, if $WA - X\delta_X^\top$ is skew-symmetric, then A satisfies Sylvester (4) at symmetric W with symmetric right-hand side, hence it is unique and symmetric. □

By applying Proposition 3 at the identity tangent vector one obtains the following corollary.

Corollary 4. *Let $X = WR \in \mathrm{GL}(n,\mathbb{R})$ with $W \in \mathscr{S}_+(n)$ and $R \in \mathscr{O}_n$. There exists a unique symmetric matrix A such that $WA - X$ is skew-symmetric, given by $A = \mathbf{T}_W^{-1}\left(X^\top + X\right)$.*

2.2. *Solving Sylvester Equation for Symmetric Matrices*

In practice, the solution to a Sylvester equation $\mathbf{T}_E^{-1}(Z)$ can be easily computed by the diagonalization of $E \in \mathscr{S}_+(n)$. Indeed, consider an eigenvalue decomposition $U\Lambda U^\top$ of E, the Sylvester equation $EX + XE = Z$ then becomes $U\Lambda U^\top X + XU\Lambda U^\top = Z$ where $\Lambda = \mathrm{Diag}(\lambda_1,\ldots,\lambda_n)$. Left and right multiplying this equation by $U^\top$ and U yields

$$\Lambda U^\top XU + U^\top XU\Lambda = U^\top ZU \tag{5}$$

which can be easily solved for $\widetilde{X} := U^\top XU$ and $\widetilde{Z} := U^\top ZU$, as noted in [12] and [2]. Denoting the matrix coefficients $\widetilde{X}_{ij}$ and $\widetilde{Z}_{ij}$ of $\widetilde{X}$ and $\widetilde{Z}$, respectively, (5) becomes

$$\lambda_i\widetilde{X}_{ij} + \widetilde{X}_{ij}\lambda_j = \widetilde{Z}_{ij}. \tag{6}$$

Since both X and Z are symmetric (and so are $\widetilde{X}$ and $\widetilde{Z}$), the system only needs to be solved on the $\frac{n(n+1)}{2}$-dimensional linear subspace of vectorized symmetric matrices. This amounts to restricting the computations to the components $\widetilde{X}_{ij}$ and $\widetilde{Z}_{ij}$ with $i \le j$. Since $E \in \mathscr{S}_+(n)$, Λ is invertible and (6) rewrites

$$\widetilde{X}_{ij} = \frac{\widetilde{Z}_{ij}}{\lambda_i + \lambda_j}.$$

The classical Jacobi method is an extremely efficient algorithm to perform an eigenvalue decomposition of a symmetric positive definite matrix with asymptotic quadratic convergence and high accuracy [7].

2.3. *Formula in Dimension 2 and 3*

For the sake of completeness, let us show the coincidence of Formula 1 with the derivative of the right stretch tensor derived in [14] and [3] for dimensions 2 and 3, respectively.

Proposition 5. *Let $X = WR \in \mathrm{GL}(2,\mathbb{R})$ be a 2-dimensional invertible matrix, with $W \in \mathscr{S}_+(2)$ and $R \in \mathscr{O}_2$. For all $\delta_X \in \mathscr{T}_X \mathrm{GL}(2,\mathbb{R})$, the following formulas for $d_X(s\circ\pi)(\delta_X)$ are equal:*

$$d_X(s\circ\pi)(\delta_X) = \mathbf{T}_W^{-1}\left(\delta_X X^\top + X\delta_X^\top\right) \tag{7}$$

$$= \delta_X R^\top - \mathrm{tr}(W)^{-1} W\left(\delta_X R^\top - R\delta_X^\top\right). \tag{8}$$

In dimension 3, the formula still coincides with the following derived in the literature:

$$d_X(s\circ\pi)(\delta_X) = \mathbf{T}_W^{-1}(\delta_X X^\top + X\delta_X^\top) \tag{9}$$

$$= \delta_X R^\top - \det(Z)^{-1} WZ(\delta_X R^\top - R\delta_X^\top)Z, \tag{10}$$

where $Z := \mathrm{tr}(W)\,\mathrm{Id} - W$ is a 3-dimensional symmetric matrix. (8) *is derived from [14] and* (10) *is first seen in [3].*

Proof of Proposition 5. Let us show the equivalence of the formulas at any tangent vector. Since W is symmetric positive definite, it does not share any eigenvalue with $-W$. Consequently, the Sylvester equation

$$YW + WY = \delta_X X^\top + X\delta_X^\top \tag{11}$$

admits a *unique* solution. Morevoer, the solution Y to (11) is symmetric. Indeed, since $\delta_X X^\top + X\delta_X^\top$ and W are symmetric matrices, Y and $Y^\top$ are both solutions of (11), hence, by unicity, they must be equal. Theorem 1 yields

$$d_X(s\circ\pi)(\delta_X) = \mathbf{T}_W^{-1}\left(\delta_X X^\top + X\delta_X^\top\right)$$

as the unique solution Y to Sylvester (11). We show that the derivative of the *2-dimensional* stretch tensor derived from [14] satisfies this equation, and by uniqueness, we infer the equality of (7) and (8). The main difficulty is to show that $\delta_X R^\top - \mathrm{tr}(W)^{-1} W(\delta_X R^\top - R\delta_X^\top)$ is symmetric, which has been done in [14], that is

$$\delta_X R^\top - \mathrm{tr}(W)^{-1} W(\delta_X R^\top - R\delta_X^\top) = R\delta_X^\top - \mathrm{tr}(W)^{-1}(R\delta_X^\top - \delta_X R^\top)W. \tag{12}$$

Thus, plugging both side of (12) as Y into the expression $YW + WY$ we get:

$$\begin{aligned}
&\left(\delta_X R^\top - \mathrm{tr}(W)^{-1} W(\delta_X R^\top - R\delta_X^\top)\right)W + W\left(R\delta_X^\top - \mathrm{tr}(W)^{-1}(R\delta_X^\top - \delta_X R^\top)W\right)\\
&\quad= \delta_X R^\top W - \mathrm{tr}(W)^{-1} W(\delta_X R^\top - R\delta_X^\top)W + WR\delta_X^\top + \mathrm{tr}(W)^{-1} W(\delta_X R^\top - R\delta_X^\top)W\\
&\quad= \delta_X R^\top W + WR\delta_X^\top\\
&\quad= \delta_X X^\top + X\delta_X^\top.
\end{aligned}$$

So $Y = \delta_X R^\top - \operatorname{tr}(W)^{-1} W(\delta_X R^\top - R\delta_X^\top)$ is the unique solution of (11), and the formulas coincide at any tangent vector δ_X at X. Since this is true for any $X = WR \in \mathrm{GL}(2,\mathbb{R})$, we conclude.

The proof for the *3-dimensional* formula of the right stretch tensor follows the exact same path. Set $Z := \operatorname{tr}(W)\operatorname{Id} - W$, the main difficulty is to show that $\delta_X R^\top - \det(Z)^{-1} WZ(\delta_X R^\top - R\delta_X^\top)Z$ is symmetric, which has been done in [14], that is

$$\delta_X R^\top - \det(Z)^{-1} WZ(\delta_X R^\top - R\delta_X^\top)Z = R\delta_X^\top - \det(Z)^{-1} Z(R\delta_X^\top - \delta_X R^\top)ZW. \tag{13}$$

Thus, plugging both sides of (13) as Y into the expression $YW + WY$ we get:

$$\begin{aligned}
&\left(\delta_X R^\top - \det(Z)^{-1} WZ(\delta_X R^\top - R\delta_X^\top)Z\right)W + W\left(R\delta_X^\top - \det(Z)^{-1} Z(R\delta_X^\top - \delta_X R^\top)ZW\right)\\
&\quad = \delta_X R^\top W - \det(Z)^{-1} WZ(\delta_X R^\top - R\delta_X^\top)ZW + WR\delta_X^\top + \det(Z)^{-1} WZ(\delta_X R^\top - R\delta_X^\top)ZW\\
&\quad = \delta_X R^\top W + WR\delta_X^\top\\
&\quad = \delta_X X^\top + X\delta_X^\top.
\end{aligned}$$

So $Y = \delta_X R^\top - \det(Z)^{-1} WZ(\delta_X R^\top - R\delta_X^\top)Z$ is the unique solution of (11), and the formulas coincide at any tangent vector δ_X at X. Since this is true for any $X = WR \in \mathrm{GL}(3,\mathbb{R})$, we conclude. □

3. Quotient Geodesics in $\mathrm{GL}(n,\mathbb{R})/\mathcal{O}_n$

Let us consider the $\mathcal{O}_n$-principal bundle $\pi : \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})/\mathcal{O}_n$ induced by the *smooth*, *proper*, and *free* right-action of the orthogonal group $\mathcal{O}_n$ on the general linear group $\mathrm{GL}(n,\mathbb{R})$,

$$R_g : X \in \mathrm{GL}(n,\mathbb{R}) \longmapsto Xg \in \mathrm{GL}(n,\mathbb{R}), \quad \forall g \in \mathcal{O}_n.$$

For all $X \in \mathrm{GL}(n,\mathbb{R})$, let $\mathcal{V}_X \subset \mathcal{T}_X\mathrm{GL}(n,\mathbb{R})$ denote the *vertical* space at X consisting of vectors tangent to the fiber through X. Following the definition given in [11], we recall that a *principal $\mathcal{O}_n$-connection* in $\pi : \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})/\mathcal{O}_n$ is an assignment of a *horizontal* subspace $\mathcal{H}_X$ of $\mathcal{T}_X\mathrm{GL}(n,\mathbb{R})$ to each $X \in \mathrm{GL}(n,\mathbb{R})$ such that $\mathcal{T}_X\mathrm{GL}(n,\mathbb{R})$ is the direct sum of $\mathcal{V}_X$ and $\mathcal{H}_X$, and the distribution $X \mapsto \mathcal{H}_X$ is smooth and $\mathcal{O}_n$-equivariant, i.e., $\mathcal{H}_{R_g(X)} = d_X R_g(\mathcal{H}_X)$. We denote by $\mathcal{V} = \ker(d\pi : \mathcal{T}\,\mathrm{GL}(n,\mathbb{R}) \to \mathcal{T}\,(\mathrm{GL}(n,\mathbb{R})/\mathcal{O}_n))$ and $\mathcal{H}$ the vertical and horizontal bundles of this connection, respectively. The choice of a Riemannian metric on $\mathrm{GL}(n,\mathbb{R})$ provides a canonical way to assign to each $X \in \mathrm{GL}(n,\mathbb{R})$ a horizontal subspace by taking the orthogonal complement of the vertical subspace with respect to the metric, that is, $\mathcal{H}_X := \mathcal{V}_X^\perp$. Therefore, endowing $\mathrm{GL}(n,\mathbb{R})$ with a Riemannian metric provides a principal connection in $\pi : \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})/\mathcal{O}_n$ and moreover

$$d_X\pi_{|\mathcal{H}_X} : \mathcal{H}_X \longrightarrow \mathcal{T}_{\pi(X)}\,(\mathrm{GL}(n,\mathbb{R})/\mathcal{O}_n)$$

is a linear isomorphism. In the sense of [11], we define the *horizontal lift of a vector field* $\delta \in \mathfrak{X}\,(\mathrm{GL}(n,\mathbb{R})/\mathcal{O}_n)$ to be the unique vector field $\delta^\sharp \in \mathfrak{X}\,(\mathrm{GL}(n,\mathbb{R}))$ which is horizontal and which projects onto δ, that is, for all $X \in \mathrm{GL}(n,\mathbb{R})$,

$$\delta_X^\sharp \in \mathcal{H}_X \quad \text{and} \quad d_X\pi\left(\delta_X^\sharp\right) = \delta_{\pi(X)}.$$

Hence, we make sense of a *quotient tangent vector* $\delta_{\pi(X)} \in \mathcal{T}_{\pi(X)}\,(\mathrm{GL}(n,\mathbb{R})/\mathcal{O}_n)$ as an $\mathcal{O}_n$-equivariant and horizontal vector field along the fiber $\pi(X) = X\mathcal{O}_n$. An $\mathcal{O}_n$-equivariant vector field δ along $X\mathcal{O}_n$ is entirely determined by its value at a single point in $X\mathcal{O}_n$ since for all $U \in \mathcal{O}_n$, $\delta_{XU} = \delta_X U$. Additionally, a vector field $\delta^\sharp$ is horizontal along $X\mathcal{O}_n$ if at each point XU in the fiber, $\delta_{XU}^\sharp \in \mathcal{H}_{XU}$.

Let us endow $\mathrm{GL}(n,\mathbb{R})$ with the Frobenius metric g^{Fro} and hence, with a canonical assignment of a *horizontal* subspace $\mathscr{H}_X := \mathscr{V}_X^{\perp} \in \mathscr{T}_X \mathrm{GL}(n,\mathbb{R})$ to each $X \in \mathrm{GL}(n,\mathbb{R})$. The Frobenius metric turns R_g into an *isometry* and the quotient map $\pi : \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})/\mathscr{O}_n$ into a Riemannian submersion [12], thus, provided the expression of the horizontal lift in $\mathrm{GL}(n,\mathbb{R})$ of a quotient tangent vector, we can derive the expression of the geodesics in the quotient space $\mathrm{GL}(n,\mathbb{R})/\mathscr{O}_n$. Let us first compute the horizontal lift associated to this Riemannian submersion.

Theorem 6. *Let $X = WR \in \left(\mathrm{GL}(n,\mathbb{R}), g^{\mathrm{Fro}}\right)$ with $W \in \mathscr{S}_+(n)$ and $R \in \mathscr{O}_n$. The horizontal lift $\delta_X^{\sharp}$ at X of a quotient tangent vector $\delta_{\pi(X)} \in \mathscr{T}_{\pi(X)}(\mathrm{GL}(n,\mathbb{R})/\mathscr{O}_n)$ is given by*

$$\delta_X^{\sharp} = \mathbf{T}_{WW^{\top}}^{-1}\Big(W\delta_W + \delta_W W\Big)X.$$

where $\delta_W := d_{\pi(X)} s\left(\delta_{\pi(X)}\right) \in \mathscr{T}_W \mathscr{S}_+(n)$. Likewise, we can express δ_W in terms of a solution to a Sylvester equation,

$$\delta_W = \mathbf{T}_W^{-1}\left(\delta_X^{\sharp} X^{\top} + X\left(\delta_X^{\sharp}\right)^{\top}\right).$$

The proof of Theorem 6 makes use of the following lemma.

Lemma 7. *Let $X \in \left(\mathrm{GL}(n,\mathbb{R}), g^{\mathrm{Fro}}\right)$. The horizontal spaces $\mathscr{H}_X^{\pi}$, $\mathscr{H}_X^{s\circ\pi}$ and $\mathscr{H}_X^{\phi}$ associated to the* submersions *π, $s\circ\pi$ and ϕ respectively coincide, and we write them*

$$\mathscr{H}_X = \left\{\delta_X^{\sharp} = SX : S \in \mathscr{S}(n)\right\}.$$

Proof of Lemma 7. Let $X = WR \in \left(\mathrm{GL}(n,\mathbb{R}), g^{\mathrm{Fro}}\right)$ with $W \in \mathscr{S}_+(n)$ and $R \in \mathscr{O}_n$. By definition, we write the vertical space $\mathscr{V}_X^{\phi}$ as $\ker(d_X\phi)$, and from [1] we have:

$$\begin{aligned}\mathscr{V}_X^{\phi} &= \left\{\delta_X \in \mathscr{T}_X \mathrm{GL}(n,\mathbb{R}) \mid \delta_X X^{\top} + X\delta_X^{\top} = 0\right\} \\ &= \left\{K\left(X^{-1}\right)^{\top} \,\Big|\, K \in \mathrm{Skew}(n)\right\}.\end{aligned}$$

Let us compute the orthogonal complement of this space with respect to the Frobenius inner product. Let $\delta_X^{\sharp}$ be a horizontal vector at X. For all skew-symmetric matrices K

$$0 = g_X^{\mathrm{Fro}}\left(\delta_X^{\sharp}, K\left(X^{-1}\right)^{\top}\right) = \mathrm{tr}\left(\left(\delta_X^{\sharp}\right)^{\top} K\left(X^{-1}\right)^{\top}\right) = \mathrm{tr}\left(\left(X^{-1}\right)^{\top}\left(\delta_X^{\sharp}\right)^{\top} K\right).$$

This happens if and only if $\left(X^{-1}\right)^{\top}\left(\delta_X^{\sharp}\right)^{\top}$ is symmetric, i.e. $\delta_X^{\sharp}X^{-1}$ is symmetric, [1]. So we write

$$\mathscr{H}_X^{\phi} := \mathscr{V}_X^{\phi^{\perp}} = \left\{\delta_X^{\sharp} = SX : S \in \mathscr{S}(n)\right\}.$$

Define $\mathscr{V}_X^{\pi}$ as $\ker(d_X\pi)$, and observe that

$$\begin{aligned}\mathscr{V}_X^{\pi} &= \left\{\delta_X \in \mathscr{T}_X \mathrm{GL}(n,\mathbb{R}) \mid \mathbf{T}_W^{-1}\left(\delta_X X^{\top} + X\delta_X^{\top}\right) = 0\right\} \\ &= \left\{\delta_X \mid \delta_X X^{\top} + X\delta_X^{\top} = 0\right\} \\ &= \mathscr{V}_X^{\phi}.\end{aligned}$$

So their orthogonal complements are also identical, $\mathscr{H}_X^{\pi} = \mathscr{H}_X^{\phi}$. Observe additionally that since the differential of the section s, $ds : \mathscr{T}(\mathrm{GL}(n,\mathbb{R})/\mathscr{O}_n) \to \mathscr{T}\mathscr{S}_+(n)$ is a linear isomorphism, we have that $\ker(d_X(s\circ\pi)) = \ker(d_{\pi(X)}s \circ d_X\pi) = \ker(d_X\pi)$. Therefore, $\mathscr{V}_X^{s\circ\pi} = \mathscr{V}_X^{\pi}$ and we deduce the equality of their orthogonal complement with respect to the Frobenius metric, $\mathscr{H}_X^{s\circ\pi} = \mathscr{H}_X^{\pi}$. □

Proof of Theorem 6. For computing the horizontal lift of a vector field in $\mathrm{GL}(n,\mathbb{R})/\mathscr{O}_n$ we shall use the parametrization of the quotient space $\mathrm{GL}(n,\mathbb{R})/\mathscr{O}_n$ given by the diffeomorphism $s : X\mathscr{O}_n \in \mathrm{GL}(n,\mathbb{R})/\mathscr{O}_n \mapsto W := \left(XX^\top\right)^{1/2} \in \mathscr{S}_+(n)$. Let us consider the $\mathscr{O}_n$-principal bundle and Riemannian submersion

$$\pi : \mathrm{GL}(n,\mathbb{R}) \longrightarrow \mathrm{GL}(n,\mathbb{R})/\mathscr{O}_n$$

arising when $\mathrm{GL}(n,\mathbb{R})$ is equipped with the Frobenius metric. For any vector field $\delta \in \mathfrak{X}(\mathrm{GL}(n,\mathbb{R})/\mathscr{O}_n)$ of the quotient space $\mathrm{GL}(n,\mathbb{R})/\mathscr{O}_n$, the $\mathscr{O}_n$-principal connection associated with the Frobenius metric ensures the existence and uniqueness of its vector field horizontal lift $\delta^\sharp \in \mathfrak{X}(\mathrm{GL}(n,\mathbb{R}))$ in $\mathrm{GL}(n,\mathbb{R})$. Let $X \in \mathrm{GL}(n,\mathbb{R})$, $\pi(X) \in \mathrm{GL}(n,\mathbb{R})/\mathscr{O}_n$ and denote by $\delta^\sharp_X \in \mathscr{T}_X\mathrm{GL}(n,\mathbb{R})$ the horizontal lift at X of the quotient tangent vector $\delta_{\pi(X)} \in \mathscr{T}_{\pi(X)}(\mathrm{GL}(n,\mathbb{R})/\mathscr{O}_n)$. By definition of the horizontal lift, $\delta^\sharp_X$ projects on $\delta_{\pi(X)}$ via π, that is,

$$d_X\pi\left(\delta^\sharp_X\right) = \delta_{\pi(X)}.$$

Since $d_{\pi(X)}s : \mathscr{T}_{\pi(X)}(\mathrm{GL}(n,\mathbb{R})/\mathscr{O}_n) \to T_W\mathscr{S}_+(n)$ is a linear isomorphism, there exists a unique tangent vector written $\delta_W \in T_W\mathscr{S}_+(n)$ obtained by pushing forward $\delta_{\pi(X)}$ along the section s, that is, $\delta_W := d_{\pi(X)}s(\delta_{\pi(X)})$ and we have

$$d_X(s\circ\pi)\left(\delta^\sharp_X\right) = \delta_W. \tag{14}$$

Theorem 1 provides an explicit formula for the differential of $s\circ\pi$ and Lemma 7 allows us to write $\delta^\sharp_X = SX$, $S \in \mathscr{S}(n)$, hence, (14) can be rewritten

$$\delta_W = \mathbf{T}_W^{-1}\left(\delta^\sharp_X X^\top + X\left(\delta^\sharp_X\right)^\top\right) = \mathbf{T}_W^{-1}\left(SXX^\top + XX^\top S\right).$$

Thus, the tangent vector δ_W is the unique solution of the Sylvester equation

$$W\delta_W + \delta_W W = SXX^\top + XX^\top S,$$

and we can, in turn, express S as a solution of a Sylvester equation as well,

$$S = \mathbf{T}^{-1}_{XX^\top}(W\delta_W + \delta_W W) = \mathbf{T}^{-1}_{WW^\top}(W\delta_W + \delta_W W).$$

Consequently, the horizontal lift at X of a quotient tangent vector $\delta_{\pi(X)} = d_W s^{-1}(\delta_W) \in \mathscr{T}_{\pi(X)}(\mathrm{GL}(n,\mathbb{R})/\mathscr{O}_n)$ is given by $\delta^\sharp_X = SX = \mathbf{T}^{-1}_{WW^\top}(W\delta_W + \delta_W W)X$. □

Using the $\mathscr{O}_n$-equivariance of principal connections, we deduce from Theorem 6 the horizontal lift at any point XU in the fiber $X\mathscr{O}_n$, $U \in \mathscr{O}_n$,

$$\delta^\sharp_{XU} = \delta^\sharp_X U = \mathbf{T}^{-1}_{WW^\top}\left(W\delta_W + \delta_W W\right)XU.$$

In particular, horizontally lifting at $W := \left(XX^\top\right)^{1/2}$ along the symmetric section $s : \mathrm{GL}(n,\mathbb{R})/\mathscr{O}_n \to \mathscr{S}_+(n) \subset \mathrm{GL}(n,\mathbb{R})$ yields

$$\delta^\sharp_W = \mathbf{T}^{-1}_{WW^\top}\left(W\delta_W + \delta_W W\right)W.$$

Provided the horizontal lift in $\mathrm{GL}(n,\mathbb{R})$ of any quotient tangent vector to the fibers of $\mathrm{GL}(n,\mathbb{R})$, we can derive from [12] an expression of the quotient geodesics on $\mathrm{GL}(n,\mathbb{R})/\mathscr{O}_n$.

Theorem 8. *Let $X = WR \in \left(\mathrm{GL}(n,\mathbb{R}), g^{\mathrm{Fro}}\right)$ with $s\circ\pi(X) = W \in \mathscr{S}_+(n)$ and $R \in \mathscr{O}_n$. Let us define the set*

$$\mathscr{D}_X := \left\{\delta^\sharp_X \in \mathscr{H}_X \,\middle|\, \mathrm{rank}\left(X + t\delta^\sharp_X\right) = n \quad \forall t \in [0,1]\right\}.$$

For all $\delta_{\pi(X)} \in d_X\pi(\mathscr{D}_X)$, the Riemannian exponential map on $\mathrm{GL}(n,\mathbb{R})/\mathscr{O}_n$ is given by

$$\begin{aligned}\mathrm{Exp}_{\pi(X)}(t\delta_{\pi(X)}) &= \pi\left(X + t\delta^\sharp_X\right)\\ &= \pi\left(X + t\mathbf{T}^{-1}_{WW^\top}\left(W\delta_W + \delta_W W\right)X\right),\end{aligned}$$

i.e., geodesics in $\mathrm{GL}(n,\mathbb{R})/\mathcal{O}_n$ *are images, through the quotient map* π *of straight lines*

$$t \longmapsto X + t\mathbf{T}^{-1}_{WW^\top}\Big(W\delta_W + \delta_W W\Big)X$$

in $\mathrm{GL}(n,\mathbb{R})$*, restricted to the time interval around* $t = 0$ *where* $X + t\mathbf{T}^{-1}_{WW^\top}\Big(W\delta_W + \delta_W W\Big)X$ *remains full rank. In particular, geodesics along the symmetric section* $s : \mathrm{GL}(n,\mathbb{R})/\mathcal{O}_n \to \mathcal{S}_+(n) \subset \mathrm{GL}(n,\mathbb{R})$ *are given by*

$$\begin{aligned}\mathrm{Exp}_W(t\delta_W) &= s\circ\pi\left(X + t\delta^\sharp_W\right)\\ &= s\circ\pi\left(X + t\mathbf{T}^{-1}_{WW^\top}\Big(W\delta_W + \delta_W W\Big)W\right),\end{aligned}$$

where $\delta_W := d_{\pi(X)}s\left(\delta_{\pi(X)}\right)$.

Proof of Theorem 8. From [12], geodesics in $\mathrm{GL}(n,\mathbb{R})/\mathcal{O}_n$ are images, through the quotient map π of straight lines

$$t \longmapsto X + t\delta^\sharp_X \tag{15}$$

restricted to the time interval around $t = 0$ where $X + t\delta^\sharp_X$ remains full rank. Theorem 6 provides an explicit expression of the horizontal lift of any quotient tangent vector, that we plug into (15). □

4. Acknowledgements

The authors warmly thank James Benn and the anonymous reviewer for interesting discussions and careful proofreading of the paper.

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

References

[1] R. Bhatia, T. Jain and Y. Lim, “On the Bures-Wasserstein distance between positive definite matrices”, *Expo. Math.* **37** (2019), no. 2, pp. 165–191.

[2] R. Bhatia and P. Rosenthal, “How and why to solve the operator equation $AX - XB = Y$”, *Bull. Lond. Math. Soc.* **29** (1997), no. 1, pp. 1–21.

[3] Y.-C. Chen and L. Wheeler, “Derivatives of the stretch and rotation tensors”, *J. Elasticity* **32** (1993), no. 3, pp. 175–182.

[4] F. J. Feppon, *Riemannian geometry of matrix manifolds for Lagrangian uncertainty quantification of stochastic fluid flows*, Master thesis, Massachusetts Institute of Technology, 2017.

[5] S. Gallot, D. Hulin and J. Lafontaine, *Riemannian geometry*, Second edition, Springer, 1990, pp. xiv+284.

[6] E. S. Gawlik and M. Leok, “Iterative computation of the Fréchet derivative of the polar decomposition”, *SIAM J. Matrix Anal. Appl.* **38** (2017), no. 4, pp. 1354–1379.

[7] G. H. Golub and C. F. Van Loan, *Matrix Computations*, 4th edition, Johns Hopkins University Press, 2013.

[8] Z. H. Guo, “Rates of stretch tensors”, *J. Elasticity* **14** (1984), no. 3, pp. 263–267.

[9] B. Hall, *Lie groups, Lie algebras, and representations*, Second edition, Springer, 2015, pp. xiv+449. An elementary introduction.

[10] A. Hoger and D. E. Carlson, “On the derivative of the square root of a tensor and Guo’s rate theorems”, *J. Elasticity* **14** (1984), no. 3, pp. 329–336.

[11] S. Kobayashi and K. Nomizu, *Foundations of differential geometry. I*, Interscience Publishers, 1963, pp. xi+329.

[12] E. Massart and P.-A. Absil, “Quotient geometry with simple geodesics for the manifold of fixed-rank positive-semidefinite matrices”, *SIAM J. Matrix Anal. Appl.* **41** (2020), no. 1, pp. 171–198.

[13] M. P. Nash and A. V. Panfilov, “Electromechanical model of excitable tissue to study reentrant cardiac arrhythmias”, *Prog. Biophys. Mol. Biol.* **85** (2004), no. 2, pp. 501–522.

[14] L. Rosati, “Derivatives and rates of the stretch and rotation tensors”, *J. Elasticity* **56** (1999), no. 3, pp. 213–230.