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Claire Burrin

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# Equidistribution of continuous low-lying pairs of horocycles via Ratner's theorem

## *Équidistribution de paires d'horocycles longs via le théorème de Ratner*

Claire Burrin <sup>a</sup>

<sup>a</sup> Institute of Mathematics, University of Zurich, Switzerland  
E-mail: [claire.burrin@math.uzh.ch](mailto:claire.burrin@math.uzh.ch)

**Abstract.** We record an alternative proof of a recent joint equidistribution result of Blomer and Michel, based on Ratner's topological rigidity theorem. This approach has the advantage of extending to non-uniform lattices.

**Résumé.** Nous présentons une preuve alternative d'un résultat récent d'équidistribution jointe de Blomer et Michel, basée sur le théorème de rigidité topologique de Ratner. Cette approche a l'avantage de s'étendre aux réseaux non uniformes.

**Keywords.** Joint equidistribution, measure classification, commensurator, Hecke operators, matrix coefficients.

**Mots-clés.** Équidistribution jointe, classification des mesures, commensurateur, opérateurs de Hecke, coefficients matriciels.

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Expanding closed horocycles are known to become equidistributed on any non-compact finite-area hyperbolic orbifold, such as, e.g., the low-lying horocycle  $\{\mathrm{SL}_2(\mathbb{Z})(x + \frac{i}{T}) \mid x \in [0, 1]\}$  on the modular curve  $X = \mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$  as  $T \rightarrow \infty$ . Equidistribution also holds for various arithmetically interesting growing discrete subsets of points along expanding closed horocycles (see, e.g., [2,3,5,9,12]). Most recently Blomer and Michel [1] investigated the joint distribution properties of sets of the shape

$$\left\{ \left( \mathrm{SL}_2(\mathbb{Z}) \left( \frac{a+i}{q} \right), \mathrm{SL}_2(\mathbb{Z}) \left( \frac{ab+i}{q} \right) \right) \mid a \pmod{q} \right\} \subset X \times X \quad (1)$$

as  $q \rightarrow \infty$  while  $b \in (\mathbb{Z}/q\mathbb{Z})^*$  varies with  $q$  in a prescribed manner. In parallel, they consider a “continuous version” of this setup, which establishes the joint equidistribution of two low-lying horocycles of different speeds under a weak diophantine condition; see Theorem 1 below. By Dirichlet's theorem, given any real numbers  $y$  and  $Q > 0$ , there is a rational number  $\frac{a}{q}$  so that  $|y - \frac{a}{q}| < \frac{1}{qQ}$  and  $q \leq Q$ . In the following statement,  $y$  is irrational and  $Q$  is picked to depend on a growing parameter  $T$  so that as  $T \rightarrow \infty$  we can find a sequence  $q \rightarrow \infty$  for which  $y$  has a rational approximation of that order of magnitude.

**Theorem 1 (Blomer–Michel).** *Let  $X = \mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$ ,  $T > 1$ ,  $y \in [1, 2]$  and write  $y = a/q + O(1/(qQ))$  for positive coprime integers  $a, q$  with  $q \leq Q := T^{0.99}$ . Let  $I \subseteq \mathbb{R}$  be a fixed non-empty interval. Then*

$$\left\{ \left( \mathrm{SL}_2(\mathbb{Z}) \left( x + \frac{i}{T} \right), \mathrm{SL}_2(\mathbb{Z}) \left( xy + \frac{i}{T} \right) \right) \mid x \in I \right\} \subseteq X \times X$$

*equidistributes as  $T \rightarrow \infty$  for pairs  $(y, T)$  with  $q \rightarrow \infty$ .<sup>1</sup>*

The argument of proof is based on the estimation of Weyl sums via a shifted convolution problem, Sato–Tate and a sieving argument, and crucially the measure classification theorem of Einsiedler–Lindenstrauss [4, Theorem 1.4]. The purpose of this note is to record the following version of their result, using exclusively homogeneous dynamics. It would be very interesting to have such an argument for the discrete equidistribution analogue of Theorem 1 concerning sets of the shape (1), which is the main result of [1] (see [1, Theorem 1.1]).

Write

$$a_t = \begin{pmatrix} \sqrt{t} & 0 \\ 0 & \sqrt{t}^{-1} \end{pmatrix} \text{ for } t \in \mathbb{R}_{>0} \quad \text{and} \quad u_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \text{ for } x \in \mathbb{R}.$$

**Theorem 2.** *Let  $G = \mathrm{SL}_2(\mathbb{R})$ , let  $\Gamma$  be a non-uniform lattice in  $G$ , set  $L = G \times G$ ,  $\Lambda = \Gamma \times \Gamma$ . Let  $T > 1$ , and let  $y > 0$  be irrational if  $\Gamma$  is arithmetic or  $y \neq 1$  if  $\Gamma$  is non-arithmetic, and let  $I \subseteq \mathbb{R}$  be a fixed non-empty interval. Then*

$$\{(\Gamma u_x a_T^{-1}, \Gamma u_{xy} a_T^{-1}) : x \in I\} \subseteq \Lambda \backslash L$$

*equidistributes as  $T \rightarrow \infty$ .*

**Proof.** Let  $\pi$  be the composition of the smooth immersion  $\iota_y : G \rightarrow L$ ,  $\iota_y(g) = (g, a_y g a_y^{-1})$  and the canonical projection  $L \rightarrow \Lambda \backslash L$  so that for any test function  $\varphi \in C_b(\Lambda \backslash L)$ , we have

$$\int_0^1 \varphi(\pi(u_x a_{1/T})) dx = \int_0^1 \varphi(\Gamma u_x a_{1/T}, \Gamma u_{xy} a_{1/T}) dx,$$

where we use the standard fact that the expansion (or contraction) of closed horocycles by the geodesic flow is algebraically expressed by  $a_y u_x a_y^{-1} = u_{xy}$ . A result of Shah [13, Theorem 1.4] asserts that if  $\overline{\pi(G)} = \Lambda \backslash L$  then

$$\lim_{T \rightarrow \infty} \int_0^1 \varphi(\Gamma u_x a_{1/T}, \Gamma u_{xy} a_{1/T}) dx = \int_{\Lambda \backslash L} \varphi.$$

We may introduce a conjugation so that  $L = G \times G$ ,  $\Lambda = \Gamma \times a_y^{-1} \Gamma a_y$  and  $\pi$  corresponds to the diagonal embedding. Let  $\Delta(G)$  denote the diagonal embedding of  $G$  in  $L$ . By Ratner's topological rigidity theorem [11], we know that  $\overline{\pi(G)} = \pi(H)$  for some closed connected subgroup  $H < L$  containing  $\Delta(G)$ , where  $H \cap \Lambda$  is a lattice in  $H$ . Since  $G$  is simple,  $\Delta(G)$  is a maximal connected subgroup in  $L$  and so the only options are that  $H = L$  (in which case  $\overline{\pi(G)} = \Lambda \backslash L$  and equidistribution follows) or that  $H = \Delta(G)$ . In the latter case, the requirement that  $\Lambda \cap H$  be a lattice in  $H$  is equivalent to asking that  $\Gamma_y := \Gamma \cap a_y^{-1} \Gamma a_y$  be a lattice in  $G$ . This holds if and only if  $a_y \in \mathrm{Comm}(\Gamma)$ , where  $\mathrm{Comm}(\Gamma)$  is the commensurator of  $\Gamma$  in  $G$ , i.e.,

$$\mathrm{Comm}(\Gamma) = \{\alpha \in \mathrm{GL}_2^+(\mathbb{R}) \mid \Gamma \cap \alpha^{-1} \Gamma \alpha \text{ has finite index in both } \Gamma \text{ and } \alpha^{-1} \Gamma \alpha\}.$$

A non-uniform lattice  $\Gamma < G$  is arithmetic if and only if it is commensurable to  $\mathrm{SL}_2(\mathbb{Z})$ . Since  $\mathrm{Comm}(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{R}^\times \cdot \mathrm{GL}_2^+(\mathbb{Q})$ , we have  $a_y \in \mathrm{Comm}(\Gamma)$  if and only if  $y \in \mathbb{Q}$ . If  $\Gamma$  is non-arithmetic, we claim that  $a_y \in \mathrm{Comm}(\Gamma)$  if and only if  $y = 1$ .

By Margulis' arithmeticity criterium [6, Theorem 1.16, Chapter IX]  $\mathrm{Comm}(\Gamma)$  contains  $\Gamma$  as a finite-index subgroup. In particular,  $\mathrm{Comm}(\Gamma)$  is itself a lattice in  $G$  and by discreteness, its maximal unipotent subgroup is infinite cyclic. Let  $u_t$  be its generator, i.e.,  $u_{t'} \notin \mathrm{Comm}(\Gamma)$  for all

<sup>1</sup>In the statement of [1, Theorem 1.3],  $x$  is replaced by  $x/T$ . The argument of proof does however refer to the low-lying horocycles stated above; see [1, Theorem 4.2 and p. 55].

$|t'| < t$ . Let  $a_y \in \text{Comm}(\Gamma)$ ; up to replacing  $a_y$  by its inverse, we may assume that  $y > 1$ . Then the fact that  $a_y^{-1} u_t a_y = u_{t/y} \in \text{Comm}(\Gamma)$  contradicts the minimality of  $t$ .  $\square$

**Remark 3.** The statement extends to finitely many factors as follows. If  $y_1, \dots, y_n \in \mathbb{R}$  are chosen such that  $a_{y_i/y_j} \notin \text{Comm}(\Gamma)$  for  $i \neq j$ , then

$$\{(\Gamma u_{x y_1} a_T^{-1}, \dots, \Gamma u_{x y_n} a_T^{-1}) \mid x \in I\} \subset \Gamma^n \backslash G^n$$

equidistributes as  $T \rightarrow \infty$ . This can be compared to [7, Theorem 10.2] and [14, Theorem 4]. See [8] for a higher rank version, which also relies on commensurability.

In the rest of this note, we come back to the case of  $X = \text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$ . By spectral expansion and Weyl's equidistribution criterion, equidistribution in Theorem 1 holds if and only if

$$\int_0^1 \varphi \left( x + \frac{i}{T}, xy + \frac{i}{T} \right) dx = o_\varphi(1)$$

holds for all  $\varphi = f_1 \otimes f_2$ , where  $f_1$  and  $f_2$  are non-constant Hecke eigenforms. We will show that when  $y \in \mathbb{Q}$  (and some additional constraints on  $y$ ) this fails, with a limit that can be expressed as a product of Hecke eigenvalues and matrix coefficients. To proceed we first recall some notation connected to the construction of Hecke operators (see, e.g., [10]). For each element  $\alpha \in \text{Comm}(\Gamma)$  we have

$$\Gamma \alpha \Gamma = \bigsqcup_{m=1}^M \Gamma \alpha h_m$$

where the  $h_m$ 's are coset representatives  $\Gamma = \bigsqcup_{m=1}^M \Gamma \alpha h_m$ , with  $\Gamma_\alpha = \Gamma \cap \alpha^{-1} \Gamma \alpha$ . For every function  $f$  on  $\Gamma \backslash G$  we set  $(f \mid \Gamma \alpha \Gamma)(g) = \sum_{m=1}^M f(\alpha h_m g)$ . Further, if  $\alpha \in \text{GL}_2^+(\mathbb{Q})$  has integer entries, there is a uniquely determined pair of positive integers  $l, m$  such that  $l \mid m$  and  $\Gamma \alpha \Gamma = \Gamma \begin{pmatrix} l & 0 \\ 0 & m \end{pmatrix} \Gamma$ . The Hecke operator arising from the double coset  $\Gamma \alpha \Gamma$  is then given by

$$T_n f = n^{-1/2} \sum_{\substack{lm=n \\ l \mid m}} \left( f \mid \Gamma \begin{pmatrix} l & 0 \\ 0 & m \end{pmatrix} \Gamma \right).$$

We say that  $f$  is a Hecke eigenform if  $f$  is an eigenfunction of the Hecke operators  $T_n$ ,  $n \geq 1$ . We denote the corresponding Hecke eigenvalues  $\lambda_f(n)$ .

By abuse of notation, we view functions on  $X$  as functions on  $G$  that are left  $\text{SL}_2(\mathbb{Z})$ -invariant and right  $\text{SO}(2)$ -invariant. We denote by  $\sigma_1$  the usual sum-of-divisors function  $\sigma_1(n) = \sum_{d \mid n} d$ .

**Proposition 4.** *Let  $\Gamma = \text{SL}_2(\mathbb{Z})$ . Let  $y = \frac{p}{q}$  with  $p, q$  distinct, coprime and squarefree, and let  $\varphi = f_1 \otimes f_2 \in C_b(X \times X)$ , where  $f_1, f_2$  are non-constant Hecke eigenforms. Then*

$$\lim_{T \rightarrow \infty} \int_0^1 \varphi \left( x + \frac{i}{T}, \frac{xp}{q} + \frac{i}{T} \right) dx = \frac{\lambda_{f_2}(pq) \sqrt{pq}}{\sigma_1(pq)} \langle f_1, a_{p/q} f_2 \rangle = \frac{\lambda_{f_1}(pq) \sqrt{pq}}{\sigma_1(pq)} \langle a_{q/p} f_1, f_2 \rangle,$$

where  $\langle f_1, a_{p/q} f_2 \rangle = \int_{\Gamma \backslash G} f_1(g) f_2(g a_{p/q}^{-1}) d\mu(g)$  and  $\mu$  is normalized so that  $\mu(\Gamma \backslash G) = 1$ .

**Proof.** Set  $\Gamma = \text{SL}_2(\mathbb{Z})$ . Since  $a_{p/q} \in \text{Comm}(\Gamma)$ , the group  $\Gamma_{p/q} = a_{p/q}^{-1} \Gamma a_{p/q} \cap \Gamma$  is a finite index subgroup of  $\Gamma$ ; in particular  $\Gamma_{p/q} < G$  is a lattice. Then  $\varphi \circ \iota_{p/q} \in C_b(\Gamma_{p/q} \backslash G)$ , and the equidistribution of pieces of expanding closed horocycles on  $\Gamma_{p/q} \backslash G$  yields [14]

$$\lim_{T \rightarrow \infty} \int_I \varphi(u_x a_T^{-1}, u_{xp/q} a_T^{-1}) dx = \frac{1}{\mu(\Gamma_{p/q} \backslash G)} \int_{\Gamma_{p/q} \backslash G} \varphi(g, a_{p/q} g a_{p/q}^{-1}) d\mu(g),$$

where the Haar measure  $\mu$  is normalized so that  $\mu(\Gamma \backslash G) = 1$ .

For  $p, q$  distinct, coprime and squarefree, we find  $\Gamma a_{p/q} \Gamma = \delta_{\sqrt{pq}}^{-1} \Gamma \begin{pmatrix} 1 & 0 \\ 0 & pq \end{pmatrix} \Gamma$ , where  $\delta_x = \begin{pmatrix} x & \\ & x \end{pmatrix}$ , and  $\mu(\Gamma_{p/q} \backslash G) = M = \sigma_1(pq)$  [10, Lemma 4.5.6]. Let  $\mathcal{F}$  be a fundamental domain for  $\Gamma \backslash G$ ; then

$\sqcup_{m=1}^M h_m \mathcal{F}$ , where the  $h_m$ 's are coset representatives of  $\Gamma_{p/q}$  in  $\Gamma$ , is a fundamental domain for  $\Gamma_{p/q} \backslash G$ . We can now compute

$$\begin{aligned} \frac{1}{\mu(\Gamma_{p/q} \backslash G)} \int_{\Gamma_{p/q} \backslash G} f_1(g) f_2(a_{p/q} g a_{p/q}^{-1}) d\mu(g) &= \frac{1}{M} \sum_{m=1}^M \int_{h_m \mathcal{F}} f_1(g) f_2(a_{p/q} g a_{p/q}^{-1}) d\mu(g) \\ &= \frac{1}{M} \int_{\mathcal{F}} f_1(g) \sum_{m=1}^M f_2(a_{p/q} h_m g a_{p/q}^{-1}) d\mu(g) \\ &= \frac{\sqrt{pq}}{\sigma_1(pq)} \int_{\Gamma \backslash G} f_1(g) T_{pq} f_2(g a_{p/q}^{-1}) d\mu(g). \quad \square \end{aligned}$$

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## Declaration of interests

The author does not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and has declared no affiliations other than their research organizations.

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