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Comptes Rendus
Mathématique

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Volume 363 (2025), p. 7-11

Online since: 6 March 2025

https://doi.org/10.5802/crmath.694

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www.centre-mersenne.org — e-ISSN : 1778-3569

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Comptes Rendus. Mathématique
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https://doi.org/10.5802/crmath.694

Research article / *Article de recherche*
Geometry and Topology / *Géométrie et Topologie*

The second coefficient of the Alexander polynomial as a satellite obstruction

Le deuxième coefficient du polynôme d'Alexander comme obstruction satellitaire

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Abstract. A set $\mathscr{P}$ of links is introduced, containing positive braid links as well as arborescent positive Hopf plumbings. It is shown that for links in $\mathscr{P}$, the leading and the second coefficient of the Alexander polynomial have opposite sign. It follows that certain satellite links, such as $(n,1)$-cables, are not in $\mathscr{P}$.

Résumé. Un ensemble $\mathscr{P}$ de liens est introduit, contenant les clôtures de tresses positives ainsi que les plombages arborescents de bandes de Hopf positives. Il est démontré que pour les liens appartenant à $\mathscr{P}$, le premier et le deuxième coefficient du polynôme d'Alexander sont de signe opposé. Il s'ensuit que certains liens satellites, tels que les câbles $(n,1)$, n'appartiennent pas à $\mathscr{P}$.

Keywords. Alexander polynomial, Hopf plumbings, satellite knots, arborescent knots, positive braids.

Mots-clés. Polynôme d'Alexander, plombages de Hopf, nœuds satellites, nœuds arborescents, tresses positives.

2020 Mathematics Subject Classification. 57K10, 57K14.

Manuscript received 11 March 2024, revised 9 October 2024, accepted 13 October 2024.

In this short note, we introduce a certain set $\mathscr{P}$ of fibered links in S^3, which in particular contains non-split positive braid links and arborescent positive Hopf plumbings, and is closed under connected sum. We then prove the following.

Theorem 1. *For all links in $\mathscr{P}$, the second coefficient of the Alexander polynomial is negative.*

Let us briefly define the above-mentioned terms. A *link* L, i.e. a non-empty compact oriented one-dimensional smooth submanifold of S^3, is *fibered* if there exists a fibration $S^3 \setminus L \to S^1$ whose every fiber is the interior of a *Seifert surface* of L, i.e. a compact connected oriented two-dimensional smooth submanifold of S^3 with boundary L. The fiber surface of a fibered link L is uniquely determined up to isotopy, and it is the unique Seifert surface of minimal genus for L (see e.g. [12, Proposition 2.19] and [4, Lemma 5.1]). For example, a disk is the fiber surface of the unknot; and an unknotted band with a left-handed full-twist, called a *positive Hopf band*, is the fiber surface of the *positive Hopf link*. On the other hand, split links are not fibered. Here, a link L is called *split* if it admits a 2-sphere $S \subset S^3 \setminus L$ such that each component of $S^3 \setminus S$ contains at least one component of L. Further examples of fibered links may be constructed as follows: given a Seifert surface Σ_0 and a properly embedded arc $h \subset \Sigma_0$, let Σ_+ be the union of Σ_0 with a positive

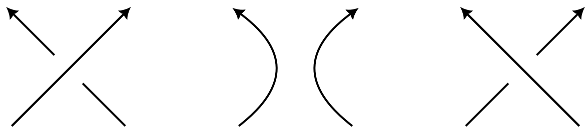

Figure 1. The links L_+, L_0, L_- appearing in the skein relation.

Hopf band H along a square S, such that $S \subset \Sigma_0$ is a product neighborhood of h, $S \subset H$ is a product neighborhood of a cocore of H, and H is contained in a small neighborhood of h in S^3, as shown in Figure 2. Then Σ_+ is a *positive Hopf plumbing* of Σ_0, and Σ_+ is a fiber surface if Σ_0 is.

Let us now define the above-mentioned two classes of fibered links. Firstly, an *arborescent positive Hopf plumbing* is associated with a plane tree T in the following way (see [3, Chapter 12], [2,5]): place a positive Hopf band at each vertex of T, and plumb any two bands whose corresponding vertices are connected by an edge; the embedding of T into $\mathbb{R}^2$ dictates the order of the (disjoint) plumbing squares around each band. Secondly, consider Artin's braid group on n strands with its standard generators $\sigma_1, \ldots, \sigma_{n-1}$ [1]. By closing off its strands, a braid β gives rise to a link called the *closure of* β. A braid is called *positive* if it can be written as a word in the generators σ_i (without their inverses). Closures of positive braids are called *positive braid links*; they are fibered if they are non-split [13].

The *Alexander polynomial* $\Delta_L(t)$ of a link L is an integer Laurent polynomial in $t^{1/2}$. We use the Conway normalization; that is to say, $\Delta_U(t) = 1$ for the unknot U, and the *skein relation*

$$\Delta_{L_+}(t) = \Delta_{L_-}(t) + \left(t^{1/2} - t^{-1/2}\right)\Delta_{L_0}$$

holds, whenever $L_\pm, L_0$ are three links as shown in Figure 1. These equations determine $\Delta_L(t)$ uniquely for all links L. Non-zero Alexander polynomials are of the form

$$\Delta_L(t) = a_d t^d + a_{d-1} t^{d-1} + \cdots + a_{-d+1} t^{-d+1} + a_{-d} t^{-d}$$

with $d(L) = d \geq 0$ an integer and $a_i = a_{-i}$ if L has an odd number of components, and $d(L) \geq 1/2$ a half-integer and $a_i = -a_{-i}$ if L has an even number of components. For $\Delta_L \neq 0$, let us call $\alpha(L) = a_d \neq 0$ the *leading coefficient* and $\beta(L) = a_{d-1}$ the *second coefficient*. It is well-known that for fibered links L (such as the links in $\mathscr{P}$), the Alexander polynomial Δ_L is non-zero, $d(L)$ equals $b_1(\Sigma)/2$ for $\Sigma \subset S^3$ a fiber surface of L, and $\alpha(L) = \pm 1$. We shall see in the proof of Theorem 1 that $\alpha(L) = 1$ for all $L \in \mathscr{P}$. Thus, if one prefers the convention that the Alexander polynomial is only well-defined up to multiplication with $\pm t^k$, one may state Theorem 1 as follows: for all links L in $\mathscr{P}$, the leading and second coefficient of Δ_L have opposite sign (i.e. $\alpha(L) \cdot \beta(L) < 0$).

Theorem 1 and its proof are inspired by Ito's theorem [6, Corollary 2] that $-\beta(L)$ equals the number of prime connected summands of L if L is a positive braid knot. Our main motivation for Theorem 1 is the satellite obstruction stated below in Corollary 2. Given a so-called *pattern* P, which is a link in the solid torus, and a *companion*, which is a knot in S^3, the *satellite link* $P(K)$ is the link obtained from P by tying the knot K into the solid torus containing P, preserving standard longitudes. The *winding number* $w(P)$ of the pattern is the algebraic intersection number of P with a meridional disk of the solid torus, or equivalently, the class of P in the first homology group of the solid torus identified with $\mathbb{Z}$.

Corollary 2. *Let P be a pattern with winding number not equal to ± 1 such that the product of the leading and the second coefficient of $\Delta_{P(U)}$ is non-negative. Then $P(K) \notin \mathscr{P}$ for all knots K.*

Proof. The Alexander polynomial of $P(K)$ equals (see e.g. [9, Theorem 6.15 and its proof])

$$\Delta_{P(K)}(t) = \Delta_K\left(t^{w(P)}\right) \cdot \Delta_{P(U)}(t).$$

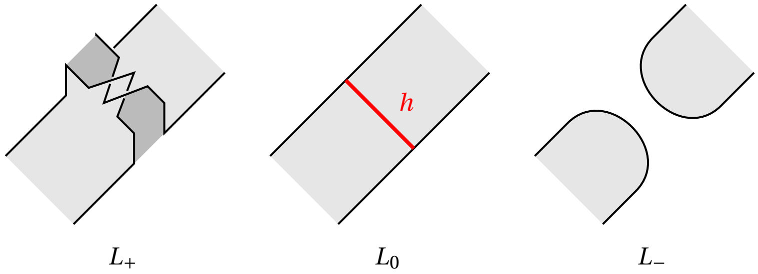

Figure 2. The links L_+, L_0, L_-, the fiber surfaces Σ_+, Σ_0, the arc h, and the Seifert surface Σ_- featuring in Definition 3.

It follows that

$$\alpha\big(P(K)\big) = \alpha(K) \cdot \alpha\big(P(U)\big),$$
$$\beta\big(P(K)\big) = \alpha(K) \cdot \beta\big(P(U)\big).$$

Multiplying the two equations, one sees that the hypothesis $\alpha(P(U)) \cdot \beta(P(U)) \geq 0$ implies that $\alpha(P(K)) \cdot \beta(P(K)) \geq 0$. By Theorem 1, it follows that $P(K) \notin \mathscr{P}$. □

The hypotheses for P in Corollary 2 are e.g. satisfied by the $(n,1)$-cable patterns for $n \geq 2$, since $P(U)$ is the unknot (which satisfies $\alpha(U) = 1, \beta(U) = 0$). So in particular, Corollary 2 recovers Krishna's recent theorem that $(n,1)$-cables of non-trivial knots are never positive braid knots [8, Theorem 1.5].

Let us now give the definition of $\mathscr{P}$ and prove our results.

Definition 3 (see Figure 2). *Let $\mathscr{P}$ be the smallest set of fibered links in S^3 that contains the positive Hopf link and satisfies the following. Let $L_0 \in \mathscr{P}$ with fiber surface Σ_0, and $h \subset \Sigma_0$ a properly embedded arc. Let Σ_- be the surface obtained from Σ_0 by cutting along h, and let Σ_+ be the surface obtained from Σ_0 by plumbing a positive Hopf band to Σ_0 along h. Let us write $L_\pm = \partial\Sigma_\pm$. If the following holds:*

$$\text{“The coefficient of } t^{b_1(\Sigma_-)/2} \text{ in } \Delta_{L_-}(t) \text{ is less than or equal to } 1.\text{”} \qquad (*)$$

then $L_+ \in \mathscr{P}$.

Let us make some remarks regarding this definition; note that $(*)$ is in particular satisfied if L_- is fibered (because then the coefficient is ± 1 or 0), or if L_- is split (because then Δ_{L_-} is zero, and so is the coefficient). Also remark that by definition, for all links $L \in \mathscr{P}$ with the exception of the positive Hopf link, there exists links L_- and L_0 such that L plays the role of L_+. Finally, note that as plumbings of positive Hopf bands, all links in $\mathscr{P}$ are strongly quasipositive [11].

Proposition 4.

(1) *$\mathscr{P}$ is closed under connected sum.*
(2) *$\mathscr{P}$ contains all non-trivial closures of non-split positive braids.*
(3) *$\mathscr{P}$ contains all non-trivial arborescent positive Hopf plumbings.*

Proof.

(1). Let $L, L' \in \mathscr{P}$. Let us show that $L\#L' \in \mathscr{P}$ (where the connected sum # is taken along any choice of components), by induction over the first Betti number b of the fiber surface of $L\#L'$. By definition of $\mathscr{P}$, the Hopf link is the only link with Betti number 1 in $\mathscr{P}$. So if $b = 2$, then L and L' are both the positive Hopf link, and so $L\#L' \in \mathscr{P}$. Otherwise, if $b > 2$, then L or L' is not the positive Hopf link; without loss of generality, suppose L is not. Then, by definition of $\mathscr{P}$, one may pick links L_- and L_0 for $L_+ = L$ as in Definition 3. Hence $L_+\#L'$ is obtained from $L_0\#L'$ by plumbing a positive Hopf band along an arc h, such that cutting along h yields $L_-\#L'$. The link $L_0\#L'$ is in $\mathscr{P}$ by induction. Since $\alpha(L') = 1$ (see proof of Theorem 1) and L_- satisfies $(*)$, so does $L_-\#L'$. It follows that $L_+\#L' = L\#L' \in \mathscr{P}$.

(2). Let L be a non-trivial non-split positive braid link. We show by induction over the first Betti number b of the fiber surface of L that $L \in \mathscr{P}$. If $b = 1$, then L is the positive Hopf link, and so $L \in \mathscr{P}$. Let us now assume $b \geq 2$. An elegant argument by Rudolph [10, proof of last proposition] shows that a positive braid link that is not an unlink can be written as the closure of a positive braid that starts with the square of a generator, i.e. is of the form $\sigma_k^2\beta$ with positive β. So let us pick such a braid $\sigma_k^2\beta$ with closure L. Let L_0 and L_- be the links obtained as the closures of $\sigma_k\beta$ and β, respectively. One may choose an arc h in the fiber surface of L_0 such that plumbing a positive Hopf band along h yields $L_+ = L$, and cutting along h yields L_-. Since $b \geq 2$, it follows that L_0 is non-trivial. Moreover, L_0 is the closure of the non-split positive braid $\sigma_k\beta$. Hence $L_0 \in \mathscr{P}$ follows by the induction hypothesis. Finally, L_- is the closure of a positive braid. Thus L_- is either split or fibered. In either case, L_- satisfies condition $(*)$. Thus $L_+ = L \in \mathscr{P}$ by definition of $\mathscr{P}$.

(3). Let T be a plane tree, and L the associated link. We proceed by induction over the number of vertices of T. If T consists of a single vertex, then L is a positive Hopf link, so $L \in \mathscr{P}$. Otherwise, pick a leaf v of T with parent w. Let L_0 be the link associated with the plane tree $T \setminus \{v\}$. There is an arc h in the fiber surface of L_0 such that plumbing a positive Hopf band along h yields the fiber surface of $L_+ = L$, and cutting along h yields a link L_- that is a connected sum of the links associated with the plane tree components of the forest $T \setminus \{v, w\}$. By induction, $L_0 \in \mathscr{P}$. Moreover, L_- satisfies $(*)$ since it is fibered (though it may not be in $\mathscr{P}$, namely if $T \setminus \{v, w\}$ is empty and L_- is thus the unknot). By definition of $\mathscr{P}$, it follows that $L_+ = L \in \mathscr{P}$. □

Proof of Theorem 1. Let $L \in \mathscr{P}$. Let us write b for the first Betti number of the fiber surface of L, so that $d(L) = b/2$. We show $\alpha(L) = a_{b/2}(L) = 1$ and $\beta(L) = a_{b/2-1}(L) \leq -1$ by induction over b. If $b = 1$, then L is the positive Hopf link, with Alexander polynomial $t^{1/2} - t^{-1/2}$ satisfying $\alpha = 1$ and $\beta = -1$. If $b \geq 2$, by definition there exists a fibered link $L_0 \in \mathscr{P}$, whose fiber surface Σ_0 contains an arc $h \subset \Sigma_0$ such that plumbing a positive Hopf band along h yields the fiber surface Σ_+ of $L_+ = L$, and cutting along h yields a Seifert surface Σ_- with $L_- = \partial\Sigma_-$ satisfying $(*)$. Since L_0 is fibered and $b_1(\Sigma_0) = b-1$, we have $d(L_0) = b/2 - 1/2$. Since L_- is the boundary of the Seifert surface Σ_- (which may or may not be a fiber surface) with $b_1(\Sigma_-) = b-2$, we have $d(L_-) \leq b/2-1$. By $(*)$, the coefficient c of $t^{b/2-1}$ in Δ_{L_-} is less than or equal to 1. Now, the skein relation holds for $L_\pm, L_0$ (compare Figures 1 and 2). Writing o for any linear combinations of powers of t with exponent less than $b/2-1$, we get

$$\alpha(L_+)\,t^{b/2} + \beta(L_+)\,t^{b/2-1} + o = c\cdot t^{b/2-1} + \left(t^{1/2} - t^{-1/2}\right)\left(\alpha(L_0)\,t^{b/2-1/2} + \beta(L_0)\,t^{b/2-3/2}\right) + o.$$

Equating the coefficients of $t^{b/2}$ and $t^{b/2-1}$ gives

$$\alpha(L_+) = \alpha(L_0) \qquad \text{and} \qquad \beta(L_+) = c - \alpha(L_0) + \beta(L_0).$$

We have $\alpha(L_0) = 1$ and $\beta(L_0) \leq -1$ by induction hypothesis, and $c \leq 1$ by $(*)$. It follows that $\alpha(L_+) = 1$ and $\beta(L_+) \leq -1$ as desired. □

While I was writing this text, Ito independently obtained an obstruction similar to Corollary 2. Namely, Theorem A.2 in [7], whose proof likewise relies on the Alexander polynomial, states that satellites with patterns satisfying a certain condition (different from the condition in Corollary 2) are never positive braid knots.

Acknowledgments

My gratitude goes to Siddhi Krishna for discussions about which cables are positive braid knots, to Filip Misev and Peter Feller for inspiring conversations, and to Peter Feller and Louis-Hadrien Robert for comments on a first version of the text.

Declaration of interests

The author does not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and has declared no affiliations other than their research organizations.

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