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The Mandelbrot–Kahane problem of Benoît Mandelbrot model of turbulence

Le problème de Mandelbrot–Kahane du modèle de turbulence de Benoît Mandelbrot

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Abstract. The purpose of this paper is to announce our solution to the Mandelbrot–Kahane problem (Mandelbrot, 1976 and Kahane, 1993) of determining the Fourier dimension of the Mandelbrot canonical cascade measure (MCCM). Specifically, we obtain the exact formula for the Fourier dimension of the MCCM with random weights W satisfying the condition $\mathbb{E}[W^t] < \infty$ for all $t \geq 1$. In addition, we show that the MCCM is Rajchman with polynomial Fourier decay whenever the random weight satisfies $\mathbb{E}[W^{1+\delta}] < \infty$ for some $\delta > 0$. In this announcement, we briefly highlight the following two applications: (1) in the Biggins–Kyprianou's boundary case, the Fourier dimension of the MCCM exhibits a second order phase transition at the inverse temperature $\beta = \frac{1}{2}$, and (2) the upper Frostman regularity and Fourier restriction estimate of the MCCM.

Résumé. Le but de cet article est d'annoncer notre résolution du problème de Mandelbrot–Kahane (Mandelbrot, 1976 et Kahane, 1993) sur la détermination de la dimension de Fourier de la mesure en cascade canonique de Mandelbrot (MCCM). Plus précisément, nous obtenons la formule exacte pour la dimension de Fourier de la MCCM avec des poids aléatoires W satisfaisant la condition $\mathbb{E}[W^t] < \infty$ pour tout $t \geq 1$. De plus, nous démontrons que la MCCM est une mesure de Rajchman avec une décroissance polynomiale de Fourier lorsque le poids aléatoire satisfait $\mathbb{E}[W^{1+\delta}] < \infty$ pour un certain $\delta > 0$. Dans cette annonce, nous soulignons brièvement les deux applications suivantes : (1) dans le cas frontière de Biggins–Kyprianou, la dimension de Fourier de la MCCM présente une transition de phase du second ordre à la température inverse $\beta = \frac{1}{2}$, et (2) la régularité de Frostman supérieure et l'estimation de restriction de Fourier pour la MCCM.

Keywords. Fourier dimension, Mandelbrot cascades, Rajchman measure, Salem measure.

Mots-clés. Dimension de Fourier, cascades de Mandelbrot, mesure de Rajchman, mesure de Salem.

2020 Mathematics Subject Classification. 60G57, 42A61, 46B09, 60J80, 60G46.

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1. The Mandelbrot–Kahane problem

Following the works of Kolmogorov [10], Landau–Lifshitz [12], Obukhov [19] and in particular the work of Yaglom [23], Mandelbrot [13] introduced the log-normal multiplicative martingales to build random measures, aiming to describe energy dissipation and explain intermittency effects in Kolmogorov's theory of turbulence. The Mandelbrot canonical cascade measure (MCCM) proposed in [14] is a simpler model, focusing on the construction of related random measures on the unit interval.

In the early stages of the theory, Mandelbrot in 1974 [14,15] proved and conjectured various fundamental fractal properties of the MCCM, many of his conjectures were rigorously established by Kahane–Peyrière in [6,20]. Kahane–Peyrière's results, with some improvements, are contained in [9] (see [17, pp. 373–388] for an English translation of [9]). At almost the same time (in 1976), Mandelbrot posed a problem to study the harmonic analysis of MCCM in [16]. Mandelbrot's question was reiterated by Kahane [8] in 1993. The Mandelbrot–Kahane problem remains open. Using ideas from modern harmonic analysis, in particular, the theory of vector-valued martingales, we are now able to solve this long-standing problem.

Fix an integer $b \geq 2$ and a random weight W, that is, W is a non-constant non-negative random variable with $\mathbb{E}[W] = 1$. Let μ_∞ denote the corresponding MCCM, in other words, μ_∞ is the corresponding random measure obtained by the action of the multiplicative cascades on the Lebesgue measure on the unit interval equipped with the standard b-adic structure. By the celebrated Kahane–Peyrière theorem, the MCCM μ_∞ is non-degenerate (that is, $\mathbb{P}(\mu_\infty \neq 0) > 0$) if and only if

$$D_H := D_H(W, b) = 1 - \mathbb{E}\big[W \log_b W\big] > 0.$$

Moreover, almost surely on $\{\mu_\infty \neq 0\}$, the measure μ_∞ is unidimensional and the exact formula of its Hausdorff dimension is given by

$$\dim_H(\mu_\infty) = D_H.$$

Mandelbrot–Kahane problem

Denote the Fourier transform of μ_∞ by

$$\widehat{\mu}_\infty(\zeta) = \int_{[0,1]} e^{-2\pi i \zeta x} \mathrm{d}\mu_\infty(x), \quad \zeta \in \mathbb{R}.$$

In 1976, Mandelbrot [16] (and in his selected works [17, p. 402]) asked *whether the Fourier coefficients of the cascade measure μ_∞ satisfy $\big|\widehat{\mu}_\infty(k)\big|^2 \sim k^{-D}$ as $k \to \infty$ for a suitable exponent D and what is the relationship between the optimal D and its Hausdorff dimension D_H.* This question was reiterated by Kahane [8] in 1993, where is included in his general open program to study the *decay behavior of Fourier transforms of natural random measures.* In particular, Kahane [8] provided detailed analysis of MCCM and noted that, *except for a few cases, the behavior of $\widehat{\mu}_\infty(\zeta)$ was not known.* To be more precise, Kahane's program on the MCCM involves the study of the following three problems:

Rajchman property: Whether $\widehat{\mu}_\infty(\zeta) \to 0$ as $\zeta \to \infty$?
Polynomial Rajchman property: Do we have $\big|\widehat{\mu}_\infty(\zeta)\big|^2 = O\big(|\zeta|^{-D}\big)$ for a certain $D > 0$?
Salem property: When does the equality $\dim_F(\mu_\infty) = \dim_H(\mu_\infty)$ hold? Here the Fourier dimension $\dim_F(\mu_\infty)$ is defined by

$$\dim_F(\mu_\infty) := \sup\Big\{D \in [0,1] : \big|\widehat{\mu}_\infty(\zeta)\big|^2 = O\big(|\zeta|^{-D}\big)\Big\}.$$

Recall that when the random weights have the specific two-point distributions (namely, there exists $a > 0$ and $\mathbb{P}(W = a) = 1 - \mathbb{P}(W = 0) \in (0,1)$), the corresponding multiplicative cascades are called the *β-models* or the *birth-and-death models* [15] (see [17, p. 359] for an English translation of [15]). The Salem property of μ_∞ in the β-model case was obtained by Shmerkin and Suomala [22] following a construction of Łaba and Pramanik [11]. By the following Corollary 2, we show that the β-model is the sole Salem measure within MCCMs.

2. The definition of the MCCM

Fix an integer $b \geq 2$ and the set of alphabets $\mathscr{A} = \{0, 1, \dots, b-1\}$. The rooted b-ary tree with the root denoted by $\emptyset$ and the convention $\mathscr{A}^0 = \{\emptyset\}$ can be canonically identified with

$$\mathscr{A}^* = \bigsqcup_{n\geq 0} \mathscr{A}^n.$$

Elements of $\mathscr{A}^*$ are written as words: if $u = x_1 x_2 \cdots x_n$ with $x_j \in \mathscr{A}$, then we set $|u| = n$, $u_j = x_j$ and $u\big|_k = x_1 \cdots x_k$ (with $u\big|_0 = \emptyset$). Recall that the b-adic intervals on the unit interval $[0,1]$ are defined for any $n = 1, 2, \dots$ and $u \in \mathscr{A}^*$ with $|u| = n$ by

$$I_u = \left[\sum_1^n u_k b^{-k}, \sum_1^n u_k b^{-k} + b^{-n}\right).$$

By a random weight, we mean a non-constant non-negative random variable W with $\mathbb{E}[W] = 1$. Let $\big(W(u)\big)_{u\in\mathscr{A}^*\setminus\{\emptyset\}}$ be independent copies of W, and as usual, on the root vertex, we set $W(\emptyset) \equiv 1$. Then, for any integer $n \geq 0$, Mandelbrot [15] defined the random measure μ_n on $[0,1]$ by

$$\mu_n(\mathrm{d}t) := \sum_{|u|=n} \left(\prod_{j=0}^n W\big(u\big|_j\big)\right) \mathbb{1}_{I_u}(t)\mathrm{d}t \quad \text{for all } n \geq 0.$$

Let $\mathscr{F}_n$ denote the sigma-algebra defined by

$$\mathscr{F}_n := \sigma\left(\left\{W(u) \,\middle|\, u \in \mathscr{A}^* \text{ with } |u| \leq n\right\}\right) \quad \text{for all } n \geq 0. \tag{1}$$

Clearly, $(\mu_n)_{n\geq 0}$ is a measure-valued martingale with respect to the filtration $(\mathscr{F}_n)_{n\geq 0}$ and almost surely, the sequence converges weakly to a limiting random measure μ_∞:

$$\mu_n \xrightarrow[a.s.]{n\to\infty} \mu_\infty.$$

The random measure μ_∞ is referred to as the Mandelbrot canonical cascade measure (MCCM).

3. Main results

Now we proceed to state our main results. Set $W^{(2)} = W^2/\mathbb{E}[W^2]$ and define

$$D_F := D_F(W,b) = \begin{cases} 1 - \dfrac{\log \mathbb{E}[W^2]}{\log b} & \text{if } \mathbb{E}\big[W^{(2)} \log W^{(2)}\big] \leq \log b, \\ 1 - \inf\limits_{\frac{1}{2}\leq t\leq 1} \dfrac{\log \mathbb{E}\big[b^{1-t} W^{2t}\big]}{t \log b} & \text{if } \mathbb{E}\big[W^{(2)} \log W^{(2)}\big] > \log b. \end{cases} \tag{2}$$

Theorem 1 (Exact Fourier dimension). *Assume that* $\mathbb{E}[W \log W] < \log b$ *and* $\mathbb{E}[W^t] < \infty$ *for all* $t > 0$. *Then* $0 < D_F < 1$ *and almost surely on* $\{\mu_\infty \neq 0\}$ *we have*

$$\dim_F(\mu_\infty) = D_F.$$

Corollary 2 (Salem property). *Assume that* $\mathbb{E}[W \log W] < \log b$ *and* $\mathbb{E}[W^t] < \infty$ *for all* $t > 0$. *Then almost surely on* $\{\mu_\infty \neq 0\}$, *the measure* μ_∞ *has Salem property if and only if* W *has a two-point distribution :*

$$\mathbb{P}(W = x^{-1}) = 1 - \mathbb{P}(W = 0) = x \quad \textit{with} \quad b^{-1} < x \leq 1.$$

Corollary 3 (Log-normal weights). *For the log-normal random weights* $W = e^{\sigma N - \frac{\sigma^2}{2}}$ *with* $\sigma^2 < 2\log b$*, almost surely, we have*

$$\dim_F(\mu_\infty) = \begin{cases} 1 - \dfrac{\sigma^2}{\log b} & \text{if } \dfrac{\sigma^2}{\log b} \le \dfrac{1}{2}, \\ 1 - \dfrac{\sigma^2}{\log b} + \left(1 - \sqrt{\dfrac{2\sigma^2}{\log b}}\right)^2 & \text{if } \dfrac{1}{2} < \dfrac{\sigma^2}{\log b} < 2. \end{cases}$$

Among all the applications of Theorem 1, we briefly introduce the following two, while others are in [4, Subsection 1.2].

Common second order phase transition of the Fourier dimension. When the random weights are in the Biggins–Kyprianou's boundary case:

$$W = \frac{e^{-\beta\xi}}{\mathbb{E}\big[e^{-\beta\xi}\big]} \quad \text{with} \quad \mathbb{E}\big[\xi e^{-\xi}\big] = 0, \quad \mathbb{E}\big[e^{-\xi}\big] = \frac{1}{b} \quad \text{and} \quad \mathbb{P}\big(\xi \in (-\infty, 0)\big) > 0.$$

Assume that $\mathbb{E}\big[e^{-t\xi}\big] < \infty$ for all $t > 0$. Then

$$D_F = D_F(\beta, \xi) = \begin{cases} \dfrac{2\psi(\beta) - \psi(2\beta)}{\log b} & \text{if } 0 < \beta \le \dfrac{1}{2}, \\ \dfrac{2\psi(\beta)}{\log b} & \text{if } \dfrac{1}{2} < \beta < 1, \end{cases}$$

where ψ is the strictly convex function defined by $\psi(t) = \psi_\xi(t) = \log \mathbb{E}\big[be^{-t\xi}\big]$. By the elementary properties of the function ψ (see [4, Lemma 7.3]), we show that, for fixed ξ, the map $\beta \mapsto D_F(\beta, \xi)$ exhibits a second order phase transition at $\beta = \frac{1}{2}$.

Upper Frostman regularity and Fourier restriction estimate. By Theorem 1, we show that the measure μ_∞ is γ-upper Frostman regular for any $0 \le \gamma < \frac{D_F}{2}$. That is, almost surely,

$$\sup_{I \subset [0,1]} \frac{\mu_\infty(I)}{|I|^\gamma} < \infty,$$

where the supremum runs over all sub-intervals $I \subset [0,1]$. Moreover, this upper Frostman regularity and Theorem 1 combined with the celebrated Fourier restriction estimate obtained in [18, Theorem 4.1] imply that for any $1 \le r < \frac{4}{4 - D_F}$, there exists $C(r, \mu_\infty) > 0$ such that for all $f \in L^r(\mathbb{R})$, we have (see [4, Corollary 1.6])

$$\|\widehat{f}\|_{L^2(\mu_\infty)} \le C(r, \mu_\infty) \|f\|_{L^r(\mathbb{R})}.$$

Theorem 4 (Polynomial Rajchman property or positive Fourier dimension). *Assume that the random weight* W *satisfies* $\mathbb{E}[W \log W] < \log b$ *and* $\mathbb{E}[W^{1+\delta}] < \infty$ *for some* $\delta > 0$*. Then almost surely on* $\{\mu_\infty \neq 0\}$*, we have* $\dim_F(\mu_\infty) > 0$*. In other words,* μ_∞ *has polynomial Rajchman property.*

4. Discussions and main strategy of the proofs

4.1. *Kahane's reduction to the Fourier coefficients*

Following Kahane, for any Borel measure μ on $[0,1]$, one may reduce the study of the decay behavior of the Fourier transform $\widehat{\mu}(\zeta)$ as $\zeta \to \infty$ to that of its Fourier coefficients $\widehat{\mu}(k)$ on the integers as $k \to \infty$. See Kahane [7, Chapter 17, Lemma 1] and a convenient version for us in [4, Lemma 1.8 and Corollary 1.9].

4.2. *Failure of the Kahane's moment method in MCCM*

Given a random measure μ on $[0,1]$, the standard strategy for obtaining a lower estimate of $\dim_F(\mu)$ is to apply Kahane's moment method. More precisely, one needs to compute the asymptotic order of decay of $\mathbb{E}\big[|\widehat{\mu}(k)|^{2m}\big]$ as $k\to\infty$ for *infinitely many* positive integers $m\geq 1$. However, in general, Kahane's moment method fails in predicting the Fourier dimension of MCCMs for the following two reasons:

- It is rare for the MCCM μ_∞ to have moments of all higher orders; indeed, one can show that, for any $p>1$ and any $k\geq 1$, the condition $\mathbb{E}\big[|\widehat{\mu}_\infty(k)|^p\big]<\infty$ implies $\mathbb{E}\big[\mu_\infty([0,1])^p\big]<\infty$. Then by Kahane's criterion for L^p-boundedness, it will imply that $\mathbb{E}[W^p]<b^{p-1}$. Hence, if $\widehat{\mu}_\infty(k)$ has finite moments of all higher orders, then W must satisfy $\|W\|_\infty\leq b$.
- Even under the additional assumption $\mathbb{E}[W^2]<b$, the asymptotic order of $\mathbb{E}\big[|\widehat{\mu}_\infty(k)|^2\big]$ may differ from the almost-sure asymptotic order of $|\widehat{\mu}_\infty(k)|^2$ (for instance, from our work, we know that such difference does appear in the case of the log-normal weights $W=e^{\sigma N-\frac{\sigma^2}{2}}$ with $\frac{\log b}{2}<\sigma^2<\log b$).

4.3. *Main strategy of the proofs*

The main strategy for obtaining our main results is to put the study of the Fourier coefficients of random cascade measures in the framework of multiplicative cascade actions on *finitely additive vector measures.*

One key ingredient in the proofs of Theorem 1 and Theorem 4 is a new connection between Fourier coefficients of the random cascade measures and the vector-valued martingale theory. Then the main part in the proof is to establish the sharp lower bound of $\dim_F(\mu_\infty)$, where Pisier's martingale type theory and the vector-valued martingale inequalities (including martingale type inequalities, vector-valued Burkholder inequalities, the recent ℓ^q-vector-valued Burkholder–Rosenthal inequalities of Dirksen and Yaroslavtsev, Bourgain–Stein inequalities, Kahane–Khintchine inequalities, etc., see [21] and [5]) will play natural roles.

More precisely, the key step in our proof is to find optimal τ such that for large enough $q>2$ and small enough $\varepsilon>0$,

$$\mathbb{E}\Bigg[\bigg\{\sum_{k\geq 1}\big|k^\tau\cdot\widehat{\mu}_\infty(k)\big|^q\bigg\}^{\frac{1+\varepsilon}{q}}\Bigg]<\infty.$$

Therefore, instead of considering coordinate-wisely all the random Fourier coefficients $\widehat{\mu}_\infty(k)$, we shall consider $\widehat{\mu}_\infty$ as a whole random object. Namely, $\widehat{\mu}_\infty$ is identified with the random vector:

$$\widehat{\mu}_\infty=\big(\widehat{\mu}_\infty(k)\big)_{k\geq 1}.$$

And for this random vector $\widehat{\mu}_\infty$, we can define a series of norms as follows. For any $\alpha\geq 0$ and $p,q\geq 1$, define the (α,p,q)-norm of μ_∞ by

$$\mathcal{N}^{(\alpha,p,q)}(\mu_\infty):=\Bigg(\mathbb{E}\Bigg[\bigg\{\sum_{k\geq 1}\big|k^\alpha\cdot\widehat{\mu}_\infty(k)\big|^q\bigg\}^{\frac{p}{q}}\Bigg]\Bigg)^{\frac{1}{p}}=\Big\|\big(k^\alpha\cdot\widehat{\mu}_\infty(k)\big)_{k\geq 1}\Big\|_{L^p(\mathbb{P};\,\ell^q)}\in[0,+\infty].$$

In the next step, we define a critical exponent α_c by

$$\alpha_c:=\sup\Big\{\alpha\in\mathbb{R}:\mathcal{N}^{(\alpha,p,q)}(\mu_\infty)<\infty\ \text{ for some }\ 1<p<2<q<\infty\Big\}.$$

To obtain the value of α_c, we shall apply various vector-valued martingale inequalities for the following vector-valued martingale in ℓ^q (with respect to the filtration $\big(\mathscr{F}_n\big)_{n\geq 0}$ defined in (1)):

$$M_n:=\big(k^\alpha\widehat{\mu}_n(k)\big)_{k\geq 1}.$$

Indeed, under the assumption of Theorem 1, we shall establish (see [4, Theorem 6.1]) sharp two-sided bounds for $\mathscr{N}^{(\alpha,p,q)}(\mu_\infty)$ when $0<\alpha<1$ and $1<p<2<q<\infty$. Then we obtain the equality $\alpha_c = \frac{D_F}{2}$. As a consequence, almost surely on $\{\mu_\infty \neq 0\}$, we have

$$\dim_F(\mu_\infty) \geq 2\alpha_c = D_F.$$

The upper bound of $\dim_F(\mu_\infty)$ relies on the fluctuations of some natural scalar martingales arising in the theory of branching random walks (see [1], [2], etc.). And we prove that almost surely on $\{\mu_\infty \neq 0\}$, for any $\varepsilon > 0$, there exists some subsequence $(k_m)_{m\geq 1} \subset \mathbb{N}$ depending on ε, such that

$$\lim_{m\to\infty} b^{k_m(D_F+\varepsilon)} \left|\widehat{\mu}_\infty\left(b^{k_m}\right)\right|^2 = \infty.$$

The details of this part are given in [4, Section 9].

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During the preparation of our manuscript, we discovered that Changhao Chen, Bing Li, and Ville Suomala [3] were also investigating from the multifractal analysis perspective the Fourier dimension of Mandelbrot cascades, a topic closely related to Theorem 1 in our paper. The two groups exchanged preliminary manuscripts, though without full details. Through these communications, we noted that their approach and ideas differ significantly from ours. We appreciate their openness to academic dialogue and value the gracious exchange of progress in their work.

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

References

[1] E. Aïdékon, "Convergence in law of the minimum of a branching random walk", *Ann. Probab.* **41** (2013), no. 3A, pp. 1362–1426.

[2] E. Aïdékon and Z. Shi, "The Seneta–Heyde scaling for the branching random walk", *Ann. Probab.* **42** (2014), no. 3, pp. 959–993.

[3] C. Chen, B. Li and V. Suomala, "Fourier dimension of Mandelbrot multiplicative cascades". Online at https://arxiv.org/abs/2409.13455.

[4] X. Chen, Y. Han, Y. Qiu and Z. Wang, "Harmonic analysis of Mandelbrot cascades – in the context of vector-valued martingales". Online at https://arxiv.org/abs/2409.13164.

[5] S. Dirksen and I. Yaroslavtsev, "L^q-valued Burkholder-Rosenthal inequalities and sharp estimates for stochastic integrals", *Proc. Lond. Math. Soc.* **119** (2019), no. 6, pp. 1633–1693.

[6] J.-P. Kahane, "Sur le modèle de turbulence de Benoît Mandelbrot", *C. R. Math. Acad. Sci. Paris* **278** (1974), pp. 621–623.

[7] J.-P. Kahane, *Some random series of functions*, Second edition, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1985, pp. xiv+305.

[8] J.-P. Kahane, "Fractals and random measures", *Bull. Sci. Math.* **117** (1993), no. 1, pp. 153–159.

[9] J.-P. Kahane and J. Peyrière, "Sur certaines martingales de Benoit Mandelbrot", *Adv. Math.* **22** (1976), no. 2, pp. 131–145.

[10] A. N. Kolmogorov, "A refinement of previous hypotheses concerning the local structure of turbulence in a viscous incompressible fluid at high Reynolds number", *J. Fluid Mech.* **13** (1962), pp. 82–85.

[11] I. Łaba and M. Pramanik, "Arithmetic progressions in sets of fractional dimension", *Geom. Funct. Anal.* **19** (2009), no. 2, pp. 429–456.

[12] L. D. Landau and E. M. Lifshitz, *Fluid mechanics*, Course of Theoretical Physics, Pergamon Press, 1959, pp. xii+536. Translated from the Russian by J. B. Sykes and W. H. Reid.

[13] B. B. Mandelbrot, "Possible refinement of the lognormal hypothesis concerning the distribution of energy dissipation in intermittent turbulence", in *Statistical Models and Turbulence. Proceedings of a Symposium held at the University of California, San Diego (La Jolla), Calif., July 15–21, 1971* (M. Rosenblatt and C. W. Van Atta, eds.), Lecture Notes in Physics, Springer, 1972, pp. 333–351.

[14] B. B. Mandelbrot, "Intermittent turbulence in self-similar cascades: divergence of high moments and dimension of the carrier", *J. Fluid Mech.* **62** (1974), pp. 331–358.

[15] B. B. Mandelbrot, "Multiplications aléatoires itérées et distributions invariantes par moyenne pondérée aléatoire: quelques extensions", *C. R. Math. Acad. Sci. Paris* **278** (1974), pp. 355–358.

[16] B. B. Mandelbrot, "Intermittent turbulence and fractal dimension: kurtosis and the spectral exponent $5/3+B$", in *Turbulence and Navier-Stokes equations (Proc. Conf., Univ. Paris-Sud, Orsay, 1975)*, Lecture Notes in Mathematics, Springer, 1976, pp. 121–145.

[17] B. B. Mandelbrot, *Multifractals and $1/f$ noise. Wild self-affinity in physics (1963–1976)*, Selected Works of Benoit B. Mandelbrot, Springer, 1999, pp. viii+442. With contributions by J. M. Berger, J.-P. Kahane and J. Peyrière, Selecta Volume N.

[18] G. Mockenhaupt, "Salem sets and restriction properties of Fourier transforms", *Geom. Funct. Anal.* **10** (2000), no. 6, pp. 1579–1587.

[19] A. M. Obukhov, "Some specific features of atmospheric turbulence", *J. Fluid Mech.* **13** (1962), pp. 77–81.

[20] J. Peyrière, "Turbulence et dimension de Hausdorff", *C. R. Math. Acad. Sci. Paris* **278** (1974), pp. 567–569.

[21] G. Pisier, *Martingales in Banach spaces*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2016, pp. xxviii+561.

[22] P. Shmerkin and V. Suomala, "Spatially independent martingales, intersections, and applications", *Mem. Am. Math. Soc.* **251** (2018), no. 1195, pp. v+102.

[23] A. M. Yaglom, "The influence of fluctuations in energy dissipation on the shape of turbulence characteristics in the inertial interval", *Dokl. Akad. Nauk SSSR* **16** (1966), pp. 49–52.