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Fractional dimension of some exceptional sets in continued fractions

Dimension fractionnaire de certains ensembles exceptionnels dans les fractions continues

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Abstract. In this paper, we calculate the Hausdorff dimension of some exceptional sets that emerge from specific constraints imposed on the partial quotients of continued fractions. In particular, we calculate the Hausdorff dimension of the sets

$$\Lambda_1 = \left\{x \in (0,1) : a_{n+1}(x) \geq \sum_{i=1}^{n} a_i(x), \text{ for all } n \in \mathbb{N}\right\},$$

and

$$\Lambda_2 = \left\{x \in (0,1) : a_{n+1}(x) \geq \sum_{i=1}^{n} a_i(x), \text{ for infinitely many } n \in \mathbb{N}\right\}.$$

We prove that the Hausdorff dimensions of Λ_1 and Λ_2 are $1/2$ and 1 respectively. The Hausdorff dimension of some other related sets, obtained by considering different faster growth rates such as replacing the growth rate of sums of partial quotients with the product of partial quotients in the above sets, is also calculated with the dimension bounds $1/3$ and at least $2/3$.

Résumé. Dans cet article, nous calculons la dimension de Hausdorff de certains ensembles exceptionnels qui émergent de contraintes spécifiques imposées aux quotients partiels des fractions continues. En particulier, nous calculons la dimension de Hausdorff des ensembles

$$\Lambda_1 = \left\{x \in (0,1) : a_{n+1}(x) \geq \sum_{i=1}^{n} a_i(x), \text{ for all } n \in \mathbb{N}\right\},$$

et

$$\Lambda_2 = \left\{x \in (0,1) : a_{n+1}(x) \geq \sum_{i=1}^{n} a_i(x), \text{ for infinitely many } n \in \mathbb{N}\right\}.$$

Nous prouvons que les dimensions de Hausdorff de Λ_1 et Λ_2 sont respectivement $1/2$ et 1. La dimension de Hausdorff de certains autres ensembles apparentés, obtenus en considérant différents taux de croissance plus rapides tels que le remplacement du taux de croissance des sommes de quotients partiels par le produit des quotients partiels dans les ensembles ci-dessus, est également calculée avec les bornes de dimension $1/3$ et au moins $2/3$.

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Keywords. Continued fractions, growth rate, Hausdorff dimension.

Mots-clés. Fractions continues, ensembles exceptionnels, dimension de Hausdorff.

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1. Introduction

It is well-known that any irrational number $x \in (0,1)$ admits a unique *continued fraction expansion* of the form

$$x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{\ddots + \cfrac{1}{a_n(x) + \ddots}}}} = \left[a_1(x), a_2(x), \ldots, a_n(x), \ldots\right],$$

where $a_1(x), a_2(x), a_3(x), \ldots$ are positive integers, called the *partial quotients* of the continued fraction expansion of x. For any $n \geq 1$, the truncation produces rational fractions

$$\frac{p_n(x)}{q_n(x)} := \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{\ddots + \cfrac{1}{a_n(x)}}}} = \left[a_1(x), a_2(x), \ldots, a_n(x)\right]$$

called the nth convergents of x. In the metrical theory of continued fractions, the fundamental set of interest is

$$\Lambda(\Phi) = \left\{x \in (0,1) : a_{n+1}(x) \geq \Phi(n), \text{ for infinitely many } n \in \mathbb{N}\right\},$$

where $\Phi\colon \mathbb{N} \to \mathbb{R}^+$ is any positive function. There have been a number of Hausdorff dimension results for some particular choices of Φ but, in full generality, the Hausdorff dimension of this set was determined by Wang–Wu [18]. The metrical properties of continued fractions play a pivotal role in understanding the approximation properties of real numbers by rational numbers, see for example [8,15]. It is worth noting that the growth properties of the product of partial quotients give information on the set of Dirichlet improvable numbers, that is, the sets of numbers admitting improvements to Dirichlet's approximation theorem. We refer the reader to [1,7–10,14] for a selection of results in this direction.

In this paper, we prove Hausdorff dimension results for certain interesting sets. The motivation for considering this problem comes from the following open problem related to Minkowski's question mark function. Minkowski's question mark function $?\colon [0,1] \to [0,1]$, introduced in 1904, can be defined as

$$?(x) = \sum_k \frac{(-1)^{k-1}}{2^{(a_1+\cdots+a_k)-1}}.$$

Some of the properties of this function are that:

(i) it is strictly increasing;

(ii) if x is rational then it is of the form $k/2^s$, where $k, s \in \mathbb{Z}$;
(iii) if x is a quadratic irrational then the continued fraction is periodic, hence the function is rational;
(iv) the function is singular.

The derivative of the function, if it exists, can take only two values: 0 and $+\infty$, and this value is concerned with the limiting behaviour of the sum $\sum_{i=1}^{t} \frac{a_i}{t}$. Note that $?(0) = 0, ?(1/2) = 1/2, ?(1) = 1$. Since the function is increasing, it is conjectured that there are exactly 5 fixed points, that is, apart from the three trivial zeros $0, 1/2, 1$, there are two other non-trivial zeros.

This conjecture is as yet unproven, however, there are some results on solutions of this equation. Namely, Gayfulin and Shulga [5] proved the following theorem:

Theorem 1 ([5]). *Let $x = [a_1, \dots, a_n, \dots]$ be the smallest or the greatest fixed point of the Minkowski question mark function on the interval $(0, \frac{1}{2})$. Then $a_1 = 2$ and*

$$a_{n+1} \le \sum_{i=1}^{n} a_i \tag{1}$$

for all $n \in \mathbb{N}$.

Hence, it is natural to estimate the size of the following exceptional sets. Let

$$\Lambda_1 = \left\{ x \in (0,1) : a_{n+1}(x) \ge \sum_{i=1}^{n} a_i(x), \text{ for all } n \in \mathbb{N} \right\},$$

and

$$\Lambda_2 = \left\{ x \in (0,1) : a_{n+1}(x) \ge \sum_{i=1}^{n} a_i(x), \text{ for infinitely many } n \in \mathbb{N} \right\}.$$

Throughout, for any set A, $\dim_H A$ denotes the Hausdorff dimension of the set A. We refer to the standard text [2] for the definition of Hausdorff dimension and measure. In our first two results, we calculate the Hausdorff dimension of Λ_1 and Λ_2 respectively.

Theorem 2. $\dim_H \Lambda_1 = \frac{1}{2}$.

Theorem 3. $\dim_H \Lambda_2 = 1$.

Similar to the above sets, it is natural to consider some other related sets. For instance, we consider sets that emerge when we substitute the growth rate of the sums of partial quotients with the product of partial quotients. This leads us to define the sets Λ_3 and Λ_4 as follows:

$$\Lambda_3 = \left\{ x \in (0,1) : a_{n+1}(x) \ge \prod_{i=1}^{n} a_i(x), \text{ for all } n \in \mathbb{N} \right\},$$

and

$$\Lambda_4 = \left\{ x \in (0,1) : a_{n+1}(x) \ge \prod_{i=1}^{n} a_i(x), \text{ for infinitely many } n \in \mathbb{N} \right\}.$$

Intuitively, one might anticipate that the Hausdorff dimension of Λ_3 and Λ_4 would not surpass that of Λ_1 and Λ_2 respectively. This expectation arises from the observation that the product of partial quotients exhibits a significantly faster growth rate compared to the sums of partial quotients. As we shall demonstrate, this intuition aligns with the following theorems:

Theorem 4. $\dim_H \Lambda_3 = \frac{1}{3}$.

Theorem 5. $\dim_H \Lambda_4 \ge \frac{2}{3}$.

In [19, Corollary 7.4], Wang, Wu, and Xu expressed the dimension of Λ_4 in terms of a pressure function. Here, we present their result in an alternative form, devoid of the notion of a pressure function.

Theorem 6 ([19]). *The Hausdorff dimension of Λ_4 is the infimum s_0 of the numbers $s \geq 0$ such that*

$$\lim_{n\to\infty} \frac{1}{n} \log \sum_{(a_1,\ldots,a_n)\in\mathbb{N}^n} \left(\frac{1}{a_1\cdots a_n q_n^2(a_1,\ldots,a_n)}\right)^s \leq 0.$$

An implication of Theorem 6 is that the Hausdorff dimension of this set lies within the range of 1/2 to 1. However, our Theorem 5 provides a more robust and optimal lower bound compared to Theorem 6. Proving this bound to be sharp is an exciting open problem.

One may ponder the implications of incorporating both the sums and the products of partial quotients into the growth rate of the aforementioned sets. Consequently, we define the sets Λ_5 and Λ_6 as follows:

$$\Lambda_5 = \left\{x \in (0,1) : a_{n+1}(x) \geq \sum_{i=1}^{n} a_i(x) + \prod_{i=1}^{n} a_i(x), \text{ for all } n \in \mathbb{N}\right\},$$

and

$$\Lambda_6 = \left\{x \in (0,1) : a_{n+1}(x) \geq \sum_{i=1}^{n} a_i(x) + \prod_{i=1}^{n} a_i(x), \text{ for infinitely many } n \in \mathbb{N}\right\}.$$

We prove the following theorems.

Theorem 7. $\dim_{\mathrm{H}} \Lambda_5 = \frac{1}{3}$.

Theorem 8. $\dim_{\mathrm{H}} \Lambda_6 \geq \frac{2}{3}$.

Remarkably, despite the significantly faster growth rates in Λ_5 compared to Λ_3, the Hausdorff dimensions are unexpectedly identical, as demonstrated in Section 6. We suspect the same behaviour for Theorems 5 and 8.

Finally, it is important to note that the growth rates given in the above sets are not exhaustive. Some further restrictions may be imposed for example restricting the partial quotients from infinite subsets of natural numbers such as primes or only considering those partial quotients that arise from arithmetic progressions. We believe similar results as above can be obtained for such settings. We leave this for an interested reader.

2. Preliminaries

In this section, we first collect some notations and basic properties and then present some useful lemmas for calculating the Hausdorff dimension of sets in continued fractions.

For any $n \geq 1$ and $(a_1,\ldots,a_n) \in \mathbb{N}^n$, we call

$$I_n(a_1,\ldots,a_n) := \{x \in [0,1) : a_1(x) = a_1,\ldots,a_n(x) = a_n\}$$

a *basic interval of order n* of continued fractions. Note that all the points in $I_n(a_1,\ldots,a_n)$ have a continued fraction expansion beginning by $a_1,\ldots,a_n$ and thus the same for $p_n(x)$ and $q_n(x)$. If there is no confusion, we write $p_n(a_1,\ldots,a_n) = p_n = p_n(x)$ and $q_n(a_1,\ldots,a_n) = q_n = q_n(x)$. It is well known (see [13, p. 4]) that p_n and q_n satisfy the following recursive formula:

$$\begin{cases} p_{-1} = 1, \;\; p_0 = 0, \;\; p_n = a_n p_{n-1} + p_{n-2} \quad (n \geq 1), \\ q_{-1} = 0, \;\; q_0 = 1, \;\; q_n = a_n q_{n-1} + q_{n-2} \quad (n \geq 1). \end{cases} \tag{2}$$

As a consequence, we have the following results.

Proposition 9 ([11, p. 18]). *For any $(a_1,\ldots,a_n) \in \mathbb{N}^n$, the interval $I_n(a_1,\ldots,a_n)$ has the endpoints p_n/q_n and $(p_n + p_{n-1})/(q_n + q_{n-1})$. More precisely,*

$$I_n(a_1,\ldots,a_n) = \begin{cases} \left[\dfrac{p_n}{q_n}, \dfrac{p_n + p_{n-1}}{q_n + q_{n-1}}\right), & \text{if } n \text{ is even,} \\ \left(\dfrac{p_n + p_{n-1}}{q_n + q_{n-1}}, \dfrac{p_n}{q_n}\right], & \text{if } n \text{ is odd.} \end{cases}$$

As a result,

$$\left|I_n(a_1,\dots,a_n)\right| = \frac{1}{q_n(q_n+q_{n-1})}.$$

Lemma 10 ([13, p. 13]). *For any* $(a_1,\dots,a_n)\in\mathbb{N}^n$*, we have*

$$q_n \geq 2^{\frac{n-1}{2}} \quad \text{and} \quad \prod_{k=1}^{n} a_k \leq q_n \leq \prod_{k=1}^{n}(a_k+1).$$

Lemma 11 ([20, Lemma 2.1]). *For any* $n\geq 1$ *and* $1\leq k\leq n$*, we have*

$$\frac{a_k+1}{2} \leq \frac{q_n(a_1,\dots,a_{k-1},a_k,a_{k+1},\dots,a_n)}{q_{n-1}(a_1,\dots,a_{k-1},a_{k+1},\dots,a_n)} \leq a_k+1.$$

Lemma 12 ([3]). *Let* $\{t_n\}_{n\geq 1}$ *be a sequence of positive integers tending to infinity with* $t_n\geq 3$ *for any* $n\geq 1$*. For any positive number* $N\geq 2$*,*

$$\dim_{\mathrm{H}}\left\{x\in(0,1): t_n\leq a_n(x)<Nt_n, \forall\, n\geq 1\right\} = \liminf_{n\to\infty}\frac{\log(t_1t_2\dots t_n)}{2\log(t_1t_2\dots t_n)+\log t_{n+1}}.$$

Lemma 13 ([17]). *For any* $\beta>0$*, let*

$$F_*(\beta) = \left\{x\in[0,1): \liminf_{n\to\infty}\frac{\log a_{n+1}(x)}{\log q_n(x)}\geq\beta\right\}.$$

Then $\dim_{\mathrm{H}} F_*(\beta) = \frac{1}{\beta+2}$.

Lemma 14 ([4,16]). *For any* $a,b>1$*, we have:*

$$\begin{aligned}&\dim_{\mathrm{H}}\left\{x\in[0,1): a_n(x)\geq a^{b^n} \text{ for infinitely many } n\geq 1\right\}\\ &\qquad = \dim_{\mathrm{H}}\left\{x\in[0,1): a_n(x)\geq a^{b^n} \text{ for all } n\geq 1\right\} = \frac{1}{1+b}.\end{aligned}$$

Lemma 15 ([6]).

$$\dim_{\mathrm{H}}\left\{x\in[0,1): a_n(x)\to\infty \text{ as } n\to\infty\right\} = \frac{1}{2}.$$

Lemma 16 ([6]). *Let*

$$J_\beta = \left\{x\in[0,1): a_{n+1}(x)\geq q_n(x)^\beta \text{ for infinitely many } n\in\mathbb{N}\right\}.$$

Then $\dim_{\mathrm{H}} J_\beta = \frac{2}{\beta+2}$.

For any $m\in\mathbb{N}$, let

$$\mathscr{E}_m = \left\{x\in[0,1): 1\leq a_n(x)\leq m \text{ for any } n\geq 1\right\}. \tag{3}$$

Jarník proved the Hausdorff dimension of E_m in his celebrated paper [12] from which one can conclude that the set of badly approximable numbers, numbers which have bounded partial quotients, has full Hausdorff dimension.

Lemma 17 ([12]). *For any* $m\geq 8$*,*

$$1-\frac{1}{m\log 2} \leq \dim_{\mathrm{H}}\mathscr{E}_m \leq 1-\frac{1}{8m\log m}.$$

In particular, the set

$$\mathscr{E} = \left\{x\in[0,1): \sup_{n\geq 1} a_n(x)<\infty\right\}$$

has Hausdorff dimension 1.

Next, we present the mass distribution principle, which is a useful classical tool to obtain the lower bound of the Hausdorff dimension of a set.

Lemma 18 ([2, Proposition 2.3]). *Let $E \subseteq (0,1)$ be a Borel set and let μ be a finite measure with $\mu(E) > 0$. If*

$$\liminf_{r\to 0} \frac{\log \mu\big(B(x,r)\big)}{\log r} \geq s \ \text{ for all } x \in E,$$

where $B(x,r)$ denotes the open ball with center at x and radius r, then we have $\dim_{\mathrm{H}} E \geq s$.

To end this section, we state an auxiliary lemma that relates the Hausdorff dimension of a set and that of its image under a Hölder map.

Lemma 19 ([2]). *Let F be a subset of $\mathbb{R}$. Let $f\colon F \to \mathbb{R}$ be a map for which there exist $c > 0$ and $0 < \alpha \leq 1$ such that*

$$\big|f(x) - f(y)\big| \leq c|x-y|^{\alpha}, \ \text{ for all } x, y \in F.$$

Then we have

$$\dim_{\mathrm{H}} f(F) \leq \frac{1}{\alpha} \dim_{\mathrm{H}} F.$$

3. Proof of Theorem 2

For the upper bound of the Hausdorff dimension, it is clear that

$$\Lambda_1 \subseteq \big\{x \in (0,1) : a_{n+1}(x) \geq n, \text{ for all } n \geq 1\big\}.$$

Then by Lemma 15, we have that

$$\dim_{\mathrm{H}} \Lambda_1 \leq \frac{1}{2}.$$

For the lower bound of the Hausdorff dimension, fix an integer $m \geq 3$. Then,

$$\Lambda_1 \supseteq \big\{x \in (0,1) : m^n \leq a_n(x) < 2m^n, \text{ for all } n \geq 1\big\}$$

since

$$2(m + m^2 + \cdots + m^n) = \frac{2m(m^n - 1)}{m-1} < m^{n+1}.$$

Hence, by Lemma 12, we have

$$\dim_{\mathrm{H}} \Lambda_1 \geq \frac{1}{2}.$$

4. Proof of Theorem 3

For a given integer $m \geq 8$, let $\mathscr{E}_m$ be the set of real numbers in $[0,1)$ whose partial quotients are at most equal to m, as defined in (3). Let $n_k = (k+1)^3$ for any $k \geq 1$.

$$E_m = \left\{x \in [0,1) : a_{n_k}(x) = \sum_{i=1}^{n_k - 1} a_i(x) \text{ for any } k \geq 1 \text{ and } 1 \leq a_i(x) \leq m \text{ for } i \neq n_k \text{ for any } k \geq 1\right\}.$$

In the following, we shall illustrate that the constructed set E_m is large enough in the sense of the Hausdorff dimension by establishing the relationship between E_m and $\mathscr{E}_m$. For brevity, we shall make use of a kind of symbolic space described below.

For any integer $n \geq 1$, set

$$D_n = \left\{(a_1, a_2, \ldots, a_n) \in \mathbb{N}^n : \begin{array}{l} a_{n_k} = \sum_{i=1}^{n_k - 1} a_i \text{ for any } n_k \leq n, \\ \quad \text{and } 1 \leq a_i \leq m \text{ for } 1 \leq i \neq n_k \leq n \text{ for any } k \geq 1 \end{array}\right\},$$

and let

$$D = \bigcup_{n=0}^{\infty} D_n, \ \ (D_0 := \emptyset).$$

For any $n \geq 1$ and $(a_1, a_2, \ldots, a_n) \in D_n$, we define

$$J_n(a_1, a_2, \ldots, a_n) = \bigcup_{a_{n+1}} I_{n+1}(a_1, a_2, \ldots, a_n, a_{n+1}),$$

where the union is taken over all a_{n+1} such that $(a_1, a_2, \ldots, a_n, a_{n+1}) \in D_{n+1}$. We call $J_n(a_1, a_2, \ldots, a_n)$ a *fundamental interval* of order n. Then,

$$\begin{aligned} E_m &= \bigcap_{n\geq 1} \bigcup_{(a_1,\ldots,a_n)\in D_n} I_n(a_1, a_2, \ldots, a_n) \\ &= \bigcap_{n\geq 1} \bigcup_{(a_1,\ldots,a_n)\in D_n} J_n(a_1, a_2, \ldots, a_n). \end{aligned}$$

For $\epsilon > 0$. Let k_0 be an integer such that for all integer $k \geq k_0$, we have

$$2^{\epsilon((k+1)^3-k)} \geq 16(2m)^{k+4} 2^{2k(k+5)} k!.$$

We define a map $f : E_m \to \mathscr{E}_m,\ y \mapsto x$ where x is obtained from y by eliminating the terms $\{a_{n_k}, k \geq 1\}$ from its continued fraction. Clearly, f is bijective. Besides, we shall show that the map f has the Hölder property.

Lemma 20. *The map f is a $\frac{1}{(1+\epsilon)^2}$ −Hölder function.*

Proof. Fix $(a_1, a_2, \ldots, a_{k_0}) \in D_{k_0}$. Let $t(n) = \max\{k \in \mathbb{N} : n_k \leq n\}$. Call $(b_1, \ldots, b_{n-t(n)})$ the sequence obtained from $(a_1, a_2, \ldots, a_{k_0})$ by deleting the terms $a_{n_1}, \ldots, a_{n_{t(n)}}$. Let

$$E(a_1, a_2, \ldots, a_{k_0}) = I_{n-k}(b_1, \ldots, b_{k_0-k}) \cap \mathscr{E}_m.$$

Consider $y_1, y_2 \in J(a_1, a_2, \ldots, a_{k_0})$ with $y_1 \neq y_2$ and $x_1 = f(y_1)$ and $x_2 = f(y_2)$. Denote by $\overline{n}$ the smallest integer n such that $a_{n+1}(y_1) \neq a_{n+1}(y_2)$. Then $\overline{n} \geq k_0$. Write $k = t(\overline{n})$, so

$$(k+1)^3 = n_k \leq \overline{n} < n_{k+1} = (k+2)^3.$$

First of all, we shall give an estimation for the largest partial quotient. For every integer $l \geq k_0$, we have

$$\begin{aligned} a_{n_l} &= \sum_{i=1}^{n_l-1} a_i \\ &\leq (l+1)^3 m + \sum_{i=1}^{l-1} a_{n_i} \\ &\leq (l+1)^3 m + l^3 m + 2\sum_{i=1}^{l-2} a_{n_i} \\ &\leq \ldots \\ &\leq m(l+1)^3 + m\sum_{i=1}^{l-2} 2^{i-1}(l+1-i)^3 + 2^{l-2} a_{n_1} \\ &\leq 2^{l-2}(l+1)^4 m + 2^{l+1} m \leq 2^{2l+5}. \end{aligned}$$

Secondly, we estimate the gap between y_1 and y_2. It is easy to find that

$$y_1 \in J_{\overline{n}+1}\big(a_1(y_1), a_2(y_1), \ldots, a_{\overline{n}+1}(y_1)\big) \subset I_{\overline{n}+1}\big(a_1(y_1), a_2(y_1), \ldots, a_{\overline{n}+1}(y_1)\big)$$

and

$$y_2 \in J_{\overline{n}+1}\big(a_1(y_2), a_2(y_2), \ldots, a_{\overline{n}+1}(y_2)\big) \subset I_{\overline{n}+1}\big(a_1(y_2), a_2(y_2), \ldots, a_{\overline{n}+1}(y_2)\big).$$

Without loss of generality, we assume that $\overline{n}$ is even and $a_{\overline{n}+1}(y_1) < a_{\overline{n}+1}(y_2)$. According to the position of the largest term, we will distinguish two cases.

If $\overline{n}+2 \neq n_{k+1}$, we have

$$y_2 \in \bigcup_{\sigma=1}^{m} I_{\overline{n}+2}\big(a_1(y_2), a_2(y_2), \ldots, a_{\overline{n}+1}(y_2), \sigma\big),$$

then the gap between y_1 and y_2 is not less than the distance between the right endpoint of

$$I_{\overline{n}+1}\big(a_1(y_2),a_2(y_2),\dots,a_{\overline{n}+1}(y_2)\big)$$

and the right endpoint of $I_{\overline{n}+2}\big(a_1(y_2),a_2(y_2),\dots,a_{\overline{n}+1}(y_2),m\big)$. By the definition of $\overline{n}$, we have that $\overline{n}+1\neq n_{k+1}$, then

$$\begin{aligned}|y_1-y_2| &\geq \left|\frac{(m+1)p_{\overline{n}+1}(y_2)+p_{\overline{n}}(y_2)}{(m+1)q_{\overline{n}+1}(y_2)+q_{\overline{n}}(y_2)}-\frac{p_{\overline{n}+1}(y_2)}{q_{\overline{n}+1}(y_2)}\right|\\ &\geq \frac{1}{\big[(m+1)q_{\overline{n}+1}(y_2)+q_{\overline{n}}(y_2)\big]q_{\overline{n}+1}(y_2)}\\ &\geq \frac{1}{\big[(m+1)(m+1)q_{\overline{n}}(y_2)+q_{\overline{n}}(y_2)\big](m+1)q_{\overline{n}(y_2)}}\\ &\geq \frac{1}{2(m+2)^3q_{\overline{n}}^2(y_2)}.\end{aligned}$$

By Lemma 10 and the conditions on k_0, we have

$$q_{\overline{n}}^{2\epsilon}(y_2)>2^{\epsilon(\overline{n}-1)}\geq 2^{\epsilon[(k+1)^3-1]}\geq 2(m+2)^3,$$

hence

$$|y_1-y_2|\geq \frac{1}{2(m+2)^3q_{\overline{n}}^2(y_2)}\geq \frac{1}{q_{\overline{n}}^{2(1+\epsilon)}(y_2)}.$$

If $\overline{n}+2=n_{k+1}$, then the gap between y_1 and y_2 is not less than the distance between the right endpoint of $I_{\overline{n}+1}\big(a_1(y_2),a_2(y_2),\dots,a_{\overline{n}+1}(y_2)\big)$ and the right endpoint of $I_{\overline{n}+2}\big(a_1(y_2),a_2(y_2),\dots,a_{\overline{n}+1}(y_2),a_{n_{k+1}}(y_2)\big)$. Hence

$$\begin{aligned}|y_1-y_2| &\geq \left|\frac{\big(a_{n_{k+1}}(y_2)+1\big)p_{\overline{n}+1}(y_2)+p_{\overline{n}}(y_2)}{\big(a_{n_{k+1}}(y_2)+1\big)q_{\overline{n}+1}(y_2)+q_{\overline{n}}(y_2)}-\frac{p_{\overline{n}+1}(y_2)}{q_{\overline{n}+1}(y_2)}\right|\\ &\geq \frac{1}{\big[\big(a_{n_{k+1}}(y_2)+1\big)q_{\overline{n}+1}(y_2)+q_{\overline{n}}(y_2)\big]q_{\overline{n}+1}(y_2)}\\ &\geq \frac{1}{\big[\big(a_{n_{k+1}}(y_2)+1\big)(m+1)q_{\overline{n}}(y_2)+q_{\overline{n}}(y_2)\big](m+1)q_{\overline{n}(y_2)}}\\ &\geq \frac{1}{16m^2a_{n_{k+1}}(y_2)q_{\overline{n}}^2(y_2)}.\end{aligned}$$

By Lemma 10 and the conditions on k_0, we have

$$q_{\overline{n}}^{2\epsilon}(y_2)>2^{\epsilon(\overline{n}-1)}\geq 2^{\epsilon[(k+1)^3-1]}\geq 16m^2 2^{2k+7}\geq 16m^2a_{n_{k+1}}(y_2),$$

hence

$$|y_1-y_2|\geq \frac{1}{16m^2a_{n_{k+1}(y_2)}q_{\overline{n}}^2(y_2)}\geq \frac{1}{q_{\overline{n}}^{2(1+\epsilon)}(y_2)}.$$

Thirdly, we show the Hölder property of the map f. Since $a_j(x_1)=a_j(x_2)$ for any $1\leq j\leq \overline{n}-k$,

$$|x_1-x_2|\leq \frac{1}{q_{\overline{n}-k}^2(x_2)}.$$

By Lemma 11, using the estimation of the largest partial quotient above, we have

$$\begin{aligned}q_{\overline{n}}(y_2) &\leq q_{\overline{n}-k}(x_2)\big(a_{n_1}(y_2)+1\big)\big(a_{n_2}(y_2)+1\big)\cdots\big(a_{n_k}(y_2)+1\big)\\ &\leq 2^k\Pi_{j=1}^k a_{n_j}(y_2)q_{\overline{n}-k}(x_2)\\ &\leq 2^k a_{n_k}^k(y_2)q_{\overline{n}-k}(x_2)\\ &\leq 2^{2k^2+6k}q_{\overline{n}-k}(x_2).\end{aligned}$$

By Lemma 10, we obtain

$$q_{n_k-k}(x_2) \geq 2^{\frac{(k+1)^3-k-1}{2}}.$$

Hence,

$$q_{\bar{n}}(y_2) \leq q_{\bar{n}-k}^{1+\epsilon}(x_2).$$

Combining these inequalities, it follows that

$$\left|f(y_1)-f(y_2)\right| = |x_1-x_2| \leq |y_1-y_2|^{\frac{1}{(1+\epsilon)^2}}. \qquad \square$$

Finally, since E_m is a countable union of sets $J(a_1,a_2,\ldots,a_{k_0})$, by Lemma 19 and the countable stability of Hausdorff dimension, we have

$$\dim_{\mathrm{H}} E_m \geq \frac{1}{(1+\epsilon)^2}\dim_{\mathrm{H}} \mathscr{E}_m.$$

On account of the arbitrariness of ϵ, we deduce Theorem 3.

5. Proofs of Theorems 4, 5 and 8

5.1. *Proof of Theorem 4*

For any $x \in \Lambda_3$, except the point $x_0 = [1,1,\ldots]$ whose partial quotients are all the integer 1, there is an integer N such that $a_N(x) \geq 2$, then we have for any $n > N$, $a_n(x) \geq 2^{2^{n-N-1}}$. Thus we have

$$\Lambda_3 \subset \bigcup_{N=1}^{\infty}\left\{x \in (0,1) : a_n(x) \geq 2^{2^{n-N-1}} \text{ for all } n \in \mathbb{N}\right\}.$$

By Lemma 14, we have the upper bound

$$\dim_{\mathrm{H}} \Lambda_3 \leq \frac{1}{3}.$$

For the lower bound, by Lemma 10, it is obvious that

$$F_*(1) \subset \Lambda_3,$$

where $F_*(\beta)$ is defined in Lemma 13 and from this lemma we readily have

$$\dim_{\mathrm{H}} \Lambda_3 \geq \frac{1}{3}.$$

5.2. *Proof of Theorems 5 and 8*

Since $q_n(x) \geq \sum_{i=1}^{n} a_i(x)$ holds for any $x \in (0,1)$, by Lemma 10, it is obvious that $J_1 \subset \Lambda_4$ and $J_1 \subset \Lambda_6$, where J_β is defined in Lemma 16, from which it follows that

$$\dim_{\mathrm{H}} \Lambda_4 \geq \frac{2}{3} \quad \text{and} \quad \dim_{\mathrm{H}} \Lambda_6 \geq \frac{2}{3}.$$

6. Proof of Theorem 7

Recall that

$$\Lambda_5 := \left\{x \in (0,1) : \forall\, n \in \mathbb{N} \quad a_{n+1}(x) \geq \prod_{j=1}^{n} a_j(x) + \sum_{j=1}^{n} a_j(x)\right\}$$

The upper bound follows from the fact that $\Lambda_5 \subseteq \Lambda_3$ and Theorem 4. To prove the lower bound, we define the subset

$$\widehat{\Lambda} := \left\{x \in (0,1) : a_1(x) \geq 2 \text{ and } \forall\, n \in \mathbb{N} \quad a_{n+1}(x) \geq 2\prod_{j=1}^{n} a_j(x)\right\}.$$

and prove the desired result via the two propositions below.

Proposition 21. *We have*

$$\widehat{\Lambda} \subseteq \Lambda_5 \cap \bigcup_{a\geq 2} I_1(a).$$

Proof. Take $x \in \widehat{\Lambda}$ and write $(a_n)_{n\geq 1} = (a_n(x))_{n\geq 1}$. It suffices to show that for all $n \in \mathbb{N}$ we have

$$\prod_{j=1}^{n} a_j \geq \sum_{j=1}^{n} a_j. \tag{4}$$

When $n = 1$, we trivially have (4). Take $n = 2$. The inequality $a_1 a_2 \leq a_1 + a_2$ occurs if and only if either $a_1 = 1$ or $a_2 = 1$. However, we have $a_1 \geq 2$ and $a_2 \geq 2a_1$, so (4) holds. If (4) did not hold for $n = 3$, we would have

$$(a_1 + a_2)a_3 < a_1 a_2 a_3 < a_1 + a_2 + a_3,$$

and we conclude the contradiction

$$2 \leq 2a_1^2 \leq a_3 < \frac{a_1 + a_2}{a_1 + a_2 - 1} \leq 2.$$

The last inequality holds because $a_1 + a_2 > 2$. Assume that we have shown (4) for some $n = N \geq 3$. If (4) was false for $n = N + 1$, we would have

$$a_{n+1} \sum_{j=1}^{n} a_j \leq a_{n+1} \prod_{j=1}^{n} a_j < \sum_{j=1}^{n+1} a_j.$$

Arguing as above, we conclude the contradiction

$$2 \leq a_{n+1} < \frac{\sum_{j=1}^{n} a_j}{\sum_{j=1}^{n} a_j - 1} < 2.$$

Therefore, (4) holds for all $n \in \mathbb{N}$ and $x \in \widehat{\Lambda}$. □

Proposition 22. *We have*

$$\dim_{\mathrm{H}} \widehat{\Lambda} = \frac{1}{3}.$$

Proof. The upper bound for $\dim_{\mathrm{H}} \widehat{\Lambda}$ follows from $\widehat{\Lambda} \subseteq \Lambda_3$. For the lower bound, consider the set

$$\widehat{\Lambda}' := \left\{ x \in [0,1) : \exists\, N \in \mathbb{N},\ \forall\, n \in \mathbb{N},\ a_{n+N+1}(x) \geq 2 \prod_{j=1}^{n} a_{N+j}(x) \right\} \cap \bigcup_{a\geq 2} I_1(a).$$

Applying Lemma 19, we may show that $\dim_{\mathrm{H}} \widehat{\Lambda}' = \dim_{\mathrm{H}} \widehat{\Lambda}$. Take any $\epsilon > 0$ and $x \in F_*(1+\epsilon)$ with $a_1(x) \geq 2$, then there is some large $M \in \mathbb{N}$ for which for all $n \in \mathbb{N}$ we have

$$\begin{aligned} a_{n+M}(x) &\geq \big(q_{n+M}(x)\big)^{1+\epsilon} \\ &\geq \big(q_M\big(a_1(x),\dots,a_M(x)\big)\big)^{1+\epsilon} \big(q_n\big(a_{M+1}(x),\dots,a_{M+n}(x)\big)\big)^{1+\epsilon} \\ &\geq 2q_n\big(a_{M+1}(x),\dots,a_{M+n}(x)\big) \\ &> 2\prod_{j=1}^{n} a_{M+j}(x). \end{aligned}$$

In other words, $F_*(1+\epsilon) \cap \bigcup_{a\geq 2} I_1(a) \subseteq \widehat{\Lambda}$. Since for all $n \in \mathbb{N}$ and all $(b_1,\dots,b_n) \in \mathbb{N}^n$ we have

$$\dim_{\mathrm{H}} F_*(1+\epsilon) \cap I_n(b_1,\dots,b_n) = \dim_{\mathrm{H}} F_*(1+\epsilon),$$

we may conclude

$$\frac{1}{3+\epsilon} \leq \dim_{\mathrm{H}} \widehat{\Lambda},$$

and the result follows. □

7. Final remarks

The bounds provided by Lemma 10 are too crude to apply the natural covering argument. Certainly, from Lemma 10, we conclude that for any positive integer n and any n-tuple of positive integers $(a_1,\ldots,a_n)$ we have

$$\prod_{j=1}^{n} a_j \le q_n(a_1,\ldots,a_n) \le 2^n \prod_{j=1}^{n} a_j.$$

For small $\varepsilon > 0$ write $s = \frac{2}{3}+\varepsilon$ and let $\mathscr{H}^s$ denote the Hausdorff s-measure of Λ_4. Then, the natural covering and these bounds yield a not so useful estimate

$$\begin{aligned}\mathscr{H}^s(\Lambda_4) &\le \liminf_{m\to\infty} \sum_{n\ge m} \sum_{(a_1,a_2,\ldots,a_n)} \left|J_n(a_1,a_2,\ldots,a_n\right|^s \\ &\le \liminf_{m\to\infty} \sum_{n\ge m} \sum_{(a_1,a_2,\ldots,a_n)} \left(\frac{2^n}{q_n^3}\right)^s \\ &\asymp \liminf_{m\to\infty} \sum_{n\ge m} \frac{2^{ns}}{2^n} \sum_{(a_1,a_2,\ldots,a_n)} 1 = \infty.\end{aligned}$$

However, if $a_1\cdots a_n$ and $q_n(a_1,\ldots,a_n)$ were equivalent (up to a multiplicative constant independent of n), a natural covering argument would lead to the desired conclusion. This suggests that we could try to argue as in Theorem 8 to solve our conjecture. That is, we might have to work with a suitable superset of Λ_4.

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