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Monotonicity for quasilinear elliptic problems with a sign-changing nonlinearity in half-planes

Monotonie des solutions de certains problèmes elliptiques quasi-linéaires dans le demi-plan avec non-linéarité changeant de signe

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Abstract. In this paper, we study the monotonicity of positive solutions u to the problem

$$-\Delta_p u + a(u)|\nabla u|^q = f(u) \text{ in } \mathbb{R}^2_+, \quad u = 0 \text{ on } \partial\mathbb{R}^2_+,$$

where $p > \frac{3}{2}$, $q \geq \max\{p-1, 1\}$ and a, f are locally Lipschitz continuous functions. We consider sign-changing nonlinearities in the case $\frac{3}{2} < p < 2$ and positive nonlinearities in the case $p > 2$. Without any assumptions on the boundedness of u or $|\nabla u|$, we show that u is monotone increasing with respect to the direction orthogonal to the boundary. This improves a recent result by Esposito et al. [10], where $|\nabla u|$ is assumed to be bounded in strips. Our proof combines the geometric techniques in the plane with the celebrated sliding and moving plane methods. Some analytic tools are also developed to deal with the lack of strong comparison and strong maximum principles when f changes sign.

Résumé. Dans cet article, nous étudions la monotonie des solutions positives u du problème

$$\Delta_p u + a(u)|\nabla u|^q = f(u) \text{ dans } \mathbb{R}^2_+, \quad u = 0 \text{ sur } \partial\mathbb{R}^2_+,$$

où $p > \frac{3}{2}$, $q \geq \max\{p-1, 1\}$ et a, f sont des fonctions localement Lipschitz. Nous considérons des non-linéarités qui changent de signe dans le cas $\frac{3}{2} < p < 2$, respectivement positives dans le cas $p > 2$. Sans aucune hypothèse sur le caractère borné de u ou de $|\nabla u|$, nous montrons que u est croissante par rapport à la direction orthogonale à la frontière. Ceci améliore un résultat récent d'Esposito et al. [10], où $|\nabla u|$ est supposé être borné dans chaque bande. Notre preuve combine les techniques géométriques dans le plan avec les célèbres méthodes du plan glissant et du plan mobile. Certains outils analytiques sont également développés pour traiter l'absence de principes de comparaison forte et de maximum fort lorsque f change de signe.

Keywords. p-Laplacian, quasilinear elliptic equation, strong comparison principle, monotonicity of solutions.

Mots-clés. p-Laplacien, équation elliptique quasi-linéaire, principe de comparaison forte, monotonie des solutions.

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1. Introduction

Let $\mathbb{R}^2_+ := \{(x,y) \in \mathbb{R}^2 \,|\, y > 0\}$ be the upper half-plane. We are interested in the monotonicity of weak solutions to the quasilinear elliptic problem

$$\begin{cases} -\Delta_p u + a(u)|\nabla u|^q = f(u) & \text{in } \mathbb{R}^2_+, \\ u > 0 & \text{in } \mathbb{R}^2_+, \\ u = 0 & \text{on } \partial\mathbb{R}^2_+, \end{cases} \tag{1}$$

where $p > \frac{3}{2}$, $q \geq \max\{p-1, 1\}$, $a, f \colon [0,+\infty) \to \mathbb{R}$ are locally Lipschitz continuous functions and $\Delta_p \cdot = \operatorname{div}(|\nabla \cdot|^{p-2}\nabla \cdot)$ denotes the well-known p-Laplace operator. Taking into account the well-known $C^{1,\alpha}$ regularity results in [9,19,26], we will study solutions $u \in C^{1,\alpha}_{\mathrm{loc}}(\overline{\mathbb{R}^2_+})$ which verify (1) in the weak distributional meaning. That is,

$$\int_{\mathbb{R}^2_+} |\nabla u|^{p-2}(\nabla u, \nabla\varphi)\mathrm{d}x + \int_{\mathbb{R}^2_+} a(u)|\nabla u|^q \varphi \mathrm{d}x = \int_{\mathbb{R}^2_+} f(u)\varphi \mathrm{d}x$$

for all $\varphi \in C^1_c(\mathbb{R}^2_+)$.

The monotonicity of positive solutions to semilinear elliptic problems in the N-dimensional half-space $\mathbb{R}^N_+ := \{(x', x_N) \in \mathbb{R}^N \,|\, x_N > 0\}$ with $N \geq 2$ has attracted the attention of several authors in recent decades. The most well-known technique to treat this kind of problem is the moving planes method. This method was first introduced by Alexandrov [1] and Serrin [25] in the context of differential geometry and partial differential equations, respectively (see also [4,18] for some improvements). By exploiting this method, Berestycki, Caffarelli and Nirenberg [2,3] showed that if f is a Lipschitz function with $f(0) \geq 0$ then any positive classical solution of

$$\begin{cases} -\Delta u = f(u) & \text{in } \mathbb{R}^N_+, \\ u = 0 & \text{on } \partial\mathbb{R}^N_+, \end{cases} \tag{2}$$

is strictly increasing in the x_N-direction. Pioneering results with more restrictions on f can also be found in Dancer's works [7,8]. When f is merely locally Lipschitz continuous, the monotonicity result can be established for positive solutions which are bounded on finite strips, see [11,24]. The case $f(0) < 0$ is more difficult. A complete proof of the monotonicity for solutions in this case is only known in dimension $N = 2$ in the works of Farina and Sciunzi [16,17].

Problem (1) is a special case of the corresponding problem in higher dimensions

$$\begin{cases} -\Delta_p u + a(u)|\nabla u|^q = f(u) & \text{in } \mathbb{R}^N_+, \\ u > 0 & \text{in } \mathbb{R}^N_+, \\ u = 0 & \text{on } \partial\mathbb{R}^N_+. \end{cases} \tag{3}$$

Compared to (2), this type of problem is much more difficult to deal with because of the nonlinear nature of the operator. The comparison principles are not equivalent to maximum principles for solutions of (3) in the case $p \neq 2$. Furthermore, the singularity or degeneracy of the operator (corresponding to $1 < p < 2$ and $p > 2$, respectively) also causes the lack of C^2 regularity of the weak solutions and other difficulties.

Nevertheless, several ideas have been introduced to partially overcome such difficulties. The monotonicity of weak solutions to (3) with $a \equiv 0$ was established in [13] for $\frac{2N+2}{N+2} < p < 2$ and in [14,15] for $p > 2$ via the moving plane method. The case $1 < p < 2$ and $1 < q \le p$ was studied in [12] by means of a careful analysis of the local symmetry regions of the solutions. All these results need the assumption that f is a positive function, i.e., $f(t) > 0$ for $t > 0$, besides its local Lipschitz continuity. The monotonicity of solutions to (3) with $\frac{2N+2}{N+2} < p < 2$ and sign-changing locally Lipschitz continuous function f was only studied recently in the work [10].

We emphasize that when dealing with a merely locally Lipschitz continuous nonlinearity in high dimensions, all monotonicity results in the literature require the boundedness of u or $|\nabla u|$ in strips even in the case $p = 2$. This restriction comes from the recovering compactness argument in high dimensions, in which we need some boundedness of solutions in strips, so that the sequence $u_n(x', x_N) := u(x' + x_n, x_N)$ is compact in $C^1_{\text{loc}}(\mathbb{R}^N_+)$ by the Arzelà–Ascoli theorem and its limit also satisfies the given problem.

The situation is different and more interesting in dimension two because we can exploit geometric techniques involving rotating and sliding lines to overcome the need for the boundedness assumption in strips. These techniques were first introduced in [3] and were improved in [16,17] to establish the monotonicity of solutions to the semilinear problem (2) without a priori boundedness assumption on the solutions. The method was also extended to obtain monotonicity results for the quasilinear problem (1) under the assumption that f is positive (see [6] for the case $p > \frac{3}{2}$, $a \equiv 0$ and [22] for the case $\frac{3}{2} < p < 2$, $a \not\equiv 0$). However, problem (1) with sign-changing nonlinearity f has not been studied in the literature even in the case $a \equiv 0$.

The aim of this paper is to study the monotonicity of solutions to problem (1) in the case $p > 2$ or f changes sign. As in the spirit of [6,22], we do not assume the boundedness of u or $|\nabla u|$ in any unbounded domains.

Let us now state our results. The first result concerns the monotonicity of solutions to problem (1) when f changes sign and $\frac{3}{2} < p < 2$. This is a natural extension of the previous results in previously prescribed works [6,10,22] and is not a trivial step due to the presence of f sign-changing nonlinearity.

Theorem 1. *Assume $\frac{3}{2} < p < 2$, $q \ge 1$ and $a, f\colon [0,+\infty) \to \mathbb{R}$ are locally Lipschitz continuous functions such that*

$$f(0) \ge 0 \text{ and } Z_f := \left\{ t \in [0,+\infty) \,\middle|\, f(t) = 0 \right\} \text{ is a discrete set.}$$

Let $u \in C^{1,\alpha}_{\text{loc}}(\overline{\mathbb{R}^2_+})$ be a solution to (1). Then u is monotone increasing with respect to the y-direction. Furthermore,

$$\frac{\partial u}{\partial y} > 0 \text{ in } \overline{\mathbb{R}^2_+} \setminus \left\{ x \in \mathbb{R}^2_+ \,\middle|\, u(x) \in Z_f \right\}.$$

Remark 2. Theorem 1 extends the main result in [22] to the sign-changing nonlinearities. It also improves the main result in [10] in dimension two by removing the boundedness assumption of $|\nabla u|$. Notice also that only the case $a \equiv 0$ was considered in [10].

Though the basic and deep ideas in Theorem 1 come from [6,22], some crucial improvements are necessary because of the sign-changing nonlinearity of f. Two improvements are listed as follows.

(I) To ensure the rotating and sliding lines technique functions properly, it is necessary to demonstrate that $u < u_{x_0,s,\theta}$ in $\mathcal{T}_{x_0,s,\theta}$. This is done by the use of a strong comparison principle. Such a principle is available in the case that f is positive as in [6,22]. Since the validity of strong comparison principles is still an open question in the case that f changes sign, we have to take another approach. Indeed, instead of mixing rotating and sliding the line $L_{x_0,s,\theta}$ as in [6,22], we only slide this line upward by letting $s \to \lambda^-$. This allows us to gain more information about the monotonicity of u in $\mathcal{T}_{x_0,s,\theta}$ in the direction V_θ. We exploit this and a sliding ball (see Figure 2) argument to show that $u < u_{x_0,s,\theta}$ inside $\mathcal{T}_{x_0,s,\theta}$ but outside the critical set $\{|\nabla u| = |\nabla u_{x_0,s,\theta}| = 0\}$. Since this kind of strong comparison principle is weaker, we need to derive a stronger weak comparison principle that holds for domains with narrow and small gradient parts, instead of small domains only as in [6,22].

(II) We use a sequence of sliding balls (see Figure 3) to show that $\frac{\partial u}{\partial y} > 0$ on the line segment $\{\overline{x}\} \times [0, \lambda)$ and that such a line segment can be chosen in any rectangle $(x_0 - r_0, x_0 + r_0) \times [0, \lambda)$. This information is weaker than that in [6,22], but we show that it is enough for us to get a comparison of u and $u_{x_0,s,\theta}$ on the boundary of $\mathcal{T}_{x_0,s,\theta}$. Moreover, this improvement helps us overcome the difficulty posed by the lack of a strong maximum principle in the case of sign-changing nonlinearity. The key step is to deduce $\frac{\partial u}{\partial y} > 0$ in Σ_λ from $\frac{\partial u}{\partial y} \geq 0$ in Σ_λ, which allows us to increase λ and ensure that $u < u_{x_0,s,\theta}$ on $\partial\mathcal{T}_{x_0,s,\theta} \setminus L_{x_0,s,\theta}$ for $s \leq \lambda$. If f is positive as discussed in [6,22], then utilizing a strong maximum principle for the linearized equation would suffice. However, such a principle is not available for the case of sign-changing nonlinearity.

Our second result extends the main result in [22] to the case $p > 2$.

Theorem 3. *Assume $p > 2$, $q \geq p - 1$ and $a, f\colon [0, +\infty) \to \mathbb{R}$ are locally Lipschitz continuous functions such that*

$$f(t) > 0 \text{ for } t > 0.$$

Let $u \in C^{1,\alpha}_{\mathrm{loc}}(\overline{\mathbb{R}^2_+})$ be a solution to (1). Then u is monotone increasing with respect to the y-direction. Furthermore,

$$\frac{\partial u}{\partial y} > 0 \ \text{ in } \overline{\mathbb{R}^2_+}.$$

Unlike Theorem 1, the proof of Theorem 3 is straightforward since we only treat positive nonlinearities there. Such a proof, therefore, is similar to that of [22, Theorem 1.1]. We only need to replace the weak comparison principle in [22] with a new one for $p > 2$ so that everything works. The case that $p > 2$ and f changes sign seems to be much more difficult and is left as an open question.

The rest of this paper is organized as follows. We recall some known results in Section 2. In Section 3 we prove a weak comparison principle for $1 < p < 2$ and two important lemmas which will be used in the proof of Theorem 1. A weak comparison principle for $p > 2$ is given in Section 4, where we also prove Theorem 3.

2. Preliminaries

In this section, we recall some known results on quasilinear elliptic equations, which will be used in the rest of this paper. We consider the equation

$$-\Delta_p w + a(w)|\nabla w|^q = f(w) \quad \text{in } \Omega, \tag{4}$$

where $p > 1$, $q \geq \max\{p-1, 1\}$, $a, f\colon [0, +\infty) \to \mathbb{R}$ are locally Lipschitz continuous functions and Ω is a domain of $\mathbb{R}^N$ with $N \geq 1$.

The following theorem extends the strong maximum principle and the Hopf lemma of Vázquez [27] to quasilinear equations with gradient terms.

Theorem 4 ([23, Theorems 2.5.1 and 5.5.1]). *Let $u \in C^1(\Omega)$ be a non-negative weak solution to*

$$-\Delta_p u + a(u)|\nabla u|^q + cu^r = g \geq 0 \quad \text{in } \Omega,$$

where $p > 1$, $q, r \geq p - 1$, $c \geq 0$, $g \in L^\infty_{\mathrm{loc}}(\Omega)$, a, f are locally Lipschitz continuous functions and Ω is a connected domain of $\mathbb{R}^N$. If $u \not\equiv 0$ in Ω, then $u > 0$ in Ω. Moreover for any point $x_0 \in \partial\Omega$ where *the interior sphere condition is satisfied, and such that* $u \in C^1\big(\Omega \cup \{x_0\}\big)$ *and* $u(x_0) = 0$*, we have that* $\frac{\partial u}{\partial \nu} > 0$ *for any inward directional derivative (this means that if* x *approaches* x_0 *in a ball* $B \subset \Omega$ *that has* x_0 *on its boundary, then* $\lim_{x \to x_0} \frac{u(x)-u(x_0)}{|x-x_0|} > 0$*).*

In the case $p > 2$ and f is positive, we have the following weighted Poincaré's inequality.

Theorem 5 ([20, Theorem 2.2]). *Let* $p > 2$*,* $q \geq p-1$ *and let* $u \in C^{1,\alpha}_{\text{loc}}(\Omega)$ *be a non-negative weak solution of* (4)*, where* a *and* f *are locally Lipschitz continuous functions and* Ω *is a domain of* $\mathbb{R}^N$ *such that* $f(t) > 0$ *for* $t > 0$*. Let* $\Omega' \subset \Omega$ *be a bounded domain and define* $\rho = |\nabla u|^{p-2}$*. Then there exists* $C_P > 0$ *such that the following weighted Poincaré's inequality holds*

$$\int_{\Omega'} v^2 \mathrm{d}x \leq C_P \int_{\Omega'} \rho |\nabla v|^2 \mathrm{d}x \quad \text{for all } v \in H^1_0(\Omega', \rho),$$

where the space $H^1_0(\Omega', \rho)$ *is endowed with the norm*

$$\|v\|_{H^1_0(\Omega',\rho)} := \left(\int_{\Omega'} v^2 \mathrm{d}x + \int_{\Omega'} \rho |\nabla v|^2 \mathrm{d}x\right)^{\frac{1}{2}}.$$

Moreover, $C_P \to 0$ *as* $|\Omega'| \to 0$*.*

In the quasilinear case, the maximum principle is not equivalent to the comparison one. Therefore, we also recall the classical version of the strong comparison principle.

Theorem 6 ([23, Theorem 2.5.2]). *Let* $p > 1$*,* $q \geq \max\{p-1, 1\}$ *and let* $u, v \in C^1(\Omega)$ *be two solutions to* (4) *such that* $u \leq v$ *in* Ω*, where* a *and* f *are locally Lipschitz continuous functions and* Ω *is a smooth domain of* $\mathbb{R}^N$*. We denote*

$$Z = \big\{x \in \Omega \,\big|\, |\nabla u(x)| + |\nabla v(x)| = 0\big\}.$$

If $x_0 \in \Omega \setminus Z$ *and* $u(x_0) = v(x_0)$*, then* $u = v$ *in the connected component of* $\Omega \setminus Z$ *containing* x_0*.*

One important ingredient of our later use is the following strong comparison principle for problem (1), which holds on the entire domain Ω.

Theorem 7 ([20, Theorem 1.2]). *Let* $p > \frac{2N+2}{N+2}$*,* $q \geq \max\{p-1, 1\}$ *and* $u, v \in C^{1,\alpha}_{\text{loc}}(\overline{\Omega})$*, where* Ω *is a smooth connected domain of* $\mathbb{R}^N$*. Suppose that either* u *or* v *is a weak solution of* (4)*, where* a *and* f *are locally Lipschitz continuous functions. Moreover, assume that*

$$-\Delta_p u + a(u)|\nabla u|^q - f(u) \leq -\Delta_p v + a(v)|\nabla v|^q - f(v) \quad \text{and} \quad u \leq v \quad \text{in } \Omega$$

in the weak distributional sense and at least one of the following two conditions holds:

(i) *either*

$$f\big(u(x)\big) > 0 \quad \text{in } \overline{\Omega} \quad \text{or} \quad f\big(u(x)\big) < 0 \quad \text{in } \overline{\Omega},$$

(ii) *either*

$$f\big(v(x)\big) > 0 \quad \text{in } \overline{\Omega} \quad \text{or} \quad f\big(v(x)\big) < 0 \quad \text{in } \overline{\Omega}.$$

Then either $u = v$ *in* Ω *or* $u < v$ *in* Ω*.*

Next, we recall that the linearized operator $L_u(v, \varphi)$ at a fixed solution $u \in C^{1,\alpha}_{\text{loc}}(\Omega)$ of

$$-\Delta_p u + a(u)|\nabla u|^q = f(u) \quad \text{in } \Omega$$

is defined for every $v, \varphi \in H^1(\Omega, \rho)$ with $\rho = |\nabla u|^{p-2}$ by

$$\begin{aligned} L_u(v,\varphi) := &\int_\Omega |\nabla u|^{p-2}(\nabla v, \nabla \varphi)\mathrm{d}x + (p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u, \nabla v)(\nabla u, \nabla \varphi)\mathrm{d}x \\ &+ \int_\Omega a'(u)|\nabla u|^q v\varphi \mathrm{d}x + q\int_\Omega a(u)|\nabla u|^{q-2}(\nabla u, \nabla v)\varphi \mathrm{d}x - \int_\Omega f'(u) v\varphi \mathrm{d}x. \end{aligned}$$

Moreover, $v \in H^1(\Omega,\rho)$ is called a weak solution of the linearized equation if

$$L_u(v,\varphi)=0 \tag{5}$$

for all $\varphi \in H_0^1(\Omega,\rho)$. Here, the weighted Sobolev space $H^1(\Omega,\rho)$ is defined as the space of functions v such that

$$\|v\|_{H^1(\Omega,\rho)} := \|v\|_{L^2(\Omega)} + \|\nabla v\|_{L^2(\Omega,\rho)} < \infty.$$

It can be also defined as the completion of smooth functions under the norm above. The space $H_0^1(\Omega,\rho)$ is obtained by taking the closure of $C_c^\infty(\Omega)$ under the same norm and $\|\nabla v\|_{L^2(\Omega,\rho)}$ is an equivalent norm in $H_0^1(\Omega,\rho)$.

By [21] we have $\frac{\partial u}{\partial x_i} \in H^1(\Omega,\rho)$ for $i=1,\dots,N$ and

$$L_u\left(\frac{\partial u}{\partial x_i},\varphi\right)=0$$

for all $\varphi \in H_0^1(\Omega,\rho)$. In other words, the derivatives of u are weak solutions to the linearized equation. Furthermore, the following strong maximum principle for the linearized equation can be deduced from [21].

Theorem 8 ([21, Theorem 1.3]). *Let $p > \frac{2N+2}{N+2}$, $q \ge \max\{p-1,1\}$ and let $u \in C^{1,\alpha}_{\mathrm{loc}}(\overline{\Omega})$ be a weak solution of* (4)*, where a and f are locally Lipschitz continuous functions and Ω is a smooth connected domain of $\mathbb{R}^N$. Assume that either*

$$f\big(u(x)\big) > 0 \quad \text{in } \overline{\Omega} \quad \text{or} \quad f\big(u(x)\big) < 0 \quad \text{in } \overline{\Omega}.$$

Let v be a solution of (5) *such that*

$$v \ge 0 \quad \text{in } \Omega.$$

Then either $v \equiv 0$ in Ω or $v > 0$ in Ω.

The following 1D weighted Poincaré's inequality plays an essential role in our proof of the weak comparison principle in the case $p<2$:

Lemma 9 ([12, Lemma 2.2]). *Let I be an open bounded subset of $\mathbb{R}$ and assume that $I = A\cup B$, where A and B are measurable subsets of I. Let $\rho\colon I \to \mathbb{R}\cup\{\infty\}$ be a measurable function such that $\inf_I \rho > 0$. Then for any $w \in H_0^1(I)$, the following inequality holds*

$$\int_I w^2(t)\,\mathrm{d}t \le 2|I|\max\left\{|A|\sup_A \frac{1}{\rho}, |B|\sup_B \frac{1}{\rho}\right\}\int_I \rho(t)\big|w'(t)\big|^2\mathrm{d}t.$$

Last but not least, we also recall the following important elementary inequalities which will be used later: there exist positive constants C_1, C_2 depending only on $N, p>1$ such that for all $\xi,\xi' \in \mathbb{R}^N$ with $|\xi|+|\xi'|>0$, it holds

$$\begin{aligned}\big(|\xi|^{p-2}\xi - |\xi'|^{p-2}\xi', \xi-\xi'\big) &\ge C_1\big(|\xi|+|\xi'|\big)^{p-2}\big|\xi-\xi'\big|^2,\\ \big||\xi|^{p-2}\xi - |\xi'|^{p-2}\xi'\big| &\le C_2\big(|\xi|+|\xi'|\big)^{p-2}\big|\xi-\xi'\big|.\end{aligned} \tag{6}$$

We refer to [5] for a proof of (6).

3. The case $\frac{3}{2} < p < 2$ and f is sign-changing

This section aims to prove Theorem 1. The weak comparison principle below is stated for dimension $N \ge 2$. In what follows, we write a point $x \in \mathbb{R}^N$ as $x=(x',x_N) \in \mathbb{R}^{N-1}\times\mathbb{R}$. For any set $\Omega \subset \mathbb{R}^N$ we denote Ω' the projection of Ω on $\mathbb{R}^{N-1}$ in the x_N-direction, i.e.,

$$\Omega' := \big\{x' \in \mathbb{R}^{N-1} \,\big|\, (x',y) \in \Omega \text{ for some } y \in \mathbb{R}\big\}.$$

The open ball of center x_0 with radius $r>0$ is always denoted as $B_r(x_0)$.

Theorem 10. *Assume $1 < p < 2$, $q \geq 1$ and $a, f : \mathbb{R} \to \mathbb{R}$ are locally Lipschitz continuous functions. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $L, M > 0$ and $u, v \in C^1(\overline{\Omega})$ be such that*

$$\Omega \subset \left\{ x \in \mathbb{R}^N \,\middle|\, |x_N| \leq L \right\},$$

$$\|u\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} + \|\nabla u\|_{L^\infty(\Omega)} + \|\nabla v\|_{L^\infty(\Omega)} \leq M$$

and

$$\begin{cases} -\Delta_p u + a(u)|\nabla u|^q \leq f(u) & \text{in } \Omega, \\ -\Delta_p v + a(v)|\nabla v|^q \geq f(v) & \text{in } \Omega, \\ u > 0, v > 0 & \text{in } \Omega, \\ u \leq v & \text{on } \partial\Omega. \end{cases} \tag{7}$$

Assume further that

$$\Omega = \bigcup_{x' \in \Omega'} \{x'\} \times \left(A_{x'} \cup B_{x'}\right),$$

where the measurable sets $A_{x'}, B_{x'} \subset (-L, L)$ satisfy

$$|A_{x'}| \leq \delta \quad \text{and} \quad B_{x'} \subset \left\{ x_N \in (-L, L) \,\middle|\, \left|\nabla u(x', x_N)\right| + \left|\nabla v(x', x_N)\right| \leq \delta \right\}.$$

Then there exists a constant

$$\delta_0 = \delta_0(N, p, q, a, f, M, L)$$

such that if we assume $\delta \leq \delta_0$, then it holds

$$u \leq v \text{ in } \Omega.$$

Proof. Since $u \leq v$ on $\partial\Omega$, the function $(u-v)^+$ belongs to $W_0^{1,p}(\Omega)$. Therefore, we may use it as a test function in the first two inequalities of (7). Then subtracting, we get

$$\begin{aligned} &\int_\Omega \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v, \nabla(u-v)^+\right) \mathrm{d}x \\ &\qquad \leq -\int_\Omega \left(a(u)|\nabla u|^q - a(v)|\nabla v|^q\right)(u-v)^+ \mathrm{d}x + \int_\Omega \left(f(u) - f(v)\right)(u-v)^+ \mathrm{d}x \\ &\qquad = -\int_\Omega a(u)\left(|\nabla u|^q - |\nabla v|^q\right)(u-v)^+ \mathrm{d}x - \int_\Omega \left(a(u) - a(v)\right)|\nabla v|^q (u-v)^+ \mathrm{d}x \\ &\qquad\quad + \int_\Omega \left(f(u) - f(v)\right)(u-v)^+ \mathrm{d}x \\ &\qquad \leq \int_\Omega |a(u)| \left||\nabla u|^q - |\nabla v|^q\right| (u-v)^+ \mathrm{d}x + \int_\Omega \left|a(u) - a(v)\right| |\nabla v|^q (u-v)^+ \mathrm{d}x \\ &\qquad\quad + \int_\Omega \left|f(u) - f(v)\right| (u-v)^+ \mathrm{d}x \\ &\qquad \leq M_a \int_\Omega \left||\nabla u|^q - |\nabla v|^q\right| (u-v)^+ \mathrm{d}x + (L_a M^q + L_f) \int_\Omega \left((u-v)^+\right)^2 \mathrm{d}x, \end{aligned}$$

where $M_a = \max_{[-M,M]}|A|$ and L_a, L_f denote the Lipschitz constants of a and f on $[-M, M]$, respectively. Combining this with (6), we deduce

$$\begin{aligned} &C_1 \int_\Omega \left(|\nabla u| + |\nabla v|\right)^{p-2} \left|\nabla(u-v)^+\right|^2 \mathrm{d}x \\ &\qquad \leq M_a \int_\Omega \left||\nabla u|^q - |\nabla v|^q\right| (u-v)^+ \mathrm{d}x + (L_a M^q + L_f) \int_\Omega \left((u-v)^+\right)^2 \mathrm{d}x. \end{aligned} \tag{8}$$

By the mean value theorem and taking into account that $q \geq 1 > \frac{p}{2}$, we have

$$
\begin{aligned}
M_a \int_\Omega \big||\nabla u|^q - |\nabla v|^q\big|(u-v)^+ \mathrm{d}x
&\leq q M_a \int_\Omega \big(|\nabla u| + |\nabla v|\big)^{q-1} \left|\nabla (u-v)^+\right| (u-v)^+ \mathrm{d}x \\
&\leq q M_a M^{q-\frac{p}{2}} \int_\Omega \big(|\nabla u| + |\nabla v|\big)^{\frac{p-2}{2}} \left|\nabla (u-v)^+\right| (u-v)^+ \mathrm{d}x \\
&\leq \frac{C_1}{2} \int_\Omega \big(|\nabla u| + |\nabla v|\big)^{p-2} \left|\nabla (u-v)^+\right|^2 \mathrm{d}x + \frac{q^2 M_a^2 M^{2q-p}}{2C_1} \int_\Omega \big((u-v)^+\big)^2 \mathrm{d}x.
\end{aligned}
$$

Substituting this into (8), we obtain

$$
\int_\Omega \big(|\nabla u| + |\nabla v|\big)^{p-2} \left|\nabla (u-v)^+\right|^2 \mathrm{d}x \leq C \int_\Omega \big((u-v)^+\big)^2 \mathrm{d}x, \tag{9}
$$

where $C = C(N, p, q, a, f, M) > 0$.

Now we estimate the integral on the right-hand side of (9) by following an idea in [12]. More precisely, setting $\rho = \big(|\nabla u| + |\nabla v|\big)^{p-2}$ and applying the 1D weighted Poincaré's inequality (Lemma 9) in the set $I_{x'} = A_{x'} \cup B_{x'} \subset (-L, L)$ for each $x' \in \Omega'$, we have

$$
\begin{aligned}
\int_\Omega \big((u-v)^+\big)^2 \mathrm{d}x &= \int_{\Omega'} \int_{A_{x'} \cup B_{x'}} \big((u-v)^+\big)^2 \mathrm{d}x_N \mathrm{d}x' \\
&\leq 2 \int_{\Omega'} \left|A_{x'} \cup B_{x'}\right| \max\left\{|A_{x'}| \sup_{A_{x'}} \frac{1}{\rho}, |B_{x'}| \sup_{B_{x'}} \frac{1}{\rho}\right\} \int_{A_{x'} \cup B_{x'}} \rho \left|\frac{\partial (u-v)^+}{\partial x_N}\right|^2 \mathrm{d}x_N \mathrm{d}x' \\
&\leq 4L \max\left\{\delta M^{2-p}, 2L\delta^{2-p}\right\} \int_\Omega \rho \left|\nabla (u-v)^+\right|^2 \mathrm{d}x.
\end{aligned}
$$

Plugging this into (9), we obtain

$$
\int_\Omega \rho\big((u-v)^+\big)^2 \mathrm{d}x \leq 4CL \max\left\{\delta M^{2-p}, 2L\delta^{2-p}\right\} \int_\Omega \rho\big((u-v)^+\big)^2 \mathrm{d}x.
$$

Hence for all $\delta < \delta_0$, where $\delta_0 > 0$ is sufficiently small, we have $\int_\Omega \rho\big((u-v)^+\big)^2 \mathrm{d}x = 0$. This implies that $(u-v)^+$ is constant in Ω. On the other hand, $(u-v)^+ = 0$ on $\partial\Omega$. Hence $(u-v)^+ = 0$ in Ω, which means $u \leq v$ in Ω. □

To continue, we recall some notations used in the method of moving planes with geometric techniques in dimension two, see [6].

Notations

For given $(x_0, s, \theta) \in \mathbb{R} \times \mathbb{R}_+ \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, let $L_\theta = (\cos\theta, \sin\theta)$ and let V_θ be the unit vector which is orthogonal to L_θ and satisfies $(V_\theta, e_2) \geq 0$. Besides, we denote by $L_{x_0,s,\theta}$ the line which is parallel to L_θ and passes through (x_0, s). We also denote by $\mathcal{T}_{x_0,s,\theta}$ the triangle delimited by the three lines $L_{x_0,s,\theta}$, $\{x_0\} \times \mathbb{R}$ and $\mathbb{R} \times \{0\}$ (see Figure 1).

Furthermore, for any $x \in \mathcal{T}_{x_0,s,\theta}$, we define

$$
u_{x_0,s,\theta}(x) = u\big(T_{x_0,s,\theta}(x)\big),
$$

where $T_{x_0,s,\theta}(x)$ is the symmetric point of x with respect to $L_{x_0,s,\theta}$.

Since the first equation of (1) is invariant under reflection, we know that $u_{x_0,s,\theta}$ fulfills

$$
-\Delta_p u_{x_0,s,\theta} + a(u_{x_0,s,\theta})\left|\nabla u_{x_0,s,\theta}\right|^q = f(u_{x_0,s,\theta}) \quad \text{in } \mathcal{T}_{x_0,s,\theta}
$$

in the weak sense. For convenience, we shall denote

$$
u_s = u_{x_0,s,0}.
$$

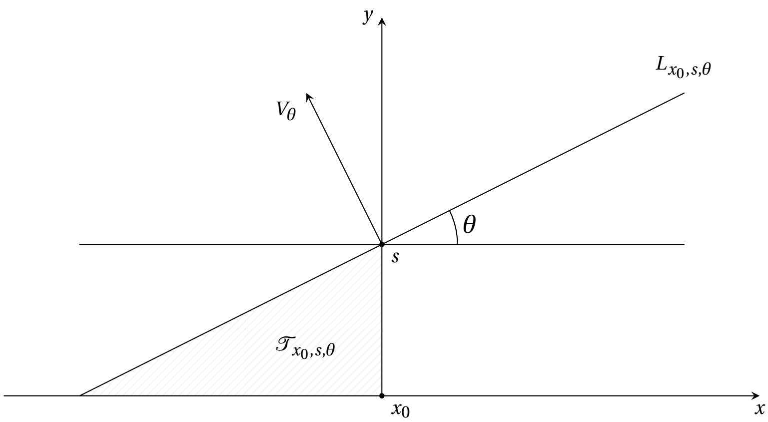

Figure 1. The triangle $\mathcal{T}_{x_0,s,\theta}$.

We also use the following notation

$$Z_{f(u)} := \{z \in \mathbb{R}^2_+ \mid f\big(u(z)\big) = 0\}.$$

The following lemma compensates for the lack of a strong comparison principle for quasilinear elliptic equations with sign-changing nonlinearity. Together with Theorem 10, it will play a vital role in our later argument.

Lemma 11. *Under the assumptions of Theorem 1, let us assume that*

$$\begin{cases} \frac{\partial u}{\partial V_\theta} \ge 0 & \text{in } \mathcal{T}_{x_0,s,\theta}, \\ u \le u_{x_0,s,\theta} & \text{in } \mathcal{T}_{x_0,s,\theta}, \\ u < u_{x_0,s,\theta} & \text{on } \partial\mathcal{T}_{x_0,s,\theta} \setminus L_{x_0,s,\theta} \end{cases} \tag{10}$$

for some $(x_0, s, \theta) \in \mathbb{R} \times \mathbb{R}_+ \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. *Then*

$$u < u_{x_0,s,\theta} \quad \text{in } \mathcal{T}_{x_0,s,\theta} \setminus \big(Z_u \cap Z_{u_{x_0,s,\theta}}\big),$$

where

$$Z_u = \{z \in \mathbb{R}^2_+ \mid |\nabla u(z)| = 0\}$$

and

$$Z_{u_{x_0,s,\theta}} = \{z \in T_{x_0,s,\theta}(\mathbb{R}^2_+) \mid |\nabla u_{x_0,s,\theta}(z)| = 0\}.$$

Proof. If $Z_f = \emptyset$, then the lemma follows from the strong comparison principle (Theorem 7). Hence in what follows we may assume $Z_f \neq \emptyset$.

Since Z_f is a discrete set, we can denote all zeroes of f in $(0, +\infty)$ by

$$\mu_1 < \mu_2 < \mu_3 < \dots$$

For convenience, we also denote $\mu_0 = 0$ (μ_0 may or may not be a zero of f).

By contradiction, assume that there exists

$$\bar{z} \in \mathcal{T}_{x_0,s,\theta} \setminus \big(Z_u \cap Z_{u_{x_0,s,\theta}}\big)$$

such that $u(\bar{z}) = u_{x_0,s,\theta}(\bar{z})$. Since $u \in C^1(\overline{\mathbb{R}^2_+})$, we deduce $|\nabla u| + |\nabla u_{x_0,s,\theta}| > 0$ in $B_\varepsilon(\bar{z}) \subset \mathcal{T}_{x_0,s,\theta}$ for some $\varepsilon > 0$ sufficiently small. Recall that both u and $u_{x_0,s,\theta}$ weakly solve

$$-\Delta_p w + a(w)|\nabla w|^q = f(w) \quad \text{in } B_\varepsilon(\bar{z}).$$

Hence we can apply the strong comparison principle (Theorem 6) to obtain

$$u = u_{x_0,s,\theta} \quad \text{in } B_\varepsilon(\overline{z}).$$

This in turn implies $|\nabla u(\overline{z})| = |\nabla u_{x_0,s,\theta}(\overline{z})| > 0$. Since u is not constant in $B_\varepsilon(\overline{z})$ and Z_f is a discrete set, we can find $z_0 \in B_\varepsilon(\overline{z})$ such that $f\big(u(z_0)\big) \neq 0$. That is,

$$\mu_k < u(z_0) < \mu_{k+1} \quad \text{for some } k \geq 0.$$

Let Ω_0 be the connected component of $\mathcal{T}_{x_0,s,\theta} \setminus Z_{f(u)}$ which contains z_0. By the strong comparison principle (Theorem 7), since $u(z_0) = u_{x_0,s,\theta}(z_0)$, we have

$$u = u_{x_0,s,\theta} \quad \text{in } \Omega_0. \tag{11}$$

Because Ω_0 is open, there exists $r_0 > 0$ such that

$$\overline{B_{r_0}(z_0)} \subset \Omega_0.$$

Now we slide the ball $B_{r_0}(z_0)$ in direction $-V_\theta$, keeping its center on the ray $\{z_0 - tV_\theta \mid t \geq 0\}$, until it touches for the first time $\partial\Omega_0$ at some point $\widehat{z}_0 \in \partial\Omega_0$. Then either $u(\widehat{z}_0) \in \{\mu_k, \mu_{k+1}\}$ or $\widehat{z}_0 \in \{0\} \times (0,s)$. We denote by $\widetilde{z}_0 = z_0 - t_0 V_\theta$ the new center of the slid ball (see Figure 2).

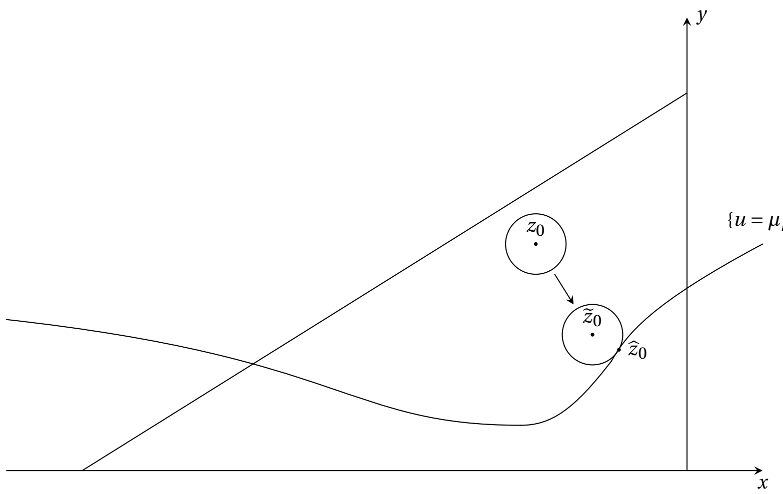

Figure 2. The sliding ball.

Since $\frac{\partial u}{\partial V_\theta} \geq 0$ in $\mathcal{T}_{x_0,s,\theta}$, for all $z \in B_{r_0}(\widetilde{z}_0)$ we have

$$\mu_k < u(z) \leq u(z + t_0 V_\theta) \leq \max_{B_{r_0}(x_0)} u < \mu_{k+1}.$$

Therefore, the touching point $\widehat{z}_0$ must satisfy either $u(\widehat{z}_0) = \mu_k$ or $\widehat{z}_0 \in \{0\} \times (0,s)$. By continuity, we also deduce from (11) that

$$u(\widehat{z}_0) = u_{x_0,s,\theta}(\widehat{z}_0) \tag{12}$$

We consider two cases.

Case 1: Either $u(\widehat{z}_0) = 0$ or $\widehat{z}_0 \in \{0\} \times (0,s)$. This implies $\widehat{z}_0 \in \partial\mathcal{T}_{x_0,s,\theta} \setminus L_{x_0,s,\theta}$. Then the last inequality in (10) gives $u(\widehat{z}_0) < u_{x_0,s,\theta}(\widehat{z}_0)$. However, this contradicts (12).

Case 2: $u(\widehat{z}_0) = \mu_k$ for some $k \ge 1$. Let us define the function

$$w(z) := u(z) - \mu_k \quad \text{for } z \in B_{r_0}(\widetilde{z}_0).$$

Since $1 < p < 2$ and f is locally Lipschitz continuous, we have

$$Cw^{p-1} + f(u) = Cw^{p-1} + f(u) - f(\mu_k) \ge Cw^{p-1} - L_f w \ge 0 \quad \text{in } B_{r_0}(\widetilde{z}_0)$$

for sufficiently large C. Hence w satisfies

$$\begin{cases} -\Delta_p w + a(w+\mu_k)|\nabla w|^q + Cw^{p-1} \ge 0 & \text{in } B_{r_0}(\widetilde{z}_0), \\ w > 0 & \text{in } B_{r_0}(\widetilde{z}_0), \\ w(\widehat{z}_0) = 0. \end{cases}$$

By the Hopf lemma (Theorem 4), we have

$$\frac{\partial u}{\partial \nu}(\widehat{z}_0) = \frac{\partial w}{\partial \nu}(\widehat{z}_0) < 0, \tag{13}$$

where $\nu = \frac{\hat{z}_0 - \tilde{z}_0}{|\hat{z}_0 - \tilde{z}_0|}$ is the outward normal at $\widehat{z}_0$. In particular, $\left|\nabla u(\widehat{z}_0)\right| \ne 0$. Since $u \in C^1(\overline{\mathbb{R}^{\mathbb{N}}_+})$, there exists a ball $B_{\rho_0}(\widehat{z}_0) \subset \mathcal{T}_{x_0,s,\theta}$ such that $|\nabla u| \ne 0$ in $B_{\rho_0}(\widehat{z}_0)$. By the strong comparison principle (Theorem 6), since $u(\widehat{z}_0) = u_{x_0,s,\theta}(\widehat{z}_0)$, we have

$$u = u_{x_0,s,\theta} \quad \text{in } B_{\rho_0}(\widehat{z}_0).$$

From (13), we can find a point $z_1 \in \left\{\widehat{z}_0 + t\nu \,\middle|\, t > 0\right\} \cap B_{\rho_0}(\widehat{z}_0)$ which is close to $\widehat{z}_0$ such that

$$\mu_{k-1} < u(z_1) < u(\widehat{z}_0) = \mu_k.$$

Therefore, from a point $z_0 \in \mathcal{T}_{x_0,s,\theta}$ with $u(z_0) = u_{x_0,s,\theta}(z_0)$ and $\mu_k < u(z_0) < \mu_{k+1}$, we have found a new point $z_1 \in \mathcal{T}_{x_0,s,\theta}$ satisfying $u(z_1) = u_{x_0,s,\theta}(z_1)$ and $\mu_{k-1} < u(z_1) < \mu_k$. Repeating this argument in a finite number of times, we finally reach a point $z_k \in \mathcal{T}_{x_0,s,\theta}$ such that $u(z_k) = u_{x_0,s,\theta}(z_k)$ and $\mu_0 < u(z_k) < \mu_1$. Then we meet a contradiction as in Case 1. □

We recall that the strong maximum principle for the linearized operator does not hold for sign-changing nonlinearities. However, the following weaker result will suffice for our purpose.

Lemma 12. *Under the assumptions of Theorem 1, let us assume that for some $\lambda > 0$ we have*

$$u \le u_\gamma \quad \text{in } \Sigma_\gamma \text{ for all } \gamma \in (0, \lambda].$$

Then for every interval $I \subset \mathbb{R}$, there exists $\overline{x} \in I$ such that

$$\begin{aligned} \frac{\partial u}{\partial y} &> 0 \quad \text{on } \{\overline{x}\} \times [0, \lambda), \\ u &< u_\gamma \quad \text{on } \{\overline{x}\} \times [0, \gamma) \text{ for all } \gamma \in (0, \lambda]. \end{aligned}$$

Proof. By the given assumption, we deduce

$$\frac{\partial u}{\partial y} \ge 0 \quad \text{in } \Sigma_\lambda.$$

We claim that

$$\frac{\partial u}{\partial y} > 0 \quad \text{in } \Sigma_\lambda \setminus Z_{f(u)}. \tag{14}$$

To see this, let Ω be a connected component of $\Sigma_\lambda \setminus Z_{f(u)}$. By the strong maximum principle for the linearized equation (Theorem 8), we know that either $\frac{\partial u}{\partial y} = 0$ in Ω or $\frac{\partial u}{\partial y} > 0$ in Ω. Suppose by contradiction that the former case happens, i.e., $\frac{\partial u}{\partial y} = 0$ in Ω. We take any $z_0 \in \Omega$ and define

$$t_0 = \sup\left\{t \ge 0 \,\middle|\, z_0 - se_2 \in \Omega \text{ for all } 0 \le s < t\right\}.$$

Then $z_0 - t_0 e_2 \in \partial\Omega$ and u is constant on the segment $\{z_0 - te_2 \,|\, 0 \le t \le t_0\}$. In particular, either $u(z_0 - t_0 e_2) = 0$ or $f\big(u(z_0 - t_0 e_2)\big) = 0$ and

$$u(z_0) = u(z_0 - t_0 e_2).$$

This contradicts the fact that $u(z_0) > 0$ and $z_0 \in \Sigma_\lambda \setminus Z_{f(u)}$. Hence (14) must hold.

Now we fix $x_0 \in \mathbb{R}$ and consider any interval $I = (x_0 - r_0, x_0 + r_0) \subset \mathbb{R}$. We have to show that there exists $\bar{x} \in I$ such that

$$\frac{\partial u}{\partial y} > 0 \text{ on } \{\bar{x}\} \times [0, \lambda). \tag{15}$$

Clearly, it suffices to prove (15) for small $r_0 > 0$. Hence, using the boundary condition, we can choose r_0 sufficiently small such that $f(u) \neq 0$ in the open square $(x_0 - r_0, x_0 + r_0) \times (0, 2r_0) \subset \Sigma_\lambda$.

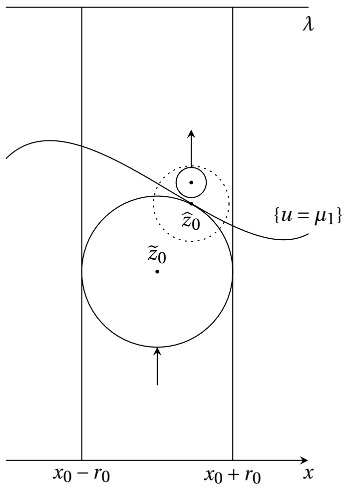

Figure 3. A sequence of sliding balls.

Therefore, the ball $B_{r_0}(z_0)$ with $z_0 = (x_0, r_0)$ is contained in a connected component Ω_0 of $\{0 < u < \mu_1\} \cap \Sigma_\lambda$. From (14), we know that $\frac{\partial u}{\partial y} > 0$ in Ω_0.

Now we slide the ball $B_{r_0}(z_0)$ in the y-direction, keeping its center on the ray $\{z_0 + te_2 \,|\, t \ge 0\}$, where $e_2 = (0,1)$, until it touches for the first time $\partial\Omega_0$ at some point $\widehat{z}_0 \in \partial\Omega_0$.

If $\widehat{z}_0 \in \mathbb{R} \times \{\lambda\}$, i.e., $z_0 = (\widehat{x}_0, \lambda)$ then (15) holds with $\bar{x} = \widehat{x}_0$.

Otherwise, we have $u(\widehat{z}_0) = \mu_1$ thanks to the assumption $\frac{\partial u}{\partial y} \ge 0$. We denote by $\widetilde{z}_0$ the new center of the slid ball (see Figure 3). We also define the function

$$w(z) := \mu_1 - u(z) \quad \text{for } z \in B_{r_0}(\widetilde{z}_0).$$

Then w satisfies

$$\begin{cases} -\Delta_p w + a(\mu_1 - w)|\nabla w|^q + Cw^{p-1} \ge 0 & \text{in } B_{r_0}(\widetilde{z}_0), \\ w > 0 & \text{in } B_{r_0}(\widetilde{z}_0), \\ w(\widehat{z}_0) = 0, \end{cases}$$

for some large $C > 0$. By the Hopf lemma (Theorem 4), we have

$$\frac{\partial u}{\partial y}(\widehat{z}_0) = -\frac{\partial w}{\partial y}(\widehat{z}_0) > 0. \tag{16}$$

Since $u \in C^1(\overline{\mathbb{R}^N_+})$, there exists a ball $B_{\rho_0}(\widehat{z}_0)$ such that $\frac{\partial u}{\partial y} > 0$ in $B_{\rho_0}(\widehat{z}_0)$. We may reduce ρ_0 if necessary so that

$$B_{\rho_0}(\widehat{z}_0) \subset \left(\bigcup_{t \geq 0} B_{r_0}(\widetilde{z}_0 + te_2) \right) \cap \Sigma_\lambda.$$

Because of (16), we can find a point $z_1 \in \left\{ \widehat{z}_0 + te_2 \,\middle|\, t > 0 \right\} \cap B_{\rho_0}(\widehat{z}_0)$ which is close to $\widehat{z}_0$ such that

$$\mu_1 < u(z_1) < \mu_2.$$

(We use the convention that $\mu_2 = +\infty$ if μ_1 is the unique zero of f.) Hence there exists $r_1 > 0$ such that

$$\mu_1 < u(z) < \mu_2 \quad \text{for } z \in B_{r_1}(z_1).$$

Moreover, we may reduce r_1 if necessary to ensure

$$\bigcup_{t \geq 0} B_{r_1}(z_1 - te_2) \subset B_{\rho_0}(\widehat{z}_0) \cup \left(\bigcup_{t \geq 0} B_{r_0}(\widetilde{z}_0 - te_2) \right).$$

By construction, $B_{r_1}(z_1)$ is contained in a connected component Ω_1 of $\left\{ \mu_1 < u < \mu_2 \right\} \cap \Sigma_\lambda$ and

$$\frac{\partial u}{\partial y} > 0 \text{ in } \mathbb{R}^2_+ \cap \left(\bigcup_{t \geq 0} B_{r_1}(z_1 - te_2) \right).$$

We can repeat the above technique by sliding the ball $B_{r_1}(z_1)$ in the y-direction until it touches $\partial\Omega_1$ and so on. Since Z_f is discrete, this procedure will stop after a finite number of steps. We will eventually find a ball $B_{r_k}(z_k)$ which can be slid in the y-direction in a connected component Ω_k of $\left\{ \mu_k < u < \mu_{k+1} \right\} \cap \Sigma_\lambda$ such that it touches $\partial\Omega_k$ at a point $\widehat{z}_k = (\widehat{x}_k, \lambda)$. Moreover, if we denote $\widetilde{z}_k$ the new center of the slid ball, then

$$\frac{\partial u}{\partial y} > 0 \text{ in } \mathbb{R}^2_+ \cap \left(\bigcup_{t \geq 0} B_{r_k}(\widetilde{z}_k - te_2) \right).$$

Hence (15) holds with $\bar{x} = \widehat{x}_k$. This completes the proof of (15).

It remains to show that

$$u < u_\gamma \text{ on } \{\bar{x}\} \times [0, \gamma) \text{ for all } \gamma \in (0, \lambda]. \tag{17}$$

To this end, let $\mathscr{C}$ be the connected component of $\left\{ x \in \Sigma_\gamma \,\middle|\, \frac{\partial u}{\partial y}(x) > 0 \right\}$ such that $\{\bar{x}\} \times [0, \gamma) \subset \mathscr{C}$. We recall that $u \leq u_\gamma$ in $\mathscr{C}$. By the classical strong comparison principle (Theorem 6), which holds now since $|\nabla u| \neq 0$ in $\mathscr{C}$, we deduce that either $u = u_\gamma$ in $\mathscr{C}$ or $u < u_\gamma$ in $\mathscr{C}$. The former case would yield $u(\bar{x}, 2\gamma) = 0$, which is a contradiction. Hence $u < u_\gamma$ in $\mathscr{C}$. In particular, (17) is verified. ☐

We are in a position to prove the main result of this section.

Proof of Theorem 1. Since $1 < p < 2$, we deduce from the local Lipschitz continuity of f and $f(0) \geq 0$ that given $t_0 > 0$, we have $f(t) + Ct^{p-1} \geq 0$ for all $0 < t < t_0$ and some $C = C(t_0) > 0$. Hence the Hopf lemma (Theorem 4) implies

$$\frac{\partial u}{\partial y} > 0 \text{ on } \partial\mathbb{R}^2_+. \tag{18}$$

To show that $\frac{\partial u}{\partial y} > 0$ in $\mathbb{R}^2_+$, we carry out the moving planes procedure in three steps.

Step 1. There exists $h > 0$ such that

$$u \leq u_\lambda \quad \text{in } \Sigma_\lambda \tag{19}$$

for every $0 < \lambda \leq h$.

To prove this, we fix some $x_0 \in \mathbb{R}$. Recalling that $\frac{\partial u}{\partial y}(x_0, 0) > \gamma > 0$ and $u \in C^{1,\alpha}_{\text{loc}}(\overline{\mathbb{R}^2_+})$. Hence there exist $h, \theta_0 > 0$ such that

$$\frac{\partial u}{\partial V_\theta} \geq \frac{\gamma}{2} > 0 \quad \text{in } (x_0 - h, x_0 + h) \times (0, 2h) \tag{20}$$

for any $|\theta| \leq \theta_0$.

For each $0 < |\theta| \le \theta_0$, there exists $s_\theta \in (0,h]$ such that

$$\mathcal{T}_{x_0,s,\theta} \cup T_{x_0,s,\theta}(\mathcal{T}_{x_0,s,\theta}) \subset (x_0-h, x_0+h) \times (0,2h) \quad \text{for all } 0 < s \le s_\theta.$$

Moreover, (20) implies

$$u < u_{x_0,s,\theta} \quad \text{in } \mathcal{T}_{x_0,s,\theta}$$

for all $0 < |\theta| \le \theta_0$ and $0 < s \le s_\theta$. Denote

$$S_\theta = \left\{ \tilde{s} \in (0,h] \,\middle|\, u \le u_{x_0,s,\theta} \text{ in } \mathcal{T}_{x_0,s,\theta} \text{ for every } 0 < s \le \tilde{s} \right\}.$$

Since $S_\theta \ne \emptyset$, we may set $\bar{s} := \sup S_\theta \le h$. Using the sliding technique, we will claim that

$$\bar{s} = h. \tag{21}$$

Assume on contrary that $\bar{s} < h$. By the definition of $\bar{s}$, we have

$$\frac{\partial u}{\partial V_\theta} \ge 0 \quad \text{and} \quad u \le u_{x_0,\bar{s},\theta} \quad \text{in } \mathcal{T}_{x_0,\bar{s},\theta}.$$

On the other hand, from (20) and the Dirichlet boundary condition, we have

$$u < u_{x_0,s,\theta} \quad \text{on } \partial\mathcal{T}_{x_0,s,\theta} \setminus L_{x_0,s,\theta} \quad \text{for all } 0 < s \le h. \tag{22}$$

Therefore, we can apply Lemma 11 to deduce

$$u < u_{x_0,\bar{s},\theta} \quad \text{in } \mathcal{T}_{x_0,\bar{s},\theta} \setminus \left(Z_u \cap Z_{u_{x_0,\bar{s},\theta}}\right). \tag{23}$$

Let $\delta_0 > 0$ satisfy Theorem 10 with $M = 2\left(\|u\|_{L^\infty(\mathcal{T}_{x_0,2h,\theta})} + \|\nabla u\|_{L^\infty(\mathcal{T}_{x_0,2h,\theta})}\right)$ and $L = h$. We choose a sufficiently large compact set $K \subset \mathcal{T}_{x_0,\bar{s},\theta} \setminus \left(Z_u \cap Z_{u_{x_0,\bar{s},\theta}}\right)$ so that

$$\Omega^{\bar{s}} = \bigcup_{x' \in (\Omega^{\bar{s}})'} \{x'\} \times \left(A^{\bar{s}}_{x'} \cup B^{\bar{s}}_{x'}\right),$$

where $\Omega^s := \mathcal{T}_{x_0,s,\theta} \setminus K$ and $A^{\bar{s}}_{x'}, B^{\bar{s}}_{x'}$ satisfy

$$\left|A^{\bar{s}}_{x'}\right| \le \tfrac{\delta_0}{2} \quad \text{and} \quad B^{\bar{s}}_{x'} \subset \left\{ x_N > 0 \,\middle|\, \left|\nabla u(x',x_N)\right| + \left|\nabla v(x',x_N)\right| \le \tfrac{\delta_0}{2} \right\}.$$

From (23), we know that

$$u \le u_{x_0,\bar{s},\theta} - C \text{ in } K \quad \text{for some } C > 0.$$

By continuity, there exists $0 < \varepsilon_0 < h - \bar{s}$ such that for any $\bar{s} < s < \bar{s} + \varepsilon_0$, we have

$$u < u_{x_0,s,\theta} \quad \text{in } K \tag{24}$$

and

$$\Omega^s = \bigcup_{x' \in (\Omega^s)'} \{x'\} \times \left(A^s_{x'} \cup B^s_{x'}\right)$$

with

$$\left|A^s_{x'}\right| \le \delta_0 \quad \text{and} \quad B^s_{x'} \subset \left\{ x_N > 0 \,\middle|\, \left|\nabla u(x',x_N)\right| + \left|\nabla v(x',x_N)\right| \le \delta_0 \right\}.$$

From (22) and (24), we have $u \le u_{x_0,s,\theta}$ on $\partial(\mathcal{T}_{x_0,s,\theta} \setminus K)$. Therefore, Theorem 10 can be applied with $v = u_{x_0,s,\theta}$ and $\Omega = \Omega^s$ to yield

$$u \le u_{x_0,s,\theta} \quad \text{in } \mathcal{T}_{x_0,s,\theta} \setminus K. \tag{25}$$

Combining (24) and (25), we get

$$u \le u_{x_0,s,\theta} \quad \text{in } \mathcal{T}_{x_0,s,\theta}$$

for all $\bar{s} < s < \bar{s} + \varepsilon_0$. This contradicts the definition of $\bar{s}$ and (21) is proved. Hence for every $0 < \lambda \le h$ and $0 < |\theta| \le \theta_0$, we have

$$u \le u_{x_0,\lambda,\theta} \quad \text{in } \mathcal{T}_{x_0,\lambda,\theta}.$$

Let $\theta \to 0^+$ and $\theta \to 0^-$, we obtain (19).

Step 2. Let us define

$$\Lambda = \{\tilde{\lambda} > 0 \,|\, u \le u_\lambda \text{ in } \Sigma_\lambda \text{ for all } 0 < \lambda \le \tilde{\lambda}\}.$$

By Step 1, $\Lambda \neq \emptyset$. Hence we can set $\bar{\lambda} := \sup \Lambda$. We claim that

$$\bar{\lambda} = \infty. \tag{26}$$

Assume by contradiction that $\bar{\lambda} < \infty$. From the definition of $\bar{\lambda}$ we have

$$u \le u_\lambda \text{ in } \Sigma_\lambda \text{ for all } \lambda \in (0, \bar{\lambda}]$$

and hence

$$\frac{\partial u}{\partial y} \ge 0 \quad \text{in } \overline{\Sigma_{\bar{\lambda}}}.$$

We show that there exists some $\bar{x} \in \mathbb{R}$ such that

$$\frac{\partial u}{\partial y}(\bar{x}, \bar{\lambda}) > 0. \tag{27}$$

Suppose by contradiction that $\frac{\partial u}{\partial y}(x, \bar{\lambda}) = 0$ for all $x \in \mathbb{R}$. There are two cases:

Case 1: $f(u(x, \bar{\lambda})) = 0$ for all $x \in \mathbb{R}$. Since Z_f is discrete, we deduce $u(x, \bar{\lambda}) = \mu$ for all $x \in \mathbb{R}$ and some $\mu \in Z_f \setminus \{0\}$. Then $w := \mu - u \ge 0$ in $\Sigma_{\bar{\lambda}}$. Moreover,

$$-\Delta_p w - a(\mu - w)|\nabla w|^q + C w^{p-1} = -f(\mu - w) + C w^{p-1} \ge 0 \quad \text{in } B_{\bar{\lambda}/2}\big((x, \bar{\lambda}/2)\big)$$

for sufficiently large C. Hence the strong maximum principle (Theorem 4) implies $w \equiv 0$ or $w > 0$ in $B_{\bar{\lambda}/2}\big((x, \bar{\lambda}/2)\big)$. The former cannot happen because $w(x, 0) = \mu - u(x, 0) = \mu > 0$. Hence $w > 0$ in $B_{\bar{\lambda}/2}\big((x, \bar{\lambda}/2)\big)$. Now the Hopf lemma yields $\frac{\partial w}{\partial y}(x, \bar{\lambda}) < 0$, which means $\frac{\partial u}{\partial y}(x, \bar{\lambda}) > 0$ for all $x \in \mathbb{R}$.

Case 2: $f(u(x_0, \bar{\lambda})) \neq 0$ for some $x_0 \in \mathbb{R}$. By continuity, there exists a small $r > 0$ such that $f(u)$ is either strictly positive or strictly negative in $B_r\big((x_0, \bar{\lambda})\big)$. Let us define

$$u_*(x, y) = \begin{cases} u(x, y) & \text{if } 0 \le y \le \bar{\lambda}, \\ u(x, 2\bar{\lambda} - y) & \text{if } \bar{\lambda} \le y \le 2\bar{\lambda}, \end{cases}$$

and

$$u^*(x, y) = \begin{cases} u(x, 2\bar{\lambda} - y) & \text{if } 0 \le y \le \bar{\lambda}, \\ u(x, y) & \text{if } \bar{\lambda} \le y \le 2\bar{\lambda}. \end{cases}$$

Since $u \in C^{1,\alpha}_{\mathrm{loc}}(\overline{\mathbb{R}^2_+})$ and $\frac{\partial u}{\partial y} = 0$ on $\mathbb{R} \times \{\bar{\lambda}\}$, we get that $u_*, u^* \in C^1(\overline{\mathbb{R}^2_+})$ and u_*, u^* are weak solutions of

$$-\Delta_p w + a(w)|\nabla w|^q = f(w) \quad \text{in } B_r\big((x_0, \bar{\lambda})\big).$$

On the other hand, $u_* \le u^*$ in $\Sigma_{2\bar{\lambda}}$ and $u_*(x, \bar{\lambda}) = u^*(x, \bar{\lambda})$ for all $x \in \mathbb{R}$. By the strong comparison principle (Theorem 7) we deduce $u_* = u^*$ in $B_r\big((x_0, \bar{\lambda})\big)$. That means $u = u_{\bar{\lambda}}$ in $B_r\big((x_0, \bar{\lambda})\big)$. However, this contradicts Lemma 12 with $I = (x_0 - r, x_0 + r)$.

Hence (27) holds. This implies $\frac{\partial u}{\partial y} > 0$ in $B_{\bar{r}}\big((\bar{x}, \bar{\lambda})\big)$ for some $\bar{r} > 0$. Exploiting this fact and Lemma 12 with $I = (\bar{x} - \bar{r}, \bar{x} + \bar{r})$, we find $x_1 \in (\bar{x} - \bar{r}, \bar{x} + \bar{r})$ and $\tilde{\varepsilon} > 0$ such that

$$\frac{\partial u}{\partial y} > 0 \text{ on } \{x_1\} \times [0, \bar{\lambda} + \tilde{\varepsilon}], \quad \text{and} \quad u < u_\lambda \text{ on } \{x_1\} \times [0, \lambda) \text{ for all } \lambda \in (0, \bar{\lambda}]. \tag{28}$$

From the second assertion in (28) we have

$$u_{\bar{\lambda}} - u > C \text{ on the compact set } \{x_1\} \times [0, \bar{\lambda} - \tilde{\varepsilon}/2]$$

for some $C > 0$. (We may reduce $\tilde{\varepsilon}$ if necessary so that $\bar{\lambda} - \tilde{\varepsilon}/2 > 0$.) Hence there exists $0 < \bar{\varepsilon} < \tilde{\varepsilon}/4$ such that

$$u_\lambda - u > \frac{C}{2} > 0 \text{ on } \{x_1\} \times [0, \bar{\lambda} - \tilde{\varepsilon}/2] \text{ for all } \lambda \in [\bar{\lambda}, \bar{\lambda} + \bar{\varepsilon}].$$

On the other hand, exploiting the first assertion in (28), we deduce

$$u_\lambda - u > 0 \text{ on } \{x_1\} \times [\bar{\lambda} - \tilde{\varepsilon}/2, \lambda) \text{ for all } \lambda \in [\bar{\lambda}, \bar{\lambda} + \bar{\varepsilon}].$$

Therefore,

$$u < u_\lambda \text{ on } \{x_1\} \times [0, \lambda) \text{ for all } \lambda \in (0, \bar{\lambda} + \bar{\varepsilon}]. \tag{29}$$

This implies the existence of some small $\theta_1 > 0$ such that

$$u < u_{x_1,s,\theta} \text{ on } \{x_1\} \times [0, s] \quad \text{for all } 0 < s \le \bar{\lambda} + \bar{\varepsilon} \text{ and } 0 < |\theta| < \theta_1. \tag{30}$$

Indeed, assume (30) does not hold. Then there exist y_n, s_n, θ_n for each $n \in \mathbb{N}$ such that

$$u(x_1, y_n) \ge u_{x_1,s_n,\theta_n}(x_1, y_n), \quad 0 < y_n \le s_n \le \bar{\lambda} + \bar{\varepsilon}, \quad 0 < |\theta_n| < \frac{1}{n}. \tag{31}$$

Up to a subsequence, we may assume $(y_n, s_n, \theta_n) \to (y_0, s_0, 0)$ with $0 \le y_0 \le s_0 \le \bar{\lambda} + \bar{\varepsilon}$. Hence the first inequality in (31) gives

$$u(x_1, y_0) \ge u_{s_0}(x_1, y_0).$$

In view of (29), this only happens if $y_0 = s_0$. The first inequality in (31) now yields

$$\frac{\partial u}{\partial y}(x_1, s_0) \le 0,$$

which is a contradiction with the first inequality of (28). Hence (30) must hold.

Combining (30) with the Dirichlet boundary condition of u, we have

$$u < u_{x_1,s,\theta} \text{ on } \partial\mathcal{T}_{x_1,s,\theta} \setminus L_{x_1,s,\theta} \quad \text{for all } 0 < s \le \bar{\lambda} + \bar{\varepsilon} \text{ and } 0 < |\theta| < \theta_1. \tag{32}$$

With the help of (32), for each $0 < |\theta| < \min\{\theta_0, \theta_1\}$, we can repeat the sliding technique in Step 1 to show that

$$u \le u_{x_1,\lambda,\theta} \text{ in } \mathcal{T}_{x_0,\lambda,\theta} \quad \text{for all } 0 < \lambda \le \bar{\lambda} + \bar{\varepsilon}.$$

By letting $\theta \to 0$, we obtain $u \le u_\lambda$ in Σ_λ for all $0 < \lambda \le \bar{\lambda} + \bar{\varepsilon}$. This contradicts the definition of $\bar{\lambda}$. Thus, (26) is proved.

Step 3: Conclusion. Step 2 implies that u is monotone increasing with respect to the y-direction. Moreover, claim (14) in the proof of Lemma 12 indicates

$$\frac{\partial u}{\partial y} > 0 \quad \text{in } \mathbb{R}^2_+ \setminus Z_{f(u)}.$$

Combining this with (18), we complete the proof of the theorem. □

4. The case $p > 2$ and f is positive

To prove Theorem 3, we start with the following weak comparison principle for small domains.

Theorem 13. *Assume $p > 2$, $q \ge \frac{p}{2}$ and $a, f\colon [0, +\infty) \to \mathbb{R}$ are locally Lipschitz continuous functions such that*

$$f(t) > 0 \textit{ for } t > 0.$$

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $L, M > 0$ and $u, v \in C^1(\bar{\Omega})$ be such that

$$\|u\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} + \|\nabla u\|_{L^\infty(\Omega)} + \|\nabla v\|_{L^\infty(\Omega)} \le M$$

and

$$\begin{cases} -\Delta_p u + a(u)|\nabla u|^q \le f(u) & \textit{in } \Omega, \\ -\Delta_p v + a(v)|\nabla v|^q \ge f(v) & \textit{in } \Omega, \\ u > 0, v > 0 & \textit{in } \Omega, \\ u \le v & \textit{on } \partial\Omega. \end{cases}$$

Then there exists a constant

$$\delta_0 = \delta_0(N, p, q, a, f, M)$$

such that if we assume $|\Omega| \le \delta_0$*, then it holds*

$$u \le v \ \text{ in } \Omega.$$

Proof. Using the same arguments as in the proof of Theorem 10, we obtain (9). That is,

$$\int_\Omega \left(|\nabla u| + |\nabla v|\right)^{p-2} \left|\nabla(u-v)^+\right|^2 \mathrm{d}x \le C \int_\Omega \left((u-v)^+\right)^2 \mathrm{d}x, \tag{33}$$

where $C = C(N, p, q, a, f, M) > 0$. Applying the weighted Poincaré inequality (Theorem 5) to the right-hand side of (33), we derive

$$\begin{aligned} \int_\Omega \left(|\nabla u| + |\nabla v|\right)^{p-2} \left|\nabla(u-v)^+\right|^2 \mathrm{d}x &\le C C_P(|\Omega|) \int_\Omega |\nabla u|^{p-2} \left|\nabla(u-v)^+\right|^2 \mathrm{d}x \\ &\le C C_P(|\Omega|) \int_\Omega \left(|\nabla u| + |\nabla v|\right)^{p-2} \left|\nabla(u-v)^+\right|^2 \mathrm{d}x, \end{aligned} \tag{34}$$

where the Poincaré constant $C_P(|\Omega|) \to 0$, as $|\Omega| \to 0$. Now, we choose $\delta_0 > 0$ sufficiently small such that the condition $|\Omega| \le \delta_0$ implies

$$C C_P(|\Omega|) < 1.$$

Then we deduce from (34) that

$$\int_\Omega \left(|\nabla u| + |\nabla v|\right)^{p-2} \left|\nabla(u-v)^+\right|^2 \mathrm{d}x = 0.$$

As in the proof of Theorem 10, this indicates $u \le v$ in Ω. □

Proof of Theorem 3. Since f is positive, the strong comparison principle (Theorem 7) and the strong maximum principle for the linearized equation (Theorem 8) hold for any domain $\Omega \subset \mathbb{R}^2_+$.

The proof of Theorem 3 is similar to that of Theorem 1, but it is easier. Instead of using Theorem 10, Lemma 11 and Lemma 12, we need to exploit Theorem 13, Theorem 7 and Theorem 8, respectively.

Alternatively, one may also repeat the same arguments in [22] using Theorem 13 above instead of [22, Proposition 2.2] there.

The detailed proof therefore will be omitted. □

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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