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Burau representation of B_4 and quantization of the rational projective plane

Représentation de Burau de B_4 et quantification du plan projectif rationnel

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Abstract. The braid group B_4 naturally acts on the rational projective plane $\mathbb{P}^2(\mathbb{Q})$, this action corresponds to the classical integral reduced Burau representation of B_4. The first result of this paper is a classification of the orbits of this action. The Burau representation then defines an action of B_4 on $\mathbb{P}^2(\mathbb{Z}(q))$, where q is a formal parameter and $\mathbb{Z}(q)$ is the field of rational functions in q with integer coefficients. We study orbits of the B_4-action on $\mathbb{P}^2(\mathbb{Z}(q))$, and show existence of embeddings of the q-deformed projective line $\mathbb{P}^1(\mathbb{Z}(q))$ that precisely correspond to the notion of q-rationals due to Morier-Genoud and Ovsienko.

Résumé. Le groupe de tresses B_4 agit naturellement sur le plan projectif rationnel $\mathbb{P}^2(\mathbb{Q})$. Cette action est donnée par la classique représentation de Burau entière de B_4. Le premier résultat de cet article consiste en une classification des orbites de cette action. La représentation de Burau permet ensuite de définir une action de B_4 sur $\mathbb{P}^2(\mathbb{Z}(q))$, où q est un paramètre formel et $\mathbb{Z}(q)$ le corps des fractions rationnelles en q, à coefficients entiers. On étudie les orbites de cette action de B_4 sur $\mathbb{P}^2(\mathbb{Z}(q))$, et on montre l'existence d'un plongement de la q-déformation de la droite projective rationnelle $\mathbb{P}^1(\mathbb{Z}(q))$ qui coïncide précisément avec la notion de q-rationnels due à Morier-Genoud et Ovsienko.

Keywords. Quantization, Burau representation, q-rational numbers, braid group, rational projective plane.

Mots-clés. Quantification, représentation de Burau, q-rationnels, groupe de tresses, plan projectif rationnel.

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1. Introduction and main results

The 4-strands Artin braid group B_4 is generated by three elements $\sigma_1, \sigma_2, \sigma_3$ with braid relations

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \qquad \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3,$$

and commutation relation $\sigma_1\sigma_3 = \sigma_3\sigma_1$.

The classical reduced *Burau representation* of B_4 is a group homomorphism

$$\rho_q\colon B_4 \longrightarrow \mathrm{GL}_3(\Lambda),$$

where $\Lambda := \mathbb{Z}[q, q^{-1}]$ is the ring of Laurent polynomials in one (formal) variable q with integer coefficients, defined by

$$\rho_q(\sigma_1) = \begin{pmatrix} q & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \rho_q(\sigma_2) = \begin{pmatrix} 1 & 0 & 0 \\ -q & q & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \rho_q(\sigma_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -q & q \end{pmatrix}. \tag{1}$$

Note that, for the sake of convenience and following [5,12] we have chosen the parameter $q = -t$, where t is a more standard choice of parameter used in the theory of braid groups.

The Burau representation goes back to Werner Burau [4] who used it to interpret the Alexander polynomial of knots in algebraic terms. Faithfulness of the representation (1) is a long standing open problem. For more details about the Burau representation, see [2,8].

The main goal of this paper is to study the natural projective version of the Burau representation, which is the action of B_4 on the projective plane $\mathbb{P}^2(\overline{\Lambda})$ with coefficients in the field $\overline{\Lambda} := \overline{\mathbb{Z}(q)}$. Recall that the field $\overline{\Lambda}$ is the same as the field $\mathbb{Q}(q)$ of rational functions in q and every $F(q) \in \overline{\Lambda}$ can be written in the form

$$F(q) = \frac{R(q)}{S(q)},$$

where R and S are polynomials in q with integer coefficients. The action of B_4 on $\mathbb{P}^2(\overline{\Lambda})$ is defined as the projectivization of (1). We will still denote by ρ_q this projective version of the Burau representation

$$\rho_q \colon B_4 \longrightarrow \mathrm{PGL}_3(\Lambda).$$

We understand this action as q-deformation, or "quantization" of the rational projective plane. Our approach is similar to that of [11] where the case of the projective line was investigated.

1.1. *The case $q = 1$, classification of orbits*

In the special case $q = 1$, the homomorphism (1) is the *integral* Burau representation,

$$\rho \colon B_4 \longrightarrow \mathrm{SL}(3, \mathbb{Z}),$$

defined for the generators by

$$\rho(\sigma_1) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \rho(\sigma_2) = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \rho(\sigma_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

Note that, unlike the Burau representation ρ_q for which the question is wide open, it is known that the integral representation ρ has a nontrivial kernel. The kernel of ρ is a normal subgroup of B_4, called a *braid Torelli group* and denoted by $\mathcal{BI}_4$. Smythe in [17] found a set of normal generators of $\mathcal{BI}_4$, i.e. a set of elements whose normal closure is $\mathcal{BI}_4$. Smythe's set of normal generators of $\mathcal{BI}_4$ is $\{\tau_1^2, \tau_3^2, \Delta^2\}$, where

$$\tau_1 = (\sigma_1\sigma_2\sigma_1)^2, \quad \tau_3 = (\sigma_3\sigma_2\sigma_3)^2 \quad \text{and} \quad \Delta = \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1. \tag{2}$$

Note that Δ is the *Garside element*, whose square $\Delta^2 = (\sigma_1\sigma_2\sigma_3)^4$ generates the center of B_4.

More recently Brendle, Margalit, and Putman gave a topological description of normal generators of $\mathcal{BI}_n$ for all n, in [3].

The identification of the kernels of specializations of Burau representation at roots of unity was raised in [18] and developped in [5].

The projective version of ρ gives rise to an action of B_4 on the rational projective plane $\mathbb{P}^2(\mathbb{Q})$ that by slightly abusing the notation we also denote by ρ

$$\rho \colon B_4 \longrightarrow \mathrm{PSL}_3(\mathbb{Z}) \curvearrowright \mathbb{P}^2(\mathbb{Q}).$$

Recall that the group $\mathrm{PSL}_3(\mathbb{Z})$ actually coincides with $\mathrm{SL}_3(\mathbb{Z})$. Let us describe the action of B_4 on $\mathbb{P}^2(\mathbb{Q})$ more explicitly. Every point $p \in \mathbb{P}^2(\mathbb{Q})$ has integer homogeneous coordinates:

$$p = [r : s : t],$$

where $r, s, t \in \mathbb{Z}$ are mutually prime. Every point has exactly two such representatives, that differ by the sign. The $\mathrm{SL}_3(\mathbb{Z})$-action preserves this convention, and so does the B_4-action we are interested in. The B_4-action on $\mathbb{P}^2(\mathbb{Q})$ is then

$$\begin{aligned}\rho(\sigma_1)&\colon [r : s : t] \longmapsto [r + s : s : t],\\ \rho(\sigma_2)&\colon [r : s : t] \longmapsto [r : s + t - r : t],\\ \rho(\sigma_3)&\colon [r : s : t] \longmapsto [r : s : t - s].\end{aligned} \tag{3}$$

While the group $\mathrm{SL}_3(\mathbb{Z})$ acts transitively on $\mathbb{P}^2(\mathbb{Q})$, the action of the braid group B_4 does not have this transitivity property. For example, the point $[1 : 0 : 1]$ is fixed by ρ and constitutes the only orbit consisting of one point.

Our first main result is a complete description of the orbits in $\mathbb{P}^2(\mathbb{Q})$ for the B_4-action (3). Despite the fact that the question has a classic nature, we did not find this statement in the literature.

Theorem 1.

(i) *Under the B_4-action, the rational projective plane is decomposed into infinitely many orbits as follows*

$$\mathbb{P}^2(\mathbb{Q}) = \big\{[1:0:1]\big\} \sqcup \mathrm{Orb}_{B_4}\big([0:1:0]\big) \sqcup \bigsqcup_{\substack{n\geq 2\\ 0<m<n/2\\ m\wedge n=1}} \mathrm{Orb}_{B_4}\big([m:n:m]\big).$$

(ii) *For every couple (m, n) of coprime integers, the orbit $\mathrm{Orb}_{B_4}\big([m : n : m]\big)$ consists of the following points*

$$\mathrm{Orb}_{B_4}\big([m:n:m]\big) = \left\{ [r:s:t] \;\middle|\; \begin{cases}\gcd(r-t, s) = n\\ r, t \equiv \pm m \mod (n)\end{cases} \right\},$$

and

$$\mathrm{Orb}_{B_4}\big([0:1:0]\big) = \big\{[r:s:t] \,\big|\, \gcd(r-t, s) = 1\big\}.$$

This theorem will be proved in Sections 2.1 and 2.3. Besides the singleton orbit $\big\{[1 : 0 : 1]\big\}$, every B_4-orbit contains infinitely many points. Moreover, we will show that every such orbit is dense in $\mathbb{P}^2(\mathbb{Q})$. However, the orbit of the point $[0 : 1 : 0]$ is (conjecturally) the "largest" orbit in the following sense. For every $N \in \mathbb{N}$, the orbit $\mathrm{Orb}_{B_4}\big([0 : 1 : 0]\big)$ contains at least three times more points in the subset $\big\{[r : s : t] \,\big|\, |r|, |s|, |t| \leq N\big\}$ of the rational plane than the union of the other orbits. Although we do not have a proof of this statement, we will give the numerical evidence for this "experimental fact". We will refer to this orbit as the "principal orbit" and use the special notation

$$\mathscr{O}_1 := \mathrm{Orb}_{B_4}\big([0:1:0]\big).$$

Let $\mathrm{Stab}_{[m:n:m]} \subset B_4$ be the stabilizer of a point $[m : n : m]$. Clearly, $\mathscr{B}\mathscr{I}_4 \subset \mathrm{Stab}_{[m:n:m]}$. The next result gives a complete description of the stabilizers modulo the braid Torelli group $\mathscr{B}\mathscr{I}_4$. Note in particular that the stabilizer $\mathrm{Stab}_{[0:1:0]}$ of the point $[0 : 1 : 0]$ is generated by σ_2, Δ and τ_1.

Theorem 2. *Let $n \in \mathbb{N}^*$, and $0 \leq m < n$ coprime to n. Then*

$$\mathrm{Stab}_{[m:n:m]} / \mathscr{B}\mathscr{I}_4 = \begin{cases} \langle \tau_1\Delta, \sigma_2\rangle & \text{if } n \geq 3,\\ \langle \tau_1\Delta, \sigma_2, \sigma_1\sigma_2^2\sigma_3\rangle & \text{if } n = 2,\\ \langle \tau_1, \Delta, \sigma_2\rangle & \text{if } n = 1.\end{cases}$$

This statement will be proved in Section 2.5.

1.2. *Quantization procedure, comparison to q-rationals*

We introduce the notion of *quantization of the rational projective plane* $\mathbb{P}^2(\mathbb{Q})$. The quantization map is a set-valued function

$$\mathcal{Q}\colon \mathbb{P}^2(\mathbb{Q}) \longrightarrow \mathcal{P}\big(\mathbb{P}^2(\overline{\Lambda})\big).$$

It associates to every point $p = [r : s : t]$ of $\mathbb{P}^2(\mathbb{Q})$ an infinite set of points $\big[R(q) : S(q) : T(q)\big]$, called the quantization of p, where R, S and T are polynomials in q with integer coefficients. The precise definition is as follows. For every $p \in \mathbb{P}^2(\mathbb{Q})$ in the orbit of $[m : n : m]$, we set

$$\mathcal{Q}(p) := \big\{\rho_q(\beta)\big([m:n:m]\big) \,\big|\, \beta \text{ s.t. } \rho(\beta)\big([m:n:m]\big) = p\big\}. \tag{4}$$

We will be mostly interested in the quantization of the principal orbit $\mathcal{O}_1$. The image of $\mathcal{O}_1$ with respect to the quantization map will be denoted by $\mathcal{O}_q$.

The above quantization procedure is analogous to the notion of *q-deformed rationals* introduced in [11]. The main difference is that the image of one point by our quantization map (4) consists of an infinite number of points. We will explain this phenomenon is Section 5.

Let us briefly describe the quantization procedure of Morier-Genoud and Ovsienko using the terms which are closest to our context. Consider the rational projective line $\mathbb{P}^1(\mathbb{Q})$ equipped with the standard transitive action of the modular group $\mathrm{PSL}(2,\mathbb{Z})$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : [r:s] \longmapsto \big[ar + bs : cr + ds\big].$$

The $\mathrm{PSL}(2,\mathbb{Z})$-action can also be considered as an action of the braid group B_3 (the center acts trivially). The Burau representation ρ_q of B_3 then defines a $\mathrm{PSL}(2,\mathbb{Z})$-action on $\mathbb{P}^1(\overline{\Lambda})$. For the generators,

$$\rho_q(\sigma_1)\colon [r:s] \longmapsto [qr + s : s], \qquad \rho_q(\sigma_2)\colon [r:s] \longmapsto [r : q(s-r)].$$

The quantization of Morier-Genoud and Ovsienko is the unique map

$$\mathcal{Q}\colon \mathbb{P}^1(\mathbb{Q}) \longrightarrow \mathbb{P}^1(\overline{\Lambda})$$

that commutes with the $\mathrm{PSL}(2,\mathbb{Z})$-action and sends $[0 : 1]$ to $[0 : 1]$ (this point remains unchanged). It associates to a point $[r : s]$ a pair of monic polynomials with positive integer coefficients $\big(R(q), S(q)\big)$. In other words, the q-rationals are defined as the orbit of the point $[0:1] \in \mathbb{P}^1(\overline{\Lambda})$ under the Burau representation.

The notion of q-rationals enjoys a number of remarkable properties, we mention only few of them:

- the "total positivity" property [11] that means roughly speaking that the topology of $\mathbb{P}^1(\mathbb{Q})$ is preserved by quantization;
- the unimodality property, conjectured in [11] and eventually proved in [13], asserts that the sequences of coefficients of the polynomials $R(q)$ and $S(q)$ are unimodal;
- a connection to the Jones polynomial of rational knots [11,16];
- the stabilization phenomenon that led to the notion of a q-deformed real number [10].

An application of q-rationals to the Burau representation of B_3 was suggested in [12], where the information about the polynomials arising in the process of quantization of $\mathbb{P}^1(\mathbb{Q})$ was sufficient to guarantee faithfulness of specializations of the Burau representation. This application is an important motivation for our study, we hope that the polynomials R, S, T will be useful for the study of the Burau representation of B_4.

1.3. *An embedding of the projective line*

Consider the following embedding of the projective line into the projective plane

$$\begin{aligned}\mathbb{P}^1(\mathbb{Q}) &\overset{\iota}{\hookrightarrow} \mathbb{P}^2(\mathbb{Q})\\ [r:s] &\longmapsto [r:s:0].\end{aligned}$$

It is easy to see that its image belongs to the principal orbit $\mathcal{O}_1$. Similarly, we define the embedding

$$\begin{aligned}\mathbb{P}^1(\overline{\Lambda}) &\overset{\iota_q}{\hookrightarrow} \mathbb{P}^2(\overline{\Lambda})\\ [R(q):S(q)] &\longmapsto [R(q):S(q):0].\end{aligned}$$

We will prove that the above embeddings commute with quantization.

Theorem 3. *The quantization of the projective line in the sense of [11] and our quantization commute with the embeddings. In other words, for every* $[r:s] \in \mathbb{P}^1(\mathbb{Q})$, *we have*

$$\iota_q\big(\mathcal{Q}([r:s])\big) \in \mathcal{Q}\big(\iota([r:s])\big).$$

This statement will be proved in Section 4.2.

The other natural embedding of the projective line into the projective plane, namely $[r:s] \mapsto [0:r:s]$ also commutes with quantization.

2. The structure of orbits

In this section, we prove Theorem 1 and Theorem 2, and we introduce the braided Euclidean algorithm.

2.1. *Proof of Theorem 1, first part*

In this section, we prove Theorem 1 in one direction. We check that the orbits $\mathrm{Orb}_{B_4}\big([m:n:m]\big)$ with different values of (coprime) m and n and $0 \le m < n/2$, are indeed disjoint.

Lemma 4.

(i) *For every* $[r:s:t] \in \mathbb{P}^2(\mathbb{Q})$, *the number*

$$n := \gcd(r-t, s)$$

is invariant under the action of B_4.

(ii) *Up to the sign, the number* $r \mod (n) = t \mod (n)$ *is invariant under the action of* B_4.

Proof.

(i). As $\gcd(r+s-t, s) = \gcd(r-t, s+r-t) = \gcd(r-t, s)$, the actions of the generators $\sigma_1, \sigma_2, \sigma_3$ given by (3) do not change the quantity $\gcd(r-t, s)$.

(ii). The quantities $r \mod (n)$ and $t \mod (n)$ coincide because n divides $r-t$ so $r \equiv t \mod (n)$. Because of the sign change $[r:s:t] = [-r:-s:-t]$, this value is only defined up to the sign. It is straightforward to check that it is invariant under the action of the generators of B_4. □

The above lemma implies that if a point $[r:s:t]$ belongs to the orbit of $[m:n:m]$, then $\gcd(r-t, s) = n$, and r and t are congruent to $\pm m$ modulo n. In particular we deduce that different points $[m:n:m]$, with $n \in \mathbb{N}^*$ and m coprime to n such that $0 \le m < n/2$, belong to different orbits.

2.2. *The condition* $0 \le m < n/2$*, further examples of orbits*

Let us now explain why the condition $0 \le m < n/2$ is necessary to have disjoint orbits. Consider a point $[m:n:m] \in \mathbb{P}^2(\mathbb{Q})$ with $m < n$. One then has

$$[m:n:m] = \rho\big(\sigma_1\sigma_2^2\sigma_3\big)\big([n-m:n:n-m]\big),$$

so that $[m:n:m]$ and $[n-m:n:n-m]$ belong to the same orbit. Therefore the condition $0 \le m < n/2$ is indeed necessary to have different orbits. Note also that when $n \ge 2$, one cannot take $m = 0$ since $[0:n:0] = [0:1:0]$, so that the condition reads $0 < m < n/2$ in this case.

Example 5.

(i) For $n = 2$, our list of orbits contains only one orbit, which is the orbit of $[1:2:1]$. More explicitly, the orbit $\mathrm{Orb}_{B_4}\big([1:2:1]\big)$ consists of the points $[r:s:t]$ such that r and t are odd, s is even and $\gcd(r-t,s) = 2$.

(ii) For $n = 3$, there is also only one orbit, the orbit of $[1:3:1]$. The point $[2:3:2]$ is recovered for instance by

$$[1:3:1] \xrightarrow{\sigma_3} [1:3:-2] \xrightarrow{\sigma_2^2} [1:-3:-2] = [-1:3:2] \xrightarrow{\sigma_1} [2:3:2].$$

The orbit $\mathrm{Orb}_{B_4}\big([1:3:1]\big)$ consists of the points $[r:s:t]$ such that s is a multiple of 3, and $\gcd(r-t,s) = 3$.

2.3. *An Euclid-like algorithm, end of the proof of Theorem 1*

In this subsection, we finish the proof of Theorem 1. We show that every point of $\mathbb{P}^2(\mathbb{Q})$ belongs to the orbit of $[m:n:m]$ for some m and n. To this end, we construct an explicit way to go from a point $[r:s:t]$, to the corresponding representative $[m:n:m]$. Note that this algorithm fits into the framework of Jacobi–Perron type multicontinued fraction algorithm as described in [15].

Braided Euclidean algorithm

Input. We start with a point $[r:s:t] \neq [1:0:1]$, with $r,s,t \in \mathbb{Z}$, mutually prime. We can assume that $s \ge 0$.

Step 1 of the algorithm. If $s = 0$, then apply σ_2 to replace s by $t-r$, and if necessary change signs to get $s > 0$.
Write the Euclidean division of r by s and t by s:

$$r = sa_1 + r' \ t = sc_1 + t'.$$

Apply $\sigma_1^{-a_1}\sigma_3^{c_1}$, so that $[r:s:t] \mapsto [r':s:t'] =: [r_1:s_1:t_1]$.

Step 2 of the algorithm. While $r_i - t_i \neq 0$, repeat:
Write the upper Euclidean division of s_i by $(r_i - t_i)$, i.e.

$$s_i = (r_i - t_i)b_{2i} + s' \quad \text{with } 0 < s' \le |r_i - t_i|.$$

Apply $\sigma_2^{b_{2i}}$ and put $s_{i+1} := s'$.
Write the Euclidean divisions of r_i and t_i by s_{i+1}:

$$r_i = s_{i+1}a_{2i+1} + r' \quad \text{and} \quad t_i = s_{i+1}c_{2i+1} + t'.$$

Apply $\sigma_1^{-a_{2i+1}}\sigma_3^{c_{2i+1}}$ and put $r_{i+1} := r'$ and $t_{i+1} := t'$.

Termination of the algorithm. If we have arrived to $r_i = t_i$, we do not proceed further. Here the algorithm terminates.

Example 6. Let us apply the algorithm to the point $x = [37:30:12]$. As $\gcd(37-12, 30) = 5$ and $37 \equiv 2 \mod (5)$, we know that x is in the orbit of $[2:5:2]$.

First step: $37 = 30 * 1 + 7$ so $x \xrightarrow{\sigma_1^{-1}} [7:30:12]$.
Second step: $30 = |7-12| * 5 + 5$ so $[7:30:12] \xrightarrow{\sigma_2^{-5}} [7:5:12]$ and then $[7:5:12] \xrightarrow{\sigma_1^{-1}\sigma_3^2} [2:5:2]$.
Finally the braid $\sigma_1^{-1}\sigma_3^2\sigma_2^{-5}\sigma_1^{-1}$ sends x to the representative of its orbit, $[2:5:2]$.

Proposition 7.

(i) *The braided Euclidean algorithm described above terminates.*
(ii) *Given a point* $p = [r:s:t] \in \mathbb{P}^2(\mathbb{Q})$, *with* $n = \gcd(r-t, s)$ *and* $m = r \mod (n)$, *the braided Euclidean algorithm provides a braid* $\beta_p \in B_4$ *such that*

$$\rho(\beta_p)(p) = [m:n:m].$$

Proof. At each step of the algorithm, by the Euclidean division property, the following inequalities hold:

(I) $|r_{i+1} - t_{i+1}| < s_{i+1} < |r_i - t_i| < s_i$;
(II) $0 \le r_i < s_i$, $0 \le t_i < s_i$.

Therefore the sequences $(s_i)_i$ and $\big(|r_i - t_i|\big)_i$ are strictly decreasing. The sequence $\big(|r_i - t_i|\big)_i$ reaches 0 in a finite number N of steps, so the algorithm terminates.

Moreover, when $i = N$, we have $r_N = t_N$ so in the end we get a point of the type $[m:n:m]$, with $0 \le m < n$ by (II).

Furthermore, Lemma 4 ensures that $\gcd(r-t, s) = \gcd(m-m, n) = n$ and $r, t \equiv m \mod (n)$.

At each step the algorithm uses the action of one elementary braid σ_i, so one can recover a braid β_p sending the starting point $[r:s:t]$ to the representative $[m:n:m]$. This braid can be expressed as

$$\beta_p = \sigma_1^{-a_{2k+1}}\sigma_3^{c_{2k+1}}\sigma_2^{b_{2k}} \cdots \sigma_1^{-a_3}\sigma_3^{c_3}\sigma_2^{b_2}\sigma_1^{-a_1}\sigma_3^{c_1}\sigma_2^{b_0},$$

where $b_0 = \delta_{s,0}$ and the a_i's and c_i's are uniquely defined by the algorithm. □

Remark 8. The algorithm does not take into account the fact that $[m:n:m]$ is in the same orbit as $[n-m:n:n-m]$. If we really want to have only the representatives of the form $[m:n:m]$ with $m < n/2$ in the end of the algorithm, we can just add $\sigma_1\sigma_2^2\sigma_3$ as a final step in case we reached the wrong representative.

Proposition 7 implies that the rational projective line $\mathbb{P}^2(\mathbb{Q})$ is indeed a union of the B_4-orbits $\mathrm{Orb}_{B_4}\big([m:n:m]\big)$. Theorem 1 is proved.

2.4. *Connection to multidimensional continued fractions*

Let us explain in which sense our braided Euclidean algorithm is a Jacobi–Perron type multidimensional continued fractions (MCF) algorithm. For a clear description of these algorithms, see [7,15].

Proposition 9. *Consider the following subsets of* $\mathbb{R}_+^3$:

$$\begin{aligned} I_0 &= \big\{(r,s,t) \,\big|\, s = \min(r,s,t)\big\}, \\ I_1 &= \big\{(r,s,t) \,\big|\, t < s \le r\big\}, \\ I_2 &= \big\{(r,s,t) \,\big|\, t < r < s\big\}, \\ I_2' &= \big\{(r,s,t) \,\big|\, r < t < s\big\}, \\ I_3 &= \big\{(r,s,t) \,\big|\, r < s \le t\big\}. \end{aligned}$$

Then the braided Euclidean algorithm is an

$$\big(\{I_0, I_1, I_2, I_2', I_3\}, \{\rho_4(\sigma_1\sigma_3^{-1}), \rho_4(\sigma_1), \rho_4(\sigma_2), \rho_4(\sigma_2^{-1}), \rho_4(\sigma_3^{-1})\}\big)$$

MCF algorithm.

Proof. It is just a restatement of the steps of the braided Euclidean algorithm. □

2.5. *Stabilizers of the points* $[m:n:m]$.

In this subsection, we give a proof of Theorem 2.

Given a point $[r:s:t] \in \mathbb{P}^2(\mathbb{Q})$, its stabilizer (for the action given by ρ) is a subgroup of B_4:

$$\mathrm{Stab}_{[r:s:t]} \subset B_4.$$

We describe stabilizers of the representative points $[m:n:m]$ of each orbit. The braid Torelli group $\mathscr{B}\mathscr{I}_4 = \ker(\rho)$ is obviously a subgroup of every stabilizer. Recall from (2) that it is normally generated by τ_1^2, τ_3^2 and Δ.

Theorem 2. *Let $n \in \mathbb{N}^*$ and $0 \le m < n$ coprime to n. Then*

$$\mathrm{Stab}_{[m:n:m]}/\mathscr{B}\mathscr{I}_4 = \begin{cases} \langle \tau_1\Delta, \sigma_2\rangle & \text{if } n \ge 3,\\ \langle \tau_1\Delta, \sigma_2, \sigma_1\sigma_2^2\sigma_3\rangle & \text{if } n = 2,\\ \langle \tau_1, \Delta, \sigma_2\rangle & \text{if } n = 1.\end{cases}$$

Proof. For convenience, column vectors of $\mathscr{M}_{3,1}(\mathbb{Q})$ will be denoted in line: (u, v, w). Let $\beta \in \mathrm{Stab}_{[m:n:m]}$, and let $M = \rho(\beta)$:

$$M = \begin{pmatrix} a & e & b\\ x & f & y\\ c & g & d\end{pmatrix}.$$

Since the action of B_4 fixes $(1,0,1)$, one has

$$b = 1-a, \qquad d = 1-c, \qquad y = -x.$$

By definition, β stabilizes $[m:n:m]$, so either $M(m,n,m) = (m,n,m)$, or $M(m,n,m) = (-m,-n,-m)$.

First, let us suppose that $M(m,n,m) = (m,n,m)$. Then the eigenspace associated to 1 has dimension more than 2, and $(0,1,0) = \frac{1}{n}(m,n,m) - m(1,0,1)$ belongs to this space. Therefore $e = g = 0$ and $f = 1$. Moreover $\det(M) = 1$ implies that $c = a-1$. The matrix M is then

$$M = \begin{pmatrix} a & 0 & 1-a\\ x & 1 & -x\\ a-1 & 0 & 2-a\end{pmatrix}.$$

Multiplying β by σ_2^x to the left, we can suppose that $x = 0$. It is straightforward to check that for all $a \in \mathbb{Z}$,

$$\rho\big((\tau_1\Delta)^{a-1}\big) = \begin{pmatrix} a & 0 & 1-a\\ 0 & 1 & 0\\ a-1 & 0 & 2-a\end{pmatrix},$$

so the matrix M is in the group generated by $\rho(\sigma_2)$ and $\rho(\tau_1\Delta)$.

Now, let us suppose that $M(m,n,m) = (-m,-n,-m)$. This means

$$\begin{cases} ma + ne + m(1-a) = -m\\ nf = -n\\ mc + ng + m(1-c) = -m\end{cases}$$

so $e = g = -2\frac{m}{n}$, and $f = -1$. Moreover, the matrix M has 1 and -1 as eigenvalues, and $\det(M) = 1$, therefore the eigenspace associated to -1 must have dimension 2. Taking the trace, we get $-1 = \mathrm{Tr}(M) = a + f + 1 - c$, so $c = a + 1$. As before, we can multiply β by σ_2^x to the left to get $x = 0$, so

$$M = \begin{pmatrix} a & -2m/n & 1-a \\ 0 & -1 & 0 \\ a+1 & -2m/n & -a \end{pmatrix}.$$

Notice that $M(0,1,0) = \left(-2\frac{m}{n}, -1, -2\frac{m}{n}\right)$, so $\rho(\beta)\big([0:1:0]\big) = [2m:n:2m]$. The points $[0:1:0]$ and $[2m:n:2m]$ are in the same orbit only if $n = 2$ or $n = 1$. So if $n \geq 3$, this case is impossible.

If $n = 2$, then $m = 1$ and

$$M = \begin{pmatrix} a & -1 & 1-a \\ 0 & -1 & 0 \\ a+1 & -1 & -a \end{pmatrix} = \rho\big((\tau_1\Delta)^{-a-1}\sigma_2(\sigma_1\sigma_2^2\sigma_3)\big).$$

Finally, if $n = 1$, then $m = 0$, and

$$M = \begin{pmatrix} a & 0 & 1-a \\ 0 & -1 & 0 \\ a+1 & 0 & -a \end{pmatrix} = \rho\big(\tau_1(\tau_1\Delta)^{a+1}\big),$$

and this completes the proof. □

3. Topology and geometry of orbits

In this section, we investigate the way the orbits fill the rational projective plane. In particular, we highlight the symmetries of the orbits, that are carried by an affine property of the principal orbit $\mathcal{O}_1$. Finally, we outline an experimental dominance property of the principal orbit that distinguishes it from the other orbits.

3.1. *Topology of orbits*

Note that the standard embedding into the real projective plane $\mathbb{P}^2(\mathbb{Q}) \subset \mathbb{RP}^2$ equips $\mathbb{P}^2(\mathbb{Q})$ with the natural topology induced by the Euclidean norm in $\mathbb{R}^3$. We will use this topology to study the question of density of orbits.

Proposition 10. *Each orbit, except for the singleton orbit $\{[1:0:1]\}$, is dense in $\mathbb{P}^2(\mathbb{Q})$.*

Proof. Let us show that the orbit of $[m_0 : n_0 : m_0]$ is dense, for $n_0 \in \mathbb{N}^*$ and m_0 coprime to n_0 such that $m_0 < n_0$. It suffices to check that for each representative $[m:n:m]$ with $m < n/2$, there exists a sequence of points in $\mathrm{Orb}_{B_4}\big([m_0 : n_0 : m_0]\big)$ converging to $[m:n:m]$.

Consider the following sequence of points in $\mathbb{P}^2(\mathbb{Q})$

$$P_k := \left[m + \frac{n_0 + m_0 - m}{n_0 k + 1} : n + \frac{n_0 - n}{n_0 k + 1} : m + \frac{m_0 - m}{n_0 k + 1}\right] \xrightarrow[k \to +\infty]{} [m:n:m].$$

For all $k \in \mathbb{N}^*$,

$$\begin{aligned} P_k &= \big[(n_0 k + 1)m + n_0 + m_0 - m : (n_0 k + 1)n + n_0 - n : (n_0 k + 1)m + m_0 - m\big] \\ &= \big[n_0(km + 1) + m_0 : n_0(kn + 1) : n_0 km + m_0\big], \end{aligned}$$

with $n_0(km + 1) + m_0$, $n_0(kn + 1)$, $n_0 km + m_0$ mutually prime because m_0 and n_0 are coprime. And

$$\gcd\big(n_0(km + 1) + m_0 - (n_0 km + m_0), n_0(kn + 1)\big) = \gcd\big(n_0, n_0(kn + 1)\big) = n_0,$$

therefore the point P_k is in the orbit of $[m_0 : n_0 : m_0]$, for all $k \in \mathbb{N}^*$. □

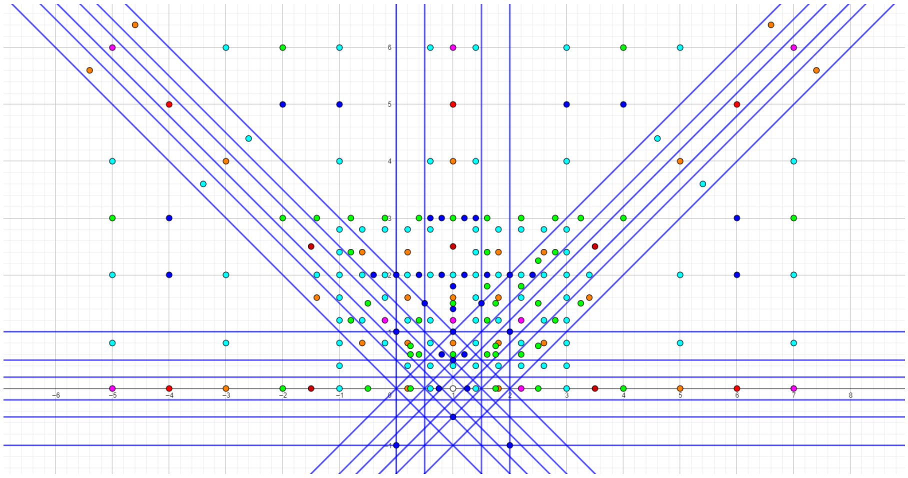

Figure 1. Sketch of the orbits in $\mathbb{P}^2(\mathbb{Q})$ with the affine chart $[r:s:1]$. Each color represents an orbit, dark blue is for the principal orbit.

Proposition 10 implies that the orbits (except for the singleton orbit $\{[1:0:1]\}$) are neither closed nor open.

3.2. *Symmetries of orbits*

We show that every orbit is stable by an action of the dihedral group D_4.

Proposition 11. *The orbits are stable under the action of the dihedral group D_4 acting on the plane with origin placed at* $(1,0)$.

Proof. The action of the dihedral group D_4 is generated by the rotation of $\pi/2$ around the point $(1,0)=[1:0:1]$ and the symmetry with respect to the horizontal axis. Therefore, it is enough to check that for every point $[x:y:1]$ the image by the rotation $[-y+1:x-1:1]$ and the image by the symmetry $[x:-y:1]$ are in the same orbit as $[x:y:1]$.

Let $x=a/b$ and $y=c/d$ be irreducible fractions, and let $\delta=\gcd(b,d)$, with $b=\delta b'$ and $d=\delta d'$, so that $[x:y:1]=\left[ad':b'c:d'b\right]$ with ad', $b'c$ and $d'b$ mutually prime. Then $[x:y:1]$ is in the orbit of the representative $[m:n:m]$ with $n=\gcd\left(a-b,c\right)$ and $m\equiv ad' \mod (n)$.

Now $[x:-y:1]=\left[ad':-b'c:d'b\right]$ so it is also in the orbit of $[m:n:m]$.

Similarly, $\left[-y+1:x-1:1\right]=\left[-bc'+bd':ad'-bd':bd'\right]$ with $\gcd\left(bc',ad'-bd'\right)=n$ and $-bc'+bd'\equiv bd'\equiv m \mod (n)$, so the point is again in the same orbit. □

3.3. *Affine lines in* $\mathscr{O}_1$

The principal orbit $\mathscr{O}_1$ has one particular property: among all orbits, it is the only one that contains infinitely many straight lines of $\mathbb{P}^2(\mathbb{Q})$, see Figure 1.

Proposition 12. *Let $r/s\in\mathbb{Q}$. The affine line of $\mathbb{Q}^2$ (embedded in $\mathbb{P}^2(\mathbb{Q})$ by $(x,y)\mapsto[x:y:1]$) having slope r/s and passing through the point $(0,c/d)$ is entirely in $\mathscr{O}_1$ if and only if*

$$cs_1+rd_1=\pm1$$

with $s_1=s/(s\wedge d)$ and $d_1=d/(s\wedge d)$.

Moreover, the only vertical lines that are in $\mathcal{O}_1$ are those of the form

$$D = \left\{ \left(\frac{r \pm 1}{r}, \lambda \right),\ \lambda \in \mathbb{Q} \right\} \quad \textit{with } r \in \mathbb{N}^*.$$

Proof. Let $c/d \in \mathbb{Q}$ in irreducible form. As $(0, c/d) \mapsto [0 : c : d]$ with $\gcd(c, d) = 1$, the point $(0, c/d)$ is already in $\mathcal{O}_1$. Let us suppose that the line passing through $(0, c/d)$ and of slope r/s is in $\mathcal{O}_1$. Then for all $\lambda \in \mathbb{Q}$, the point $\left(\lambda, r/s\lambda + c/d\right)$ is in $\mathcal{O}_1$. We have

$$\left[\lambda : \tfrac{r}{s}\lambda + \tfrac{c}{d} : 1\right] = \left[\alpha ds : r\alpha d + c\beta s : \beta ds\right] = \left[\alpha ds_1 : r\alpha d_1 + c\beta s_1 : \beta ds_1\right],$$

where $d_1 = d/(d \wedge s)$ and $s_1 = s/(d \wedge s)$. As the three coordinates of this point are not necessarily mutually prime, the fact that the point is in $\mathcal{O}_1$ means that

$$\gcd\left(\alpha ds_1, \alpha rd_1 + \beta cs_1, ds_1\beta\right) = \gcd\left(ds_1(\alpha - \beta), \alpha rd_1 + \beta cs_1\right),$$

or, since α and β are coprime,

$$\gcd\left(ds_1, \alpha rd_1 + \beta cs_1\right) = \gcd\left(ds_1(\alpha - \beta), \alpha rd_1 + \beta cs_1\right).$$

In particular, if we take $\alpha = \beta = 1$, then we get that $rd_1 + cs_1$ divides ds_1. Note that this means that $r \wedge c$ divides d or s, so actually $r \wedge c = 1$. Now take $\alpha = (c + d)s_1$ and $\beta = (s - r)d_1$, so that $\alpha rd_1 + \beta cs_1 = (rd_1 + cs_1)ds_1$. We get

$$ds_1 = \pm(rd_1 + cs_1)ds_1,$$

so $rd_1 + cs_1 = \pm 1$. For this argument to be valid, we need to check that the α and β chosen above are coprime. We check that β is coprime to $\alpha - \beta$, which is equivalent :

$$\begin{aligned}
\gcd\left(\alpha - \beta, \beta\right) &= \gcd\left(rd_1 + cs_1, (s - r)d_1\right) \\
&= \gcd\left(rd_1 + cs_1, s - r\right) && \text{because } d_1 \wedge cs_1 = 1 \\
&= \gcd\left(rd_1 + cs_1, r\right) && \text{because } rd_1 + cs_1 \text{ divides } s \\
&= \gcd\left(cs_1, r\right) \\
&= 1.
\end{aligned}$$

This concludes the argument.

Conversely, let us suppose that $rd_1 + cs_1 = \pm 1$. Let $\lambda = \alpha/\beta$ in irreducible form. Let us check the criteria for the point $\left[\alpha ds : r\alpha d + c\beta s : \beta ds\right] = \left[\alpha ds_1 : r\alpha d_1 + c\beta s_1 : \beta ds_1\right]$ to be in $\mathcal{O}_1$. First note that the greatest common divisor of αds_1, βds_1 and $r\alpha d_1 + c\beta s_1$ is $\gcd\left(ds_1, r\alpha d_1 + c\beta s_1\right)$. One has

$$\begin{aligned}
\gcd\left(\alpha ds_1 - \beta ds_1, r\alpha d_1 + c\beta s_1\right) &= \gcd\left((\alpha - \beta)ds_1, rd_1(\alpha - \beta) \pm \beta\right) && \text{because } rd_1 + cs_1 = \pm 1 \\
&= \gcd\left(ds_1, rd_1(\alpha - \beta) \pm \beta\right) \\
&= \gcd\left(ds_1, r\alpha d_1 + c\beta s_1\right).
\end{aligned}$$

Therefore, when divided by $\gcd(ds_1, r\alpha d_1 + c\beta s_1)$, we get that $\left[\alpha ds_1 : r\alpha d_1 + c\beta s_1 : \beta ds_1\right]$ is in $\mathcal{O}_1$, and so the entire line is in the principal orbit.

For the vertical lines, let us suppose that the line $\left\{(a/b, \lambda)\right\}$ is in $\mathcal{O}_1$. Then the point $[a/b : 0 : 1] = [a : 0 : b]$ is in $\mathcal{O}_1$, so $\gcd\left(a - b, 0\right) = 1$, meaning that $a - b = \pm 1$.

Conversely, it is straightforward to check that the line $\left\{((r \pm 1)/r, \lambda)\right\}$ is entirely in $\mathcal{O}_1$.

Proposition 12 is proved. □

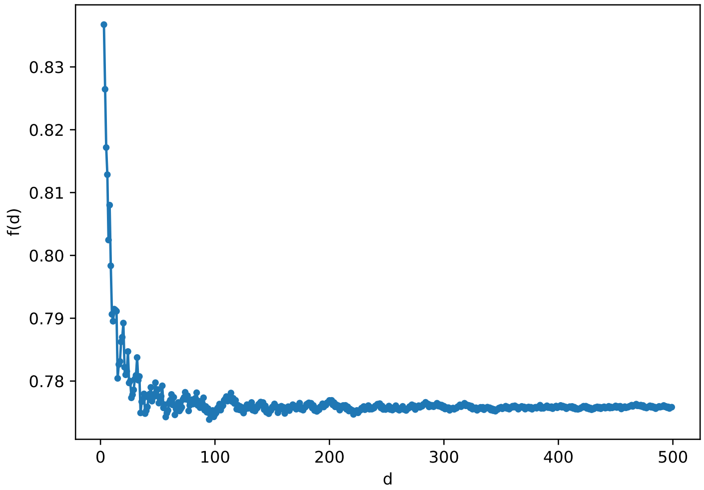

Figure 2. Partial rational density of $\mathcal{O}_1$ for $d = 1, \ldots, 500$.

3.4. *Experimental statistics: asymptotical growth of orbits*

Let us explain why in a certain sense, the principal orbit is much bigger than the others. Although we have no precise statement, computer experimentations clearly demonstrate this phenomenon.

Definition 13. *Let A be a subset of $\mathbb{Q}^2_{>0}$. Let $d \in \mathbb{N}^*$. Let S_d be the set of positive rationals with numerator and denominator lower that d (when written in irreducible form). We say that A has rational density α when:*

$$\frac{\sharp\left(A \cap S_d^2\right)}{\sharp S_d^2} \xrightarrow[d \to +\infty]{} \alpha.$$

Remark 14. As the natural density for positive integers, this rational density is a way to state the visual intuition that the principal orbit takes the major part of the projective plane (around 3/4 of the plane). This definition of rational density was inspired by the work done in [9].

The following conjecture relies on the exact computation of the values $\sharp\left(\mathcal{O}_1 \cap S_d^2\right)/\sharp S_d^2$ for $d \leq 500$, see Figure 2.

Conjecture 15. *The principal orbit $\mathcal{O}_1$ has rational density greater than* 0.75.

4. Quantization of $\mathbb{P}^2(\mathbb{Q})$

Now we can come back to the Burau representation ρ_q where q is a formal variable. The study of the case $q = 1$ in the previous section motivates us to focus on the principal orbit.

4.1. *Quantizing the principal orbit*

Definition 16. *By analogy with the case where $q = 1$, let us denote by $\mathcal{O}_q$ the orbit of the point $[0:1:0] \in \mathbb{P}^2\left(\mathbb{Z}(q)\right)$ under the action of B_4 via the reduced Burau representation ρ_q, and let us call it the quantized principal orbit.*

Remark 17. With this definition, a point $[r:s:t] \in \mathcal{O}_1$ has infinitely many quantizations, depending on the braid chosen to connect $[r:s:t]$ to $[0:1:0]$.

Given a braid β such that $\rho(\beta)\big([0:1:0]\big) = [r:s:t]$, the quantizations of the point $[r:s:t]$ are reached by $\rho_q\big(\beta\,\mathrm{Stab}_{[0:1:0]}\big)\big([0:1:0]\big)$. Yet, the generators computed in Proposition 2 have a trivial action on $[0:1:0]$ via ρ_q, as

$$\rho_q(\sigma_2) = \begin{pmatrix} 1 & 0 & 0 \\ -q & q & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho_q(\tau_1) = \begin{pmatrix} -q^3 & 0 & q+1 \\ 0 & -q^3 & -q^2+1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho_q(\Delta) = \begin{pmatrix} 0 & 0 & q \\ 0 & -q^2 & 0 \\ q^3 & 0 & 0 \end{pmatrix}.$$

Therefore, the split of the point $[r:s:t]$ in several quantizations actually comes from the braid Torelli group $\mathcal{BI}_4$. For instance, the braid $\sigma_3\tau_1^2\sigma_3^{-1}$ is quantized as

$$\rho_q(\sigma_3\tau_1^2\sigma_3^{-1}) = \begin{pmatrix} q^6 & q^5+q^4-q^2-q & -q^5-q^4+q^2+q \\ 0 & q^4+q^3-q & q^6-q^4-q^3+q \\ 0 & q^4+q^3-q-1 & q^6-q^4-q^3+q+1 \end{pmatrix}.$$

Definition 18. *The quantization map is a set-valued function defined by*

$$\begin{aligned} \mathcal{Q} : \mathcal{O}_1 &\longrightarrow \mathcal{P}\big(\mathbb{P}^2(\overline{\Lambda})\big) \\ p &\longmapsto \big\{\rho_q(\beta)\big([0:1:0]\big) \,\big|\, \beta \text{ s.t. } \rho(\beta)\big([0:1:0]\big) = p\big\}. \end{aligned}$$

For $p \in \mathcal{O}_1$, the image $\mathcal{Q}(p)$ is called the quantization *of p, and an element $[R:S:T]$ of $\mathcal{Q}(p)$ will be called a* deformation *of p.*

Remark 19. According to the previous remark, for $p \in \mathcal{O}_1$, if we denote by β_p the braid provided by the braided Euclidean algorithm such that $\rho(\beta_p)\big([0:1:0]\big) = p$, we have

$$\mathcal{Q}(p) = \big\{\rho_q(\beta_p\gamma)\big([0:1:0]\big) \,\big|\, \gamma \in \mathcal{BI}_4\big\}.$$

Example 20. Let us quantize the point $[7:18:14]$.

(i) Thanks to the braided Euclidean algorithm, we compute the braid $\beta = \sigma_2^2\sigma_1\sigma_3^{-3}\sigma_2^{-3}\sigma_1^3\sigma_3^{-2}$ such that $\rho(\beta)\big([0:1:0]\big) = [7:18:14]$. Then

$$\rho_q(\beta)\big([0:1:0]\big) = \big[R(q):S(q):T(q)\big],$$

with

$$\begin{aligned} R(q) &= q^9+2q^8+3q^7+3q^6+q^5-q^4-q^3-q^2, \\ S(q) &= -q^{11}-2q^{10}-2q^9+q^8+6q^7+11q^6+10q^5+4q^4-q^3-4q^2-3q-1, \\ T(q) &= q^8+3q^7+6q^6+6q^5+3q^4-2q^2-2q-1. \end{aligned}$$

Note that these polynomials have positive and negative coefficients, and the positive (resp. the negative) parts are unimodal.

(ii) Let us also compute $\rho_q\big(\beta\sigma_3\tau_1^2\sigma_3^{-1}\big)\big([0:1:0]\big) = \big[R'(q):S'(q):T'(q)\big]$:

$$\begin{aligned} R'(q) &= q^{15}+2q^{14}+3q^{13}+3q^{12}-3q^{10}-3q^9+3q^7+3q^6+q^5-q^4-q^3-q^2, \\ S'(q) &= -q^{17}-2q^{16}-2q^{15}+q^{14}+7q^{13}+13q^{12}+11q^{11}+q^{10}-9q^9-10q^8-2q^7+7q^6+9q^5 \\ &\qquad +4q^4-q^3-4q^2-3q-1, \\ T'(q) &= q^{14}+3q^{13}+6q^{12}+6q^{11}+2q^{10}-3q^9-5q^8-2q^7+3q^6+5q^5+3q^4-2q^2-2q-1. \end{aligned}$$

Again, these polynomials have positive and negative parts, and each part is unimodal.

4.2. *Proof of Theorem 3*

Let r and s be two coprime integers, with $s>0$, and such that

$$\mathscr{Q}\big([r:s]\big)=\big[R(q):S(q)\big]_q$$

with $R(q)$, $S(q)$ two coprime polynomials. Let us apply the braided Euclidean algorithm to the point $[r:s:0]$. As the third coordinate of the point is zero, the algorithm follows exactly the steps of the usual Euclidean algorithm, so the braid provided is

$$\beta=\sigma_1^{-a_{2k+1}}\sigma_2^{a_{2k}}\cdots\sigma_1^{-a_3}\sigma_2^{a_2}\sigma_1^{-a_1},$$

with the a_i's being the coefficients of the continued fraction:

$$\frac{r}{s}=a_1+\cfrac{1}{a_2+\cfrac{1}{\ddots+\cfrac{1}{a_{2k+1}}}}=[a_1,a_2,\dots,a_{2k+1}].$$

One of the quantizations of $[r:s:0]$ is $\rho_q(\beta^{-1})\cdot[0:1:0]$.

$$\rho_q(\beta^{-1})=\rho_q(\sigma_1)^{a_1}\rho_q(\sigma_2)^{-a_2}\cdots\rho_q(\sigma_1)^{a_{2k+1}}=\begin{pmatrix}M_q & \star\\ 0 & 1\end{pmatrix},$$

where M_q is the image of the 3-strand braid β by the Burau representation of B_3, because of the following commutative diagramme.

$$\begin{array}{ccc} B_3 & \xrightarrow{\ \rho_q\ } & \mathrm{GL}_2(\Lambda)\\ \sigma_i\mapsto\sigma_i\ \big\downarrow & & \big\downarrow\ M\mapsto\begin{pmatrix}M&0\\0&1\end{pmatrix}\\ B_4 & \xrightarrow[\ \rho_q\ ]{} & \mathrm{GL}_3(\Lambda)\end{array}$$

As a 3-strand braid, β sends $[0:1]$ on $[r:s]$, so the second column of M_q is the unique deformation of $[r:s]$ in the sense of [11], that is $\big(R(q),S(q)\big)$, and thus $\big[R(q):S(q):0\big]$ is among the deformations of the point $[r:s:0]$.

The case of the other natural embedding $[r:s]\mapsto[0:r:s]$ is exactly symmetric.

Theorem 3 is proved.

4.3. *Special specializations*

The obstruction to the unicity of quantization comes from the fact that the inclusion $\ker(\rho_q)\subset\mathscr{B}\mathscr{I}_4$ is strict (the braid Torelli group is "too big"). We investigate specializations of q at complex values for which the reverse inclusion holds.

Definition 21. *For $z\in\mathbb{C}^*$, let $\mathrm{ev}_z\colon\ \mathrm{PGL}_3(\overline{\Lambda})\to\mathrm{PGL}_3(\mathbb{C})$ be the evaluation map at $q=z$.*

Notation 22. Let $j:=\mathrm{e}^{\frac{2i\pi}{3}}$ be one of the third roots of unity.

Lemma 23. *The only complex values z for which $\mathscr{B}\mathscr{I}_4\subset\ker(\mathrm{ev}_z\,\rho_q)$ are $1,-1,j,j^2$.*

Proof. Recall that $\mathscr{B}\mathscr{I}_4$ is normally generated by τ_1^2, τ_3^2 and Δ^2 defined in (2), and that $\rho_q(\Delta^2)=q^4I$. We can compute for all $k\in\mathbb{N}$,

$$\rho_q(\tau_1^2)^k=\begin{pmatrix}q^{6k} & 0 & [k]_{q^6}f_1(q)\\ 0 & q^{6k} & [k]_{q^6}f_2(q)\\ 0 & 0 & 1\end{pmatrix},$$

$$\text{with }\begin{cases}f_1(q)=-q^4-q^3+q+1=-(q-1)(q+1)(q^2+q+1),\\ f_2(q)=q^5-q^3-q^2+1=(q-1)^2(q+1)(q^2+q+1).\end{cases}$$

Likewise,

$$\rho_q(\tau_3^2)^k = \begin{pmatrix} 1 & 0 & 0 \\ -[k]_{q^6} g_1(q) & q^{6k} & 0 \\ -[k]_{q^6} g_2(q) & 0 & q^{6k} \end{pmatrix},$$

$$\text{with } \begin{cases} g_1(q) = q^6 - q^4 - q^3 + q = q(q-1)^2(q+1)(q^2+q+1), \\ g_2(q) = q^6 + q^5 - q^3 - q^2 = -q^2(q-1)(q+1)(q^2+q+1). \end{cases}$$

If $q = z \in \{1, -1, j, j^2\}$, then $\mathrm{ev}_z\big(\rho_q(\tau_1^2)\big) = \mathrm{ev}_z\big(\rho_z(\tau_3^2)\big) = I$, so the whole normal group generated by τ_1^2, τ_3^2 and Δ^2 is in $\ker(\mathrm{ev}_z \rho_q)$.

Conversely, if $z \notin \{1, -1, j, j^2\}$, then $f_1(z) \neq 0$ so $\mathrm{ev}_z\big(\rho_q(\tau_1^2)\big) \neq I$, and $\mathscr{B}\mathscr{I}_4 \not\subset \ker(\mathrm{ev}_z \rho_q)$. □

Remark 24. For $q = 1$, the statement is obvious, as $\ker(\mathrm{ev}_1 \rho_q) = \ker(\rho) = \mathscr{B}\mathscr{I}_4$ by definition. For $q = -1$, we have $\ker(\mathrm{ev}_{-1} \rho_q) = P_4$, the pure braid group with 4 strands.

For $q = z$ a primitive third root of unity, the inclusion $\mathscr{B}\mathscr{I}_4 \subset \ker(\mathrm{ev}_z \rho_q)$ is strict. For instance, $\mathrm{ev}_z\big(\rho_q(\sigma_1^3)\big) = I$.

Definition 25. *Let $p = [r : s : t] \in \mathscr{O}_1$. The j-deformed point associated to p is defined by*

$$[r : s : t]_j := \mathrm{ev}_j\big(\rho_q(\beta)\big)\big([0:1:0]\big),$$

for any $\beta \in B_4$ such that $\rho(\beta)\big([0:1:0]\big) = p$.

Example 26. Let $p = [1 : 5 : 3]$. The braid $\beta = \sigma_2^2 \sigma_1 \sigma_3^{-3}$ satisfies $\rho(\beta)\big([0 : 1 : 0]\big) = p$. Then the j-analogue of p is

$$[1:5:3]_j = [1 : -j : 0].$$

Remark 27. During experimentations, we noticed that for every point $[r : s : t] \in \mathscr{O}_1$ we looked at, the j-analogue $[R : S : T] = [r : s : t]_j$ satisfies that R and T are either 0 or invertible in $\mathbb{Z}[j]$ (that is $\mathscr{N}(R) \leq 1$, $\mathscr{N}(T) \leq 1$) and that $\mathscr{N}(S) \in \{0, 1, 3\}$, where $\mathscr{N}$ denotes the norm of the ring of integers $\mathbb{Z}[j]$.

4.4. *Minimal unimodal quantization*

The aim of this paragraph is to choose one particular deformation of a point $p \in \mathscr{O}_1$ among the quantization $\mathscr{Q}(p)$.

Definition 28. *Let $R(q) \in \mathbb{Z}[q]$. We say that R is piecewise unimodal when its sequence of coefficients $(a_0, a_1, \ldots, a_n)$ is divided in subsequences of alternatively positive and negative coefficients $\big(+|a_0|, \ldots, +|a_{i_1}|\big), \big(-|a_{i_1+1}|, \ldots, -|a_{i_2}|\big), \ldots$, each subsequence being unimodal.*

Let $[R : S : T] \in \mathbb{P}^2(\overline{\Lambda})$, renormalized such that R, S and T are polynomials in q. We say that $[R : S : T]$ is fully piecewise unimodal when R, S and T are piecewise unimodal.

Example 29. The deformation of $[3:6:4]$ obtained via the braided Euclidean algorithm is

$$\big[q^5 + q^4 + q^3, -q^{10} - 2q^9 - 2q^8 - 2q^7 - q^6 + q^5 + 3q^4 + 4q^3 + 3q^2 + 2q + 1, q^3 + q^2 + q + 1\big].$$

This deformation is fully piecewise unimodal. In particular, in the second coordinate the sequence of coefficients is $\big(1, 2, 3, 4, 3, 1, -1, -2, -2, -2, -1\big)$, so it has two pieces and each is unimodal.

Remark 30. Some deformations of points in the principal orbit are not fully piecewise unimodal. For instance, in $\mathscr{Q}\big([21:29:11]\big)$, there is $\big[R(q) : S(q) : T(q)\big]$ with

$$S(q) = q^{13} + 3q^{12} + 6q^{11} + 8q^{10} + 8q^9 + 5q^8 + 2q^7 + q^6 + 2q^5 + 2q^4 - q^3 - 3q^2 - 3q - 2,$$

which is not piecewise unimodal.

Conjecture 31. *Let $p \in \mathscr{O}_1$. There is a unique deformation* $[R:S:T]$ *of p in $\mathscr{Q}(p)$ such that* $\deg(R)$, $\deg(S)$ *and* $\deg(T)$ *are minimal.*

Definition 32. *Assuming the conjecture above, we can define* the *quantization of a point $p \in \mathscr{O}_1$ to be the minimal (in degrees) deformation of p. If $p = [r:s:t]$, we denote this minimal deformation by $[r:s:t]_q$.*

Remark 33. This definition would match with the quantization of the projective rational line embedded in $\mathbb{P}^2(\mathbb{Q})$, because if $[r:s]_q = [R:S]$, then $[R:S:0]$ is minimal by unicity of the quantization of $[r:s]$. Moreover in this case, the deformation of $[r:s:0]$ is fully piecewise unimodal.

4.5. *Examples and experimentations*

To support our conjecture, we sum up some of the examples we computed.

Example 34 (Non unimodality). Using the braided Euclidean algorithm, we computed deformations of $[r:s:t]$ for all the triplets of nonnegative integers (r,s,t) (satisfying the condition $(r-t)\wedge s = 1$ to be in $\mathscr{O}_1$) bounded by 100. In this set of examples, only 1,518 (over 302,172) were not fully piecewise unimodular, the first one (for the lexicographic order) occuring for the triplet $(10,67,3)$, for which the polynomials are

$$R(q) = -q^{15} - 2q^{14} - q^{13} + q^{12} + 4q^{11} + 5q^{10} + 3q^9 + q^8,$$
$$S(q) = -q^{15} - 3q^{14} - 4q^{13} - 3q^{12} + q^{11} + 6q^{10} + 8q^9 + 8q^8 + 7q^7 + 8q^6 + 9q^5 + 10q^4 + 9q^3 + 7q^2 + 4q + 1,$$
$$T(q) = q^{10} + q^9 + q^8.$$

For every point whose deformation using the braided Euclidean algorithm was not fully piecewise unimodal, we found another deformation that is fully piecewise unimodal, by applying variations of the braided Euclidean algorithm. For instance, the braid $\beta = \sigma_2^{-10}\sigma_1^{-3}\sigma_3\sigma_2^2\sigma_1^{-1}$ satisfies $\rho(\beta)\big([0:1:0]\big) = [10:67:3]$, and we have $\rho_q(\beta)\big([0:1:0]\big) = \big[R'(q):S'(q):T'(q)\big]$ with

$$R'(q) = -q^{14} - 2q^{13} - 3q^{12} - 2q^{11} - q^{10} - q^9,$$
$$S'(q) = q^{15} + q^{14} - 3q^{12} - 5q^{11} - 6q^{10} - 7q^9 - 7q^8 - 7q^7 - 7q^6 - 7q^5 - 7q^4 - 6q^3 - 4q^2 - 2q - 1,$$
$$T'(q) = -q^{16} - q^{15} - q^{14},$$

which are piecewise unimodal.

However, we can find a third deformation of $[10:67:3]$ such that the degrees of the three coordinates are together minimal. Indeed, with $\beta'' = \sigma_2^{-9}\sigma_1^2\sigma_2^3\sigma_1^2\sigma_3^{-3}\sigma_2\sigma_3\tau_1^2\tau_3(\sigma_2\sigma_3)^2$, we get $\rho_q(\beta'') = [R'':S'':T'']$:

$$R''(q) = q^{12} + 2q^{11} + 3q^{10} + 3q^9 + q^8,$$
$$S''(q) = q^{12} + 3q^{11} + 6q^{10} + 8q^9 + 8q^8 + 7q^7 + 7q^6 + 7q^5 + 7q^4 + 6q^3 + 4q^2 + 2q + 1,$$
$$T''(q) = q^{10} + q^9 + q^8.$$

Examples 35 (Minimality of degrees). Let us focus on three significative examples, the points $[2:1:1]$, $[3:1:5]$ and $[21:29:11]$. For these three points, we computed many different deformations in their quantizations. Indeed, for a given point p, one can compute the braid given by the braided Euclidean algorithm, β_p, and then multiply by any element of $\mathscr{B}\mathscr{I}_4$ to get another deformation of the point.

Let us denote by $\sigma(a,b,c)$ the braid $\sigma_1^a\sigma_2^b\sigma_3^c$ with $a,b,c \in \mathbb{Z}$.

For each point p, we computed the deformations corresponding to the braids $\beta_p\gamma\tau\gamma^{-1}$, for $\tau\in\{\tau_1^2,\tau_1^2\tau_3^2\}$ and $\gamma\in\{\sigma(a_1,b_1,c_1)\sigma(a_2,b_2,c_2)\dots\sigma(a_N,b_N,c_N)\,|\,-4\le a_i,b_i,c_i<4,\ 0<N<3\}$. These parameters were chosen according to the computation power of our computer. With this method, we reach 16,514 braids.

(i) $p=[2:1:1]$

The braided Euclidean algorithm gives $\beta_p=\sigma_1^2\sigma_3^{-1}$, and the corresponding deformation is $[q+1:1:1]$. Among the 16,514 deformations we looked at, the deformation $[q+1:1:1]$ is minimal in degrees. There were 3,522 non fully piecewise unimodal deformations, the minimal one (in degrees) being

$$\begin{aligned}R(q)&=-q^{20}-4q^{19}-8q^{18}-10q^{17}-7q^{16}+7q^{14}+10q^{13}+11q^{12}+12q^{11}+11q^{10}+5q^9-3q^8\\&\quad-7q^7-6q^6-2q^5-q^4-2q^3-2q^2-q,\\S(q)&=-q^{19}-3q^{18}-5q^{17}-5q^{16}-2q^{15}+2q^{14}+4q^{13}+5q^{12}+7q^{11}+9q^{10}+7q^9-5q^7-6q^6\\&\quad-2q^5-q^3-2q^2-q,\\T(q)&=-q^{19}-4q^{18}-8q^{17}-10q^{16}-7q^{15}+6q^{13}+9q^{12}+11q^{11}+13q^{10}+12q^9+5q^8-3q^7-8q^6\\&\quad-6q^5-2q^4-q^3-2q^2-2q-1.\end{aligned}$$

(ii) $p=[3:1:5]$

We applied the same protocol to this second point. Here the braided Euclidean algorithm returns $\beta_p=\sigma_1^3\sigma_3^{-5}$, leading to the deformation $\left[q^6+q^5+q^4:q^4:q^4+q^3+q^2+q+1\right]$. There were 2,737 non fully piecewise unimodal deformations. However, we found that the braid $\beta'=\sigma_1^3\sigma_3^{-5}\sigma_2\sigma_3\tau_1^2\tau_3^2(\sigma_2\sigma_3)^2$ provides polynomials of lower degrees than with β_p, indeed

$$\rho_q(\beta')\big([0:1:0]\big)=\left[q^4+2q^3+q^2-1:q^3+q^2-1:q^3+2q^2+2q\right].$$

Therefore even if the braided Euclidean algorithm is efficient, it is not always the most efficient. It seems that $\left[q^4+2q^3+q^2-1:q^3+q^2-1:q^3+2q^2+2q\right]$ is the minimal deformation for $[3:1:5]$.

(iii) $p=[21:29:11]$

In this example, the braided Euclidean algorithm leads to a non fully piecewise unimodal deformation of p. In this example, there are 2,219 non fully piecewise unimodal deformations. The minimal deformation seems to be the one given by the braided Euclidean algorithm. The lowest deformation we found that is fully piecewise unimodal is

$$\begin{aligned}R(q)&=q^{14}+2q^{13}+4q^{12}+5q^{11}+5q^{10}+3q^9+q^8+q^6+2q^5+q^4-q^3-2q^2-q,\\S(q)&=q^{14}+3q^{13}+6q^{12}+8q^{11}+8q^{10}+5q^9+q^8-q^7+q^6+4q^5+3q^4-q^3-4q^2-4q-1,\\T(q)&=q^{12}+2q^{11}+3q^{10}+3q^9+2q^8-q^6+q^4+q^3-q.\end{aligned}$$

5. Discussion: open problems and future prospects

Several open problems and conjectures were spotted during experimentations with both actions by ρ and ρ_q on the projective rational plane. In this section, we briefly discuss the general status of the subject.

5.1. *Non-uniqueness of quantization*

In geometrical context, quantization usually leads to an extension of the quantized space. For instance, in geometric quantization the initial symplectic manifold increases its dimension by one and becomes a contact manifold. The canonical choice of the quantized rational numbers in [11] is due to the fact that the quantized space (the projective line) is one-dimensional in this case. However, even in this situation the quantization is not unique: the second, “left” quantization was developed in [1]. Non-uniqueness was also observed in the complex case [14]. The choice of a canonical representative of a quantized point $p \in \mathbb{P}^2(\mathbb{Q})$ is a challenging problem.

5.2. *Distribution of orbits*

The exact structure of the orbits of the action of B_4 on the rational projective plane is still mysterious. Based on large computations, we conjecture that the principal orbit takes around 3/4 of the plane, in the sense of Proposition 10. This experimental fact still remains to be proven.

Moreover, what happen for the other orbits is unknown, even if we suspect that the rational density decreases when n grows (where an orbit is represented by a point $[m : n : m]$). In particular, it would be interesting to study more precisely the group of symmetries for each orbit.

5.3. *Specialization at roots of unity*

The evaluation of q at a primitive third root of unity j is a particularly simple situation thanks to the inclusion $\mathscr{B}\mathscr{I}_4 \subset \ker(\mathrm{ev}_j \rho_q)$. It could be worth investigating whether evualuation at other roots of unity is far from the case of the third roots. The link between the theory of q-rationals and the Burau representation of B_3 was used with success in [12] to study the faithfulness of specializations of this representation. For the braid group B_4, we hope that our quantization could perform the same type of progress. The kernel of the specializations of Burau representation at roots of unity was studied in [5,6], following a paper of Squier [18].

5.4. *Almost unimodality*

Our computations of quantized points of the projective rational plane suggested that a large majority of the deformations are fully piecewise unimodal. We lack an explanation for this phenomenon, but it could be the sign that some deformations are better than others. The next step would be to find a combinatorial interpretation of these deformations, where unimodality would be derived from the combinatorial model.

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