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Convexity of space-like projections of submanifolds with co-dimension 2 in Lorentz–Minkowski space

Convexité des projections de sous-variétés de co-dimension 2 dans l'espace de Lorentz–Minkowski

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Abstract. In this paper, we give a necessary and sufficient condition that any space-like projection of a submanifold with co-dimension 2 in Lorentz–Minkowski space is locally strictly convex, and give its applications.

Résumé. Dans cet article, nous donnons une condition nécessaire et suffisante pour que toute projection d'une sous-variété de co-dimension 2 dans l'espace de Lorentz–Minkowski soit localement strictement convexe, et nous donnons ses applications.

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1. Introduction

Let $\mathbf{R}^{n+1}_1$ be Lorentz–Minkowski $(n+1)$-space with signature $(+\cdots+-)$ and set $\mathbf{R}^n_0 := \{(x,0)\,;\, x\in\mathbf{R}^n\}$ which can be identified with Euclidean n-space. We fix a connected $(n-1)$-manifold M^{n-1} $(n\geq 2)$ without boundary. It is well-known that an immersion $f\colon M^{n-1}\to\mathbf{R}^n_0$ is locally strictly convex at $p\in M^{n-1}$ if and only if the second fundamental form α^f_p of f at p is definite. The following fact is well-known:

Fact 1 ([3]). *If M^{n-1} $(n\geq 3)$ is compact and f is locally strictly convex at each point of M^{n-1}, then M^{n-1} is diffeomorphic to the $(n-1)$-sphere S^{n-1} and $f(M^{n-1})$ bounds a convex domain in $\mathbf{R}^n_0$.*

Let $F\colon N^m \to \mathbf{R}^{n+1}_1$ be an immersion defined on an m-manifold N^m. A point $p \in N^m$ is called *time-like*, *light-like* or *space-like* if the restriction of the canonical Lorentzian metric $\langle\cdot,\cdot\rangle$ of $\mathbf{R}^{n+1}_1$ to $\mathrm{d}F(T_pN^m)$ is Lorentzian, degenerate or Riemannian, respectively. A point $p \in N^m$ is said to be *cocausal* if it is not time-like. Moreover, the immersion F is said to be *cocausal* (resp. *space-like*) if all points on N^m are cocausal (resp. space-like). We denote by ∇ the Levi-Civita connection on $\mathbf{R}^{n+1}_1$. Let $F\colon M^{n-1} \to \mathbf{R}^{n+1}_1$ be an immersion and let $\mathscr{N}(p)$ $(\subset \mathbf{R}^{n+1}_1)$ be the normal space of F at $p \in M^{n-1}$. We set

$$\beta^F_{p,\mathbf{n}}(\mathbf{x}) := \left\langle \nabla_{\mathbf{x}} \mathrm{d}F(X), \mathbf{n} \right\rangle \qquad (\mathbf{x} \in T_pM^{n-1},\ \mathbf{n} \in \mathscr{N}(p)),$$

where X is a vector field of M^{n-1} satisfying $X_p = \mathbf{x}$. The right-hand side is independent of the choice of X. If p is not light-like, the second fundamental form α^F_p is well-defined as an $\mathscr{N}(p)$-valued bilinear form so that $\left\langle \alpha^F_p(\mathbf{x},\mathbf{x}), \mathbf{n} \right\rangle = \beta^F_{p,\mathbf{n}}(\mathbf{x})$ $(\mathbf{x} \in T_pM^{n-1},\ \mathbf{n} \in \mathscr{N}(p))$.

Definition 1. *We say that F is* essentially cocausal *at p if p is a cocausal point and each $\beta^F_{p,\mathbf{n}}(\mathbf{x})$ ($\mathbf{x} \in T_pM^{n-1} \setminus \{\mathbf{0}\}$) does not vanish for any choice of space-like vector $\mathbf{n} \in \mathscr{N}(p)$. In this setting, F is called an* essentially cocausal immersion *if it is essentially cocausal for each $p \in M^{n-1}$.*

One cannot replace "for any choice" with "for some choice": in fact, if $n = 2$ and $G(t) = (t + t^2, 0, t - t^2)$ $(t \in \mathbf{R})$, then $\beta^G_{p,\mathbf{n}}(\mathbf{x}_0) = \langle G''(0), \mathbf{n}\rangle$ $(\mathbf{x}_0 := \mathrm{d}/\mathrm{d}t\big|_{t=0})$ does not vanish when $\mathbf{n} = (1,1,1) \in \mathscr{N}(0)$ but does vanish if $\mathbf{n} = (0,1,0) \in \mathscr{N}(0)$, where $'$ means $\mathrm{d}/\mathrm{d}t$. We remark that, when $n = 2$, F is a regular curve in $\mathbf{R}^3_1$ which is essentially cocausal at $t = t_0$ if and only if the osculating plane $\mathscr{P}$ of F at $t = t_0$ exists (i.e. $F'(t_0)$ and $F''(t_0)$ are linearly independent) and the plane $\mathscr{P}$ is not time-like (in the case of G, the osculating plane $\mathscr{P}$ is time-like). At a space-like point, the following simplification is possible:

Proposition 2. *A cocausal immersion $F\colon M^{n-1} \to \mathbf{R}^{n+1}_1$ is essentially cocausal at a space-like point p if and only if $\alpha^F_p(\mathbf{x},\mathbf{x})$ does not vanish and is not time-like for each $\mathbf{x} \in T_pM^{n-1} \setminus \{\mathbf{0}\}$.*

Proof. Since $\mathbf{v} := \alpha^F_p(\mathbf{x},\mathbf{x})$ belongs to $\mathscr{N}(p)$, $\mathbf{v}$ is time-like if and only if there exists a space-like vector $\mathbf{n} \in \mathscr{N}(p)$ such that $\langle \mathbf{v}, \mathbf{n}\rangle$ $(= \beta^F_{p,\mathbf{n}}(\mathbf{x}))$ vanishes. Using this, one can easily obtain the assertion. □

Example 3. Consider a space-like immersion $F\colon M^{n-1} \to \mathbf{R}^{n+1}_1$ $(n \geq 4)$ into the light-cone Λ^n. The shape operator A satisfies $\langle A(\mathbf{x}), \mathbf{x}\rangle = -S(\mathbf{x},\mathbf{x})$ $(\mathbf{x} \in TM^{n-1})$, where S is the Schouten tensor on the conformally flat manifold M^{n-1} (cf. [2]). By [5, (1.2)], $\left\langle \alpha^F_p(\mathbf{x},\mathbf{x}), \alpha^F_p(\mathbf{x},\mathbf{x}) \right\rangle = 2\langle \mathbf{x},\mathbf{x}\rangle S(\mathbf{x},\mathbf{x})$ holds. By Proposition 2, F is essentially cocausal if only if S is positive semi-definite on M^{n-1}.

If $F(M^{n-1})$ is a strictly convex hypersurface lying in a space-like hyperplane Π in $\mathbf{R}^{n+1}_1$, then it is clear that F and any small deformation of F in $\mathbf{R}^{n+1}_1$ towards the outside of Π are essentially cocausal. So, there are many examples of essentially cocausal immersions not lying in any hyperplane in $\mathbf{R}^{n+1}_1$. For a time-like vector $\mathbf{v} \in \mathbf{R}^{n+1}_1$, $\Pi(\mathbf{v}) := \left\{\mathbf{x} \in \mathbf{R}^{n+1}_1 ; \langle \mathbf{x}, \mathbf{v}\rangle = 0\right\}$ is a space-like hyperplane in $\mathbf{R}^{n+1}_1$. We denote by $\pi_{\mathbf{v}}\colon \mathbf{R}^{n+1}_1 \to \Pi(\mathbf{v})$ the orthogonal projection. In this paper, such a projection is called a *space-like projection*. Let $F\colon M^{n-1} \to \mathbf{R}^{n+1}_1$ be an immersion. We set $f_{\mathbf{v}} := \pi_{\mathbf{v}} \circ F$.

Theorem 4. *For $p \in M^{n-1}$, F is an essentially cocausal immersion at p if and only if $f_{\mathbf{v}}$ is a locally strictly convex immersion around p for any time-like vector $\mathbf{v}$ in $\mathbf{R}^{n+1}_1$.*

Proof. We fix a unit time-like vector $\mathbf{v}$ arbitrarily. Since we can write $f_{\mathbf{v}} = F + \langle F, \mathbf{v}\rangle \mathbf{v}$,

$$\mathrm{d}f_{\mathbf{v}}(X) = \mathrm{d}F(X) + \left\langle \mathrm{d}F(X), \mathbf{v}\right\rangle \mathbf{v} \qquad \text{and} \qquad \nabla_X \mathrm{d}f_{\mathbf{v}}(X) = \nabla_X \mathrm{d}F(X) + \left\langle \nabla_X \mathrm{d}F(X), \mathbf{v}\right\rangle \mathbf{v} \qquad (1)$$

hold for any vector field X of M^{n-1}. We can give the following construction $(*)$ of a unit normal vector field $\mathbf{n}$ of $f_{\mathbf{v}}$ for the time-like vector $\mathbf{v}$:

> We assume p is not time-like. Then, there exists a neighborhood U of p such that $\Pi(\mathbf{v}) \cap \mathcal{N}(q)$ $(q \in U)$ is one dimensional, since $\Pi(\mathbf{v})$ and $\mathcal{N}(q)$ have different causalities. So there exists a space-like normal vector field $\mathbf{n}$ on U which is perpendicular to $\mathbf{v}$. We set $\mathbf{n} := \mathbf{n}/\sqrt{\langle \mathbf{n}, \mathbf{n} \rangle}$. $(*)$

Then, $\mathbf{n}$ is a unit normal vector field of $f_{\mathbf{v}}$ in $\Pi(\mathbf{v})$ (cf. the first equality of (1)). We have

$$(h(\mathbf{x},\mathbf{x}) :=) \langle \nabla_{\mathbf{x}} df_{\mathbf{v}}(X), \mathbf{n}_p \rangle = \langle \nabla_{\mathbf{x}} dF(X), \mathbf{n}_p \rangle = \beta^F_{p,\mathbf{n}_p}(\mathbf{x}) \qquad (\mathbf{x} \in T_pM^{n-1}), \tag{2}$$

where X is a vector field on U satisfying $X_p = \mathbf{x}$. We remark that h can be identified with the second fundamental form of $f_{\mathbf{v}}$.

We assume the essential cocausality of F and fix a unit time-like vector $\mathbf{v}$. Then $\Pi(\mathbf{v})$ is a space-like hyperplane in $\mathbf{R}^{n+1}_1$. Since F is essentially cocausal at p, (2) with the fact that $\mathbf{n}_p$ is a space-like vector belonging to $\mathcal{N}(p)$ implies that $h(\mathbf{x},\mathbf{x}) \neq 0$ for each $\mathbf{x} \in T_pM^{n-1} \setminus \{\mathbf{0}\}$, proving the local strict convexity.

We then assume the local strict convexity. If p is time-like, then there exists $\mathbf{x} \in T_pM^{n-1}$ such that $\mathbf{v} := dF_p(\mathbf{x})$ is a time-like vector, and $\pi_{\mathbf{v}} \circ F$ is not an immersion at p, contradiction. So we may assume p is not time-like. We fix a space-like normal vector $\mathbf{n}_0 \in \mathcal{N}(p)$ arbitrarily. Let $\mathbf{n}_0^{\perp}$ be the vector subspace of $\mathbf{R}^{n+1}_1$ consisting of the vectors perpendicular to $\mathbf{n}_0$. Since $\mathbf{n}_0^{\perp}$ is time-like, there exists a time-like vector $\mathbf{v} \in \mathbf{n}_0^{\perp}$. By $(*)$, the vector $\mathbf{n}_0$ can be considered as a normal vector of $f_{\mathbf{v}}$ at p. Since $\mathbf{n}_0$ is space-like, (2) implies the essential cocausality of F. In fact, we have $0 \neq h(\mathbf{x},\mathbf{x}) = \beta_{p,\mathbf{n}_0}(\mathbf{x})$ for each $\mathbf{x} \in T_pM^{n-1} \setminus \{\mathbf{0}\}$. □

2. Applications

Proposition 5. *Suppose that $\Gamma\colon \mathbf{R} \to \mathbf{R}^3_1$ is an l-periodic $(l > 0)$ regular curve which is cocausal. Then for any future-pointing time-like vector $\mathbf{v} \in \mathbf{R}^3_1$, the map $\gamma_{\mathbf{v}}\colon [0,l] \ni t \mapsto \pi_{\mathbf{v}} \circ \Gamma(t) \in \Pi(\mathbf{v})$ is a closed regular curve on $\Pi(\mathbf{v})$. Moreover, the value ι_Γ of the rotation index of the map $\gamma_{\mathbf{v}}$ as a plane curve is independent of the choice of $\mathbf{v}$ (called the* rotation index *of Γ).*

Proof. Since $\gamma_{\mathbf{v}}$ is a regular closed curve in $\Pi(\mathbf{v})$, the rotation index $\iota_{\gamma_{\mathbf{v}}}$ depends continuously on $\mathbf{v}$ and we obtain the conclusion. □

Remark 6. In [4], the rotation index is defined for closed space-like regular curve in $\mathbf{R}^3_1$.

As an application of Theorem 4, we give the following:

Theorem 7. *Suppose that M^{n-1} $(n \geq 2)$ is compact and $F\colon M^{n-1} \to \mathbf{R}^{n+1}_1$ is an essentially cocausal immersion. If $n = 2$, we assume also that $\iota_F = \pm 1$. Then:*

(i) *M^{n-1} is diffeomorphic to the $(n-1)$-sphere S^{n-1} and F is an embedding.*
(ii) *$\pi_{\mathbf{v}} \circ F(M^{n-1})$ bounds a convex domain in $\Pi(\mathbf{v})$ for any time-like vector $\mathbf{v}$ of $\mathbf{R}^{n+1}_1$.*

Remark 8. In Ye and Ma [6], a space-like closed regular curve such that its osculating plane at each point is space-like is said to be "strong space-like", which implies essentially cocausality, but the converse is not true. When $n = 2$ and F is strong space-like, the theorem was shown in [6, Lemma 2.1].

Proof of Theorem 7. If $n \geq 3$, then $f_{\mathbf{v}} := \pi_{\mathbf{v}} \circ F$ is an embedding and $f_{\mathbf{v}}(M^{n-1})$ is the boundary of a convex domain by Fact 1, proving the theorem. So we may assume $n = 2$ and the rotation index of F is equal to ± 1. Then $f_{\mathbf{v}}$ is locally strictly convex with index ± 1 by Theorem 4 and Proposition 5. In particular, F is an embedding. □

Example 9. If we consider the map $F_0\colon \mathbf{R}/2\pi\mathbf{Z} \ni t \mapsto \big(\cos t, \sin t, (\sin 3t)/9\big) \in \mathbf{R}^3_1$, then it satisfies the assumption of Theorem 7. In fact, F_0' and F_0'' are linearly independent at each point of $\mathbf{R}/2\pi\mathbf{Z}$. Since $\langle F_0'(t), F_0'(t)\rangle = \big(17-\cos 6t\big)/18$ and $\langle F_0''(t), F_0''(t)\rangle = \cos^2(3t)$, F_0 is essentially cocausal. For a time-like vector $\mathbf{v} := (1,0,2)/\sqrt{3}$, the image of $\pi_{\mathbf{v}} \circ F_0\big([0,2\pi]\big)$ is a strictly convex curve in the plane $\Pi(\mathbf{v})$ as in Figure 1, right.

It is well-known that a compact conformally flat space-like hypersurface in the light-cone Λ^n is diffeomorphic to the $(n-1)$-sphere S^{n-1} (cf. [1]). By Example 3, we obtain the following:

Corollary 10. *Let $F\colon M^{n-1} \to \mathbf{R}^{n+1}_1$ $(n \geq 4)$ be a space-like immersion with positive semi-definite Schouten tensor S whose image is lying on the light-cone Λ^n in $\mathbf{R}^{n+1}_1$. If M^{n-1} is compact, then $\pi_{\mathbf{v}} \circ F(M^{n-1})$ is a boundary of a convex domain in $\Pi(\mathbf{v})$ for any time-like vector $\mathbf{v} \in \mathbf{R}^{n+1}_1$.*

For example, a section of Λ^n by a space-like hyperplane satisfies the assumption. If M^{n-1} is complete and the trace of S is bounded below by a positive constant c, then the Ricci tensor of M^{n-1} satisfies $\mathrm{Ric} \geq c\,ds^2$, where ds^2 is the first fundamental form. By Myers's Theorem with Corollary 10, M^{n-1} is diffeomorphic to S^{n-1}. In [1, Prop. 5.2], another condition for a given complete hypersurface in Λ^n to be compact is given.

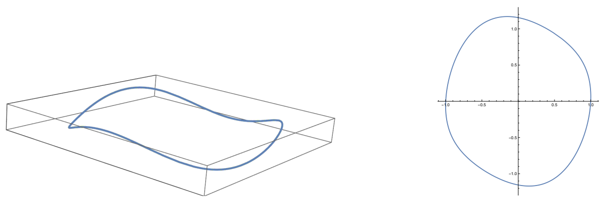

Figure 1. The image of F_0 and the image of $\pi_{\mathbf{v}} \circ F_0\big([0,2\pi]\big)$ for $\mathbf{v} := (1,0,2)/\sqrt{3}$.

Ye and Ma [6] showed the existence of a compact space-like maximal surface bounded by a given strong space-like curve of rotation index ± 1 embedded in $\mathbf{R}^3_1$. Moreover, they showed that such a surface is a graph over a convex domain, this last conclusion follows also from the following:

Corollary 11. *Let N^n $(n \geq 2)$ be a compact connected manifold with boundary M^{n-1}, and let $\Phi\colon N^n \to \mathbf{R}^{n+1}_1$ be a cocausal immersion. Suppose that the restriction $F := \Phi|_{M^{n-1}}$ is essentially cocausal. When $n = 2$, suppose also that $|\iota_F|$ is equal to one. Then M^{n-1} is diffeomorphic to S^{n-1}, and $\Phi(N^n)$ is a graph over the closed convex domain $\pi_{\mathbf{v}} \circ \Phi(N^n)$ for any choice of a time-like vector $\mathbf{v}$ of $\mathbf{R}^{n+1}_1$.*

Proof. By Theorem 7, $F(M^{n-1})$ bounds a convex domain $\mathscr{D}$ in $\Pi(\mathbf{v})$. Since Φ is cocausal, $\varphi := \pi_{\mathbf{v}} \circ \Phi\colon N^n \to \Pi(\mathbf{v})$ is an immersion. If there exists $p \in N^n \setminus M^{n-1}$ such that $\varphi(p)$ does not lie on $\mathscr{D}$, then there must exist $q \in N^n \setminus M^{n-1}$ giving the maximum of the continuous function $\delta(x) := d\big(\mathscr{D}, \varphi(x)\big)$ $(x \in N^n)$, where d is the canonical Euclidean distance function on $\Pi(\mathbf{v})$. Then q must be a critical point of the map φ, which contradicts that φ is an immersion. Thus, $\varphi\big(N^n \setminus M^{n-1}\big)$ is a subset of $\mathscr{D}$, and we can conclude that $\varphi(N^n)$ coincides with the closure $\overline{\mathscr{D}}$ in $\Pi(\mathbf{v})$. We then suppose that $\varphi^{-1}(y)$ is an infinite set for some $y \in \overline{\mathscr{D}}$. Since N^n is compact, we can find an accumulation point p of $\varphi^{-1}(y)$ in N^n. However, it contradicts the fact that φ is an immersion at p. So $\varphi^{-1}(y)$ is a finite set for each $y \in \overline{\mathscr{D}}$. Since N^n is connected and φ is an immersion, the cardinality of the set $\varphi^{-1}(y)$ does not depend on y. So we denote it by m. By Theorem 7, we know that $\varphi|_{M^{n-1}}$ is injective and M^{n-1} is diffeomorphic to S^{n-1}. In particular, we have $m = 1$, that is, φ is also an injection. Thus, $\Phi(N^n)$ is a graph over the closed domain $\overline{\mathscr{D}}$, proving the assertion. □

The authors hope that "essential cocausality" will be widely recognized by geometers and physicists as a new convexity for submanifolds of co-dimension 2. These statements are no longer expected when the ambient space is the Euclidean space (F_0 as in the left of Figure 1 into the xz-plane is not convex).

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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