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Stability estimates for solving Stokes problem with nonconforming finite elements

Estimations de stabilité pour résoudre le problème de Stokes avec des éléments finis non conformes

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Abstract. We propose to analyse the discretization of the Stokes problem with nonconforming finite elements in light of the T-coercivity. First we exhibit a family of operators to prove T-coercivity and we show that the stability constant is equal to the classical one up to a constant which depends on the Babuška–Aziz constant. Then we explicit the stability constants with respect to the shape regularity parameter for order 1 in 2 or 3 dimensions, and order 2 in 2 dimensions. In this last case, we improve the result of the original Fortin–Soulie paper. Second, we illustrate the importance of using a divergence-free velocity reconstruction on some numerical experiments.

Résumé. Nous proposons d'analyser la discrétisation du problème de Stokes avec des éléments finis non conformes à la lumière de la T-coercivité. Tout d'abord, pour prouver la T-coercivité, nous exhibons une famille d'opérateurs et nous montrons que la constante de stabilité est égale à la constante de stabilité classique, à une constante près qui dépend de la constante de Babuška–Aziz. Par la suite, nous explicitons les constantes de stabilité par rapport au paramètre de régularité de forme pour l'ordre 1 en dimension 2 ou 3, et l'ordre 2 en dimension 2. Dans ce dernier cas, nous améliorons le résultat de l'article original de Fortin–Soulie. Ensuite nous illustrons l'importance d'utiliser une méthode de projection conforme dans **H**(div) pour certaines expériences numériques.

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1. Introduction

The Stokes problem describes the steady state of incompressible Newtonian flows. It follows from the Navier–Stokes equations [32]. With regard to numerical analysis, the study of Stokes problem helps to build an appropriate approximation of the Navier–Stokes equations. We consider here a discretization with nonconforming finite elements [22,29]. We propose to state the discrete inf-sup condition in light of the T-coercivity (cf. [17] for Helmholtz-like problems, see [20,33,36] for the neutron diffusion equation), which allows to estimate the discrete error constant. In Section 2, we recall the T-coercivity theory [17], which is known to be an equivalent reformulation of the Banach–Nečas–Babuška Theorem and we apply it to the continuous Stokes problem. We give details on the triangulation, and we apply the T-coercivity to the discretization of Stokes problem with nonconforming mixed finite elements. For the Stokes problem, in the discrete case, this amounts to finding a Fortin operator. In Section 3, we precise the proof of the well-posedness in the case of order 1 and 2 nonconforming mixed finite elements. In Section 4, we illustrate the importance of using a divergence-free velocity on some numerical experiments.

2. Exact and discrete T-coercivity for Stokes problem

2.1. *T-coercivity and application to Stokes problem*

We recall here the T-coercivity theory as written in [17]. Consider first the variational problem, where V and W are two Hilbert spaces and $f \in V'$:

$$\text{Find } u \in V \text{ such that } \forall\, v \in W,\ a(u,v) = \langle f, v\rangle_V. \tag{1}$$

Classically, we know that Problem (1) is well-posed if $a(\cdot,\cdot)$ satisfies the stability and the solvability conditions of the so-called Banach–Nečas–Babuška (BNB) Theorem (see e.g. [27, Theorem 25.9]). For some models, one can also prove the well-posedness using the T-coercivity theory (cf. [17] for Helmholtz-like problems, see [20,33,36] for the neutron diffusion equation).

Definition 1. *Let V and W be two Hilbert spaces and $a(\cdot,\cdot)$ be a continuous and bilinear form over $V\times W$. It is T-coercive if*

$$\exists\, T \in \mathscr{L}(V,W),\ \textit{bijective},\ \exists\, \alpha_T > 0\ \forall\, v \in V,\ \left|a(v,Tv)\right| \geq \alpha_T \|v\|_V^2. \tag{2}$$

It is proved in [16,17] that the T-coercivity condition is equivalent to the stability and solvability conditions of the BNB Theorem. Whereas the BNB Theorem relies on an abstract inf-sup condition, T-coercivity uses explicit inf-sup operators, both at the continuous and discrete levels. Notice that if the pair (T,α_T) satisfies (2), then for all $\lambda>0$, the pair $(T_\lambda,\alpha_{T_\lambda}) := (\lambda T, \lambda\alpha_T)$ also satisfies (2). Thus, there exists an infinity of pairs (T,α_T) and $(T_\lambda,\alpha_{T_\lambda})_{\lambda>0}$ satisfying (2).

Theorem 2 (well-posedness). *Let $a(\cdot,\cdot)$ be a continuous bilinear form. Suppose that the form $a(\cdot,\cdot)$ is T-coercive. Then Problem* (1) *is well-posed.*

Let Ω be a connected bounded domain of $\mathbb{R}^d$, $d=2,3$, with a polygonal ($d=2$) or Lipschitz polyhedral ($d=3$) boundary $\partial\Omega$. We consider Stokes problem:

$$\text{Find } (\mathbf{u},p) \text{ such that } \begin{cases} -\nu\,\Delta\,\mathbf{u} + \mathbf{grad}\,p = \mathbf{f}, \\ \qquad\qquad \operatorname{div}\mathbf{u} = 0, \end{cases} \tag{3}$$

with Dirichlet boundary conditions for the velocity $\mathbf{u}$ and a normalization condition for the pressure p:

$$\mathbf{u} = 0 \ \text{ on } \ \partial\Omega,\ \int_\Omega p = 0.$$

The vector field $\mathbf{u}$ represents the velocity of the fluid and the scalar field p represents its pressure divided by the fluid density which is supposed to be constant. The first equation of (3) corresponds to the momentum balance equation and the second one corresponds to the conservation of the mass. The constant parameter $\nu > 0$ is the kinematic viscosity of the fluid. The vector field $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$ represents a body forces divided by the fluid density.

Before stating the variational formulation of Problem (3), we provide some definitions and reminders. Let us set $\mathbf{L}^2(\Omega) = \left(L^2(\Omega)\right)^d$, $\mathbf{H}_0^1(\Omega) = \left(H_0^1(\Omega)\right)^d$, $\mathbf{H}^{-1}(\Omega) = \left(H^{-1}(\Omega)\right)^d$ its dual space and $L^2_{\text{zmv}}(\Omega) = \left\{q \in L^2(\Omega) \,\middle|\, \int_\Omega q = 0\right\}$. We recall that $\mathbf{H}(\text{div};\Omega) = \left\{\mathbf{v} \in \mathbf{L}^2(\Omega) \,\middle|\, \operatorname{div}\mathbf{v} \in L^2(\Omega)\right\}$. Let h_Ω be the diameter of Ω. We recall the Poincaré–Steklov inequality [26, Lemma 3.24]:

$$\exists\, C_{\text{PS}} > 0 \ \text{ such that } \ \forall\, v \in H_0^1(\Omega),\ \|v\|_{L^2(\Omega)} \leq C_{\text{PS}}\, h_\Omega \|\mathbf{grad}\, v\|_{\mathbf{L}^2(\Omega)}. \tag{4}$$

Thanks to this result, in $H_0^1(\Omega)$, the semi-norm is equivalent to the natural norm, so that the scalar product reads $(v,w)_{H_0^1(\Omega)} = (\mathbf{grad}\, v, \mathbf{grad}\, w)_{\mathbf{L}^2(\Omega)}$ and the norm is $\|v\|_{H_0^1(\Omega)} = \|\mathbf{grad}\, v\|_{\mathbf{L}^2(\Omega)}$. Let $\mathbf{v},\mathbf{w} \in \mathbf{H}_0^1(\Omega)$. We denote by $(v_i)_{i=1}^d$ (resp. $(w_i)_{i=1}^d$) the components of $\mathbf{v}$ (resp. $\mathbf{w}$), and we set $\mathbf{Grad}\,\mathbf{v} = \left(\partial_j v_i\right)_{i,j=1}^d \in \mathbb{L}^2(\Omega)$, where $\mathbb{L}^2(\Omega) = \left(L^2(\Omega)\right)^{d\times d}$. We have:

$$\left(\mathbf{Grad}\,\mathbf{v}, \mathbf{Grad}\,\mathbf{w}\right)_{\mathbb{L}^2(\Omega)} = (\mathbf{v},\mathbf{w})_{\mathbf{H}_0^1(\Omega)} = \sum_{i=1}^d (v_i, w_i)_{H_0^1(\Omega)}$$

and

$$\|\mathbf{v}\|_{\mathbf{H}_0^1(\Omega)} = \left(\sum_{j=1}^d \|v_j\|^2_{H_0^1(\Omega)}\right)^{\frac{1}{2}} = \|\mathbf{Grad}\,\mathbf{v}\|_{\mathbb{L}^2(\Omega)}.$$

Let us set $\mathbf{V} = \left\{\mathbf{v} \in \mathbf{H}_0^1(\Omega) \,\middle|\, \operatorname{div}\mathbf{v} = 0\right\}$. The vector space $\mathbf{V}$ is a closed subset of $\mathbf{H}_0^1(\Omega)$. We denote by $\mathbf{V}^\perp$ the orthogonal of $\mathbf{V}$ in $\mathbf{H}_0^1(\Omega)$. We recall the following result.

Proposition 3 ([32, Corollary I.2.4]). *The operator* $\operatorname{div}\colon \mathbf{H}_0^1(\Omega) \to L^2(\Omega)$ *is an isomorphism of* $\mathbf{V}^\perp$ *onto* $L^2_{\mathrm{zmv}}(\Omega)$. *We call* C_{div} *the constant such that:*

$$\forall\, p \in L^2_{\mathrm{zmv}}(\Omega), \exists!\, \mathbf{v} \in \mathbf{V}^\perp \text{ such that } \operatorname{div}\mathbf{v} = p \text{ and } \|\mathbf{v}\|_{\mathbf{H}_0^1(\Omega)} \le C_{\mathrm{div}} \|p\|_{L^2(\Omega)}. \tag{5}$$

The constant C_{div} depends only on the domain Ω. Recall that we have

$$\|\mathbf{v}\|^2_{\mathbf{H}_0^1(\Omega)} = \|\mathbf{curl}\,\mathbf{v}\|^2_{\mathbf{L}^2(\Omega)} + \|\operatorname{div}\mathbf{v}\|^2_{L^2(\Omega)} \ge \|p\|^2_{L^2(\Omega)},$$

hence $C_{\mathrm{div}} \ge 1$. Using Proposition 3, we can define a lifting in $\mathbf{H}^1(\Omega)$ of a function $v \in L^2(\Omega)$:

Corollary 4. *For all* $v \in L^2(\Omega)$, *there exists* $\mathbf{s} \in \mathbf{H}^1(\Omega)$ *such that:*

$$\operatorname{div}\mathbf{s} = v \quad \text{and} \quad \|\mathbf{s}\|_{\mathbf{L}^2(\Omega)} + h_\Omega \|\mathbf{Grad}\,\mathbf{s}\|_{\mathbb{L}^2(\Omega)} \le C_\Omega h_\Omega \|v\|_{L^2(\Omega)}, \tag{6}$$

where the dimensionless constant C_Ω *depends on* C_{div} *and* C_{PS}.

Proof. Let $\underline{v} = \int_\Omega v/|\Omega|$, and $\mathbf{s}_0 \in \mathbf{H}_0^1(\Omega)$ be such that $\operatorname{div}\mathbf{s}_0 = v - \underline{v}$ and $\|\mathbf{Grad}\,\mathbf{s}_0\|_{\mathbb{L}^2(\Omega)} \le C_{\mathrm{div}} \|v\|_{L^2(\Omega)}$ (cf. Proposition 3). Let $d' \in \{1,\ldots,d\}$. We consider $\mathbf{s} := \mathbf{s}_0 + \underline{v}(x_{d'} - \underline{x}_{d'})\mathbf{e}_{d'} \in \mathbf{H}^1(\Omega)$, where $\underline{x}_{d'} = \int_\Omega x_{d'}/|\Omega|$. We have: $\operatorname{div}\mathbf{s} = v$ and $\|\mathbf{Grad}\,\mathbf{s}\|_{\mathbb{L}^2(\Omega)} \le \widetilde{C}_{\mathrm{div}} \|v\|_{L^2(\Omega)}$, where $\widetilde{C}_{\mathrm{div}} = C_{\mathrm{div}} + 1$. Using inequality (4), one can prove that $\|\mathbf{s}\|_{\mathbf{L}^2(\Omega)} \le h_\Omega \widetilde{C}_\Omega \|v\|_{L^2(\Omega)}$, where $\widetilde{C}_\Omega = C_{\mathrm{PS}}\, C_{\mathrm{div}} + 1$. Setting $C_\Omega = \widetilde{C}_\Omega + \widetilde{C}_{\mathrm{div}}$, we obtain (6). □

Actually, the constant C_{div} is such that $C_{\mathrm{div}} = 1/\beta(\Omega)$ where $\beta(\Omega)$, known as the Babuška–Aziz constant, is the inf-sup condition (or Ladyzhenskaya–Babuška–Brezzi condition):

$$\beta(\Omega) = \inf_{q \in L^2_{\mathrm{zmv}}(\Omega)\setminus\{0\}} \sup_{\mathbf{v} \in \mathbf{H}_0^1(\Omega)\setminus\{0\}} \frac{(q, \operatorname{div}\mathbf{v})_{L^2(\Omega)}}{\|q\|_{L^2(\Omega)} \|\mathbf{v}\|_{\mathbf{H}_0^1(\Omega)}}. \tag{7}$$

Generally, the value of $\beta(\Omega)$ is not known explicitly. In [6], Bernardi et al. established results on the discrete approximation of $\beta(\Omega)$ using conforming finite elements. Recently, Gallistl proposed in [30] a numerical scheme with adaptive meshes for computing approximations to $\beta(\Omega)$. In the case of $d = 2$, Costabel and Dauge [21] established the following bound:

Theorem 5 ([21, Theorem 2.3]). *Let* $\Omega \subset \mathbb{R}^2$ *be a domain contained in a ball of radius* R, *star-shaped with respect to a concentric ball of radius* ρ. *Then*

$$\beta(\Omega) \ge \frac{\rho}{\sqrt{2}R}\left(1 + \sqrt{1 - \frac{\rho^2}{R^2}}\right)^{-\frac{1}{2}} \ge \frac{\rho}{2R}. \tag{8}$$

Let us detail the bound for some remarkable domains. If Ω is a ball, $\beta(\Omega) \geq \frac{1}{2}$ and if Ω is a square, $\beta(\Omega) \geq \frac{1}{2\sqrt{2}}$. Suppose now that Ω is stretched in some direction by a factor k, then $\beta(\Omega) \geq \frac{1}{2k}$. Finally, if Ω is L-shaped (resp. cross-shaped) such that $L = kl$, where L is the largest length and l is the smallest length of an edge, then $\beta(\Omega) \geq \frac{1}{2\sqrt{2}k}$ (resp. $\beta(\Omega) \geq \frac{1}{4k}$).

The variational formulation of Problem (3) reads:

Find $(\mathbf{u}, p) \in \mathbf{H}_0^1(\Omega) \times L^2_{\text{zmv}}(\Omega)$ such that

$$\begin{cases} \nu(\mathbf{u},\mathbf{v})_{\mathbf{H}_0^1(\Omega)} - (p, \text{div}\,\mathbf{v})_{L^2(\Omega)} = \langle \mathbf{f}, \mathbf{v}\rangle_{\mathbf{H}_0^1(\Omega)} & \forall\, \mathbf{v} \in \mathbf{H}_0^1(\Omega), \\ (q, \text{div}\,\mathbf{u})_{L^2(\Omega)} = 0 & \forall\, q \in L^2_{\text{zmv}}(\Omega). \end{cases} \tag{9}$$

Classically, one proves that Problem (9) is well-posed using Poincaré–Steklov inequality (4) and Proposition 3. Check for instance the proof of [32, Theorem I.5.1].

Let us set $\mathscr{X} = \mathbf{H}_0^1(\Omega) \times L^2_{\text{zmv}}(\Omega)$, which is a Hilbert space which we endow with the following norm:

$$\left\|(\mathbf{v}, q)\right\|_{\mathscr{X}} = \left(\|\mathbf{v}\|^2_{\mathbf{H}_0^1(\Omega)} + \nu^{-2}\|q\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}. \tag{10}$$

We consider now the following bilinear symmetric and continuous form:

$$a_S: \begin{cases} \mathscr{X} \times \mathscr{X} \longrightarrow \mathbb{R} \\ (\mathbf{u}', p') \times (\mathbf{v}, q) \longmapsto \nu(\mathbf{u}', \mathbf{v})_{\mathbf{H}_0^1(\Omega)} - (p', \text{div}\,\mathbf{v})_{L^2(\Omega)} - (q, \text{div}\,\mathbf{u}')_{L^2(\Omega)}. \end{cases} \tag{11}$$

We can write Problem (3) in an equivalent way as follows:

$$\text{Find } (\mathbf{u}, p) \in \mathscr{X} \text{ such that } a_S\big((\mathbf{u}, p), (\mathbf{v}, q)\big) = \langle \mathbf{f}, \mathbf{v}\rangle_{\mathbf{H}_0^1(\Omega)} \ \forall\, (\mathbf{v}, q) \in \mathscr{X}. \tag{12}$$

Let us prove that Problem (12) is well-posed using the T-coercivity theory.

Proposition 6. *The bilinear form $a_S(\cdot,\cdot)$ is T-coercive:*

$$\begin{aligned} &\exists\, T \in \mathscr{L}(\mathscr{X}),\ \textit{bijective},\ \exists\, \alpha_T > 0, \forall\, (\mathbf{u}', p') \in \mathscr{X}, \\ &a_S\Big((\mathbf{u}', p'), T\big((\mathbf{u}', p')\big)\Big) \geq \alpha_T \left\|(\mathbf{u}', p')\right\|^2_{\mathscr{X}}. \end{aligned} \tag{13}$$

Proof. We follow here the proof given in [5,9,19]. Let us consider $(\mathbf{u}', p') \in \mathscr{X}$ and let us build $(\mathbf{v}^\star, q^\star) = T(\mathbf{u}', p') \in \mathscr{X}$ satisfying (2) (with $V = \mathscr{X}$). We need three main steps.

Step 1. According to Proposition 3, there exists $\widetilde{\mathbf{v}}_{p'} \in \mathbf{H}_0^1(\Omega)$ such that: $\text{div}\,\widetilde{\mathbf{v}}_{p'} = p'$ in Ω and $\|\widetilde{\mathbf{v}}_{p'}\|_{\mathbf{H}_0^1(\Omega)} \leq C_{\text{div}}\|p'\|_{L^2(\Omega)}$. Let us set $\mathbf{v}_{p'} = \nu^{-1}\widetilde{\mathbf{v}}_{p'}$ so that $\text{div}\,\mathbf{v}_{p'} = \nu^{-1}p'$ and

$$\|\mathbf{v}_{p'}\|_{\mathbf{H}_0^1(\Omega)} \leq \nu^{-1}C_{\text{div}}\|p'\|_{L^2(\Omega)}. \tag{14}$$

Let us set $(\mathbf{v}^\star, q^\star) := \big(\gamma\mathbf{u}' - \mathbf{v}_{p'}, -\gamma p'\big)$, with $\gamma > 0$. We obtain:

$$a_S\big((\mathbf{u}', p'), (\mathbf{v}^\star, q^\star)\big) = \nu\gamma\|\mathbf{u}'\|^2_{\mathbf{H}_0^1(\Omega)} + \nu^{-1}\|p'\|^2_{L^2(\Omega)} - \nu(\mathbf{u}', \mathbf{v}_{p'})_{\mathbf{H}_0^1(\Omega)}. \tag{15}$$

Step 2. In order to bound the last term of (15), we use Young inequality and then inequality (14), so that for all $\eta > 0$:

$$(\mathbf{u}', \mathbf{v}_{p'})_{\mathbf{H}_0^1(\Omega)} \leq \frac{\eta}{2}\|\mathbf{u}'\|^2_{\mathbf{H}_0^1(\Omega)} + \frac{\eta^{-1}}{2}\left(\frac{C_{\text{div}}}{\nu}\right)^2\|p'\|^2_{L^2(\Omega)}. \tag{16}$$

Step 3. Using the bound (16) in (15) and choosing $\eta = \gamma$, we get:

$$a_S\big((\mathbf{u}', p'), (\mathbf{v}^\star, q^\star)\big) \geq \nu\left(\frac{\gamma}{2}\|\mathbf{u}'\|^2_{\mathbf{H}_0^1(\Omega)} + \nu^{-2}\left(1 - \frac{\gamma^{-1}}{2}(C_{\text{div}})^2\right)\|p'\|^2_{L^2(\Omega)}\right).$$

Consider now $\gamma = (C_{\text{div}})^2$. We obtain:

$$a_S\big((\mathbf{u}', p'), (\mathbf{v}^\star, q^\star)\big) \geq \frac{1}{2}\nu\Big((C_{\text{div}})^2\|\mathbf{u}'\|^2_{\mathbf{H}_0^1(\Omega)} + \nu^{-2}\|p'\|^2_{L^2(\Omega)}\Big).$$

Reminding that $C_{\rm div} \geq 1$, it comes:

$$a_S\big((\mathbf{u}',p'),(\mathbf{v}^\star,q^\star)\big) \geq \frac{\nu}{2}\big\|(\mathbf{u}',p')\big\|^2_{\mathscr{X}}. \tag{17}$$

We obtain (13) with $\alpha_T = \frac{\nu}{2}$.

The operator T such that $T\big((\mathbf{u}',p')\big) = (\mathbf{v}^\star,q^\star)$ is linear and continuous. We have indeed:

$$\begin{aligned}\big\|T\big((\mathbf{u}',p')\big)\big\|^2_{\mathscr{X}} &:= \|\mathbf{v}^\star\|^2_{\mathbf{H}^1_0(\Omega)} + \nu^{-2}\|q^\star\|^2_{L^2(\Omega)}\\ &\leq 2\gamma^2\|\mathbf{u}'\|^2_{\mathbf{H}^1_0(\Omega)} + 2\|\mathbf{v}_{p'}\|^2_{\mathbf{H}^1_0(\Omega)} + \gamma^2\nu^{-2}\|p'\|^2_{L^2(\Omega)}\\ &\leq 2\gamma^2\|\mathbf{u}'\|^2_{\mathbf{H}^1_0(\Omega)} + (2(C_{\rm div})^2+\gamma^2)\nu^{-2}\|p'\|^2_{L^2(\Omega)}.\end{aligned}$$

We deduce that:

$$\big\|T\big((\mathbf{u}',p')\big)\big\|_{\mathscr{X}} \leq C_{\max}\big\|(\mathbf{u}',p')\big\|_{\mathscr{X}} \text{ where } C_{\max} = C_{\rm div}\Big(\max\big(2+(C_{\rm div})^2, 2(C_{\rm div})^2\big)\Big)^{\frac{1}{2}}. \tag{18}$$

Remark that, given $(\mathbf{v}^\star,q^\star)\in\mathscr{X}$, choosing $(\mathbf{u}',p') = \big(\gamma^{-1}\mathbf{v}^\star - \gamma^{-2}\mathbf{v}_{q^\star}, -\gamma^{-1}q^\star\big)$ yields $T\big((\mathbf{u}',p')\big) = (\mathbf{v}^\star,q^\star)$. Hence, the operator $T\in\mathscr{L}(\mathscr{X})$ is bijective. □

We can now prove the following result.

Theorem 7. *Problem* (12) *is well-posed. It admits one and only one solution such that:*

$$\forall\,\mathbf{f}\in\mathbf{H}^{-1}(\Omega),\ \begin{cases}\|\mathbf{u}\|_{\mathbf{H}^1_0(\Omega)} \leq \nu^{-1}\|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)},\\ \|p\|_{L^2(\Omega)} \leq C_{\rm div}\|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}.\end{cases} \tag{19}$$

Proof. According to Proposition 6, the continuous bilinear form $a_S(\cdot,\cdot)$ is T-coercive. Hence, according to Theorem 2, Problem (12) is well-posed. Let us prove (19). Consider $(\mathbf{u},p)$ the unique solution of Problem (12). Choosing $\mathbf{v}=0$, we obtain that $\forall\ q\in L^2_{\rm zmv}(\Omega)$, $(q,\operatorname{div}\mathbf{u})_{L^2(\Omega)} = 0$, so that $\mathbf{u}\in\mathbf{V}$. Now, choosing $\mathbf{v}=\mathbf{u}$ and using Cauchy–Schwarz inequality, we have: $\nu\|\mathbf{u}\|^2_{\mathbf{H}^1_0(\Omega)} = \langle\mathbf{f},\mathbf{u}\rangle_{\mathbf{H}^1_0(\Omega)} \leq \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}\|\mathbf{u}\|_{\mathbf{H}^1_0(\Omega)}$, so that: $\|\mathbf{u}\|_{\mathbf{H}^1_0(\Omega)} \leq \nu^{-1}\|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}$. Next, we choose in (12) $\mathbf{v} = \widetilde{\mathbf{v}}_p \in \mathbf{V}^\perp$, where $\operatorname{div}\widetilde{\mathbf{v}}_p = -p$ (see Proposition 3). Since $\mathbf{u}\in\mathbf{V}$ and $\widetilde{\mathbf{v}}_p\in\mathbf{V}^\perp$, we have $(\mathbf{u},\widetilde{\mathbf{v}}_p)_{\mathbf{H}^1_0(\Omega)} = 0$. This gives:

$$-(p,\operatorname{div}\widetilde{\mathbf{v}}_p)_{L^2(\Omega)} = \|p\|^2_{L^2(\Omega)} = \langle\mathbf{f},\widetilde{\mathbf{v}}_p\rangle_{\mathbf{H}^1_0(\Omega)} \leq \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}\|\widetilde{\mathbf{v}}_p\|_{\mathbf{H}^1_0(\Omega)} \leq C_{\rm div}\|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}\|p\|_{L^2(\Omega)},$$

so that: $\|p\|_{L^2(\Omega)} \leq C_{\rm div}\|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}$. □

2.2. *Comments on the stability constant*

Using (18) in (17), we have:

$$a_S\big((\mathbf{u}',p'),(\mathbf{v}^\star,q^\star)\big) \geq \alpha_T(C_{\max})^{-1}\big\|(\mathbf{u}',p')\big\|_{\mathscr{X}}\big\|(\mathbf{v}^\star,q^\star)\big\|_{\mathscr{X}}$$

Let us set $|||T||| := \sup_{(\mathbf{u}',p')\in\mathscr{X}\setminus(\mathbf{0},0)} \frac{\|T((\mathbf{u}',p'))\|_{\mathscr{X}}}{\|(\mathbf{u}',p')\|_{\mathscr{X}}}$. According to (18), we have the bound: $|||T||| \leq C_{\max}$.

Hence, we have:

$$a_S\big((\mathbf{u}',p'),(\mathbf{v}^\star,q^\star)\big) \geq \frac{\alpha_T}{|||T|||}\big\|(\mathbf{u}',p')\big\|_{\mathscr{X}}\big\|(\mathbf{v}^\star,q^\star)\big\|_{\mathscr{X}}$$

We recover the first Banach–Nečas–Babuška condition [27, Theorem 25.9, (BNB1)].

Thus, the T-coercivity approach gives an overestimate of the stability constant α given below:

$$\frac{\alpha_T}{|||T|||} \geq \alpha := \inf_{(\mathbf{v},q)\in\mathscr{X}\setminus(\mathbf{0},0)}\ \sup_{(\mathbf{u}',p')\in\mathscr{X}\setminus(\mathbf{0},0)} \frac{a_S\big((\mathbf{u}',p'),(\mathbf{v},q)\big)}{\big\|(\mathbf{u}',p')\big\|_{\mathscr{X}}\big\|(\mathbf{v},q)\big\|_{\mathscr{X}}}.$$

It suggests that the stability constant α is proportional to the parameter ν and depends on the constant $C_{\rm div}$, therefore on the shape of the domain. More precisely, our estimate gives:

$$\alpha \leq \frac{\nu}{2}\times\begin{cases}\left(C_{\rm div}\sqrt{2+(C_{\rm div})^2}\right)^{-1} & \text{if } 1\leq C_{\rm div}\leq\sqrt{2},\\ \left(\sqrt{2}(C_{\rm div})^2\right)^{-1} & \text{if } C_{\rm div}\geq\sqrt{2}.\end{cases}$$

In our computations, α_T depends on the choice of the parameters η and γ, so that it could be further optimized to minimize $\frac{\alpha_T}{|||T|||}$. Studying the bilinear form $a_S\Big((\mathbf{u}',p'),T\big((\mathbf{v},q)\big)\Big)$ leads to an alternative variational formulation of Stokes problem, as proposed in [19]. It does not depend on the parameters η and γ because it is coercive. However, the new variational formulation requires a specific treatment of the right-hand side.

2.3. *Conforming discretization and discrete well-posedness*

If we were using a conforming discretization to solve Problem (12) (e.g. Taylor–Hood finite elements [47]), we would use the bilinear form $a_S(\cdot,\cdot)$ to state the discrete variational formulation. Let us call the discrete spaces $\mathbf{X}_{{\rm c},h}\subset\mathbf{H}^1_0(\Omega)$ and $Q_{{\rm c},h}\subset L^2_{\rm zmv}(\Omega)$. Then to prove the discrete T-coercivity, we would need to state the discrete counterpart to Proposition 3. To do so, we can build a linear operator $\Pi_{\rm c}:\mathbf{X}\to\mathbf{X}_{{\rm c},h}$, known as Fortin operator, such that (see e.g. [8, §8.4.1]):

$$\exists\, C_{\rm c} \text{ such that } \forall\,\mathbf{v}\in\mathbf{H}^1(\Omega),\ \|\mathbf{Grad}\,\Pi_{\rm c}\mathbf{v}\|_{\mathbb{L}^2(\Omega)}\leq C_{\rm c}\|\mathbf{Grad}\,\mathbf{v}\|_{\mathbb{L}^2(\Omega)}, \tag{20}$$

$$\forall\,\mathbf{v}\in\mathbf{H}^1(\Omega),\ (\operatorname{div}\Pi_{\rm c}\mathbf{v},q_h)_{L^2(\Omega)}=(\operatorname{div}\mathbf{v},q_h)_{L^2(\Omega)},\ \forall\,q_h\in Q_{{\rm c},h}. \tag{21}$$

Using a nonconforming discretization, we will not use the bilinear form $a_S(\cdot,\cdot)$ to exhibit the discrete variational formulation, but we will need a similar operator to (20)-(21) to prove the discrete T-coercivity, which is stated in Theorem 16.

2.4. *Discretization notations*

We call $\big(O,(x_{d'})_{d'=1}^d\big)$ the Cartesian coordinates system, of orthonormal basis $(\mathbf{e}_{d'})_{d'=1}^d$. Consider $(\mathcal{T}_h)_h$ a simplicial triangulation sequence of Ω, where h denotes the mesh size. The triangulations are regular in the sense of Ciarlet. For a triangulation $\mathcal{T}_h$, we use the following index sets:

- $\mathcal{I}_K$ denotes the index set of the elements, such that $\mathcal{T}_h:=\bigcup_{\ell\in\mathcal{I}_K}K_\ell$ is the set of elements.
- $\mathcal{I}_F$ denotes the index set of the facets[1], such that $\mathcal{F}_h:=\bigcup_{f\in\mathcal{I}_F}F_f$ is the set of facets. Let $\mathcal{I}_F=\mathcal{I}_F^i\cup\mathcal{I}_F^b$, where $\forall\,f\in\mathcal{I}_F^i$, $F_f\subset\Omega$ and $\forall\,f\in\mathcal{I}_F^b$, $F_f\in\partial\Omega$.
- $\mathcal{I}_S$ denotes the index set of the vertices, such that $(S_j)_{j\in\mathcal{I}_S}$ is the set of vertices. Let $\mathcal{I}_S=\mathcal{I}_S^i\cup\mathcal{I}_S^b$, where $\forall\,j\in\mathcal{I}_S^i$, $S_j\in\Omega$ and $\forall\,j\in\mathcal{I}_S^b$, $S_j\in\partial\Omega$.

We also define the following index subsets:

- $\forall\,\ell\in\mathcal{I}_K$, $\mathcal{I}_{F,\ell}^{(i,b)}=\left\{f\in\mathcal{I}_F^{(i,b)}\,\middle|\,F_f\in K_\ell\right\}$, $\mathcal{I}_{S,\ell}=\left\{j\in\mathcal{I}_S\,\middle|\,S_j\in K_\ell\right\}$.
- $\forall\,j\in\mathcal{I}_S$, $\mathcal{I}_{K,j}=\left\{\ell\in\mathcal{I}_K\,\middle|\,S_j\in K_\ell\right\}$, $N_j:=\operatorname{card}(\mathcal{I}_{K,j})$.

For all $\ell\in\mathcal{I}_K$, we call h_ℓ the diameter of K_ℓ and ρ_ℓ the diameter of the sphere inscribed in K_ℓ, and we let: $\sigma_\ell=\frac{h_\ell}{\rho_\ell}$, $h=\max_{\ell\in\mathcal{I}_K}h_\ell$. When $(\mathcal{T}_h)_h$ is a shape-regular triangulation sequence (see e.g. [26, Definition 11.2]), there exists a constant $\sigma>1$, called the shape regularity parameter, such that for all h, for all $\ell\in\mathcal{I}_K$, $\sigma_\ell\leq\sigma$. For all $f\in\mathcal{I}_F$, M_f denotes the barycenter of F_f, and by $\mathbf{n}_f$ its unit normal (outward oriented if $F_f\in\partial\Omega$). For all $j\in\mathcal{I}_S$, for all $\ell\in\mathcal{I}_{K,j}$, $\lambda_{j,\ell}$ denotes the barycentric coordinate of S_j in K_ℓ, $F_{j,\ell}$ denotes the face opposite to vertex S_j in element

[1] The term facet stands for face (resp. edge) when $d=3$ (resp. $d=2$).

K_ℓ, and $\mathbf{x}_{j,\ell}$ denotes its barycenter. We call $\mathscr{S}_{j,\ell}$ the outward normal vector of $F_{j,\ell}$ and of norm $|\mathscr{S}_{j,\ell}| = |F_{j,\ell}|$.

Let us introduce spaces of piecewise regular elements.

We set $\mathscr{P}_h H^1 = \{v \in L^2(\Omega) \,|\, \forall\, \ell \in \mathscr{I}_K,\ v|_{K_\ell} \in H^1(K_\ell)\}$, endowed with the scalar product:

$$(v,w)_h := \sum_{\ell\in\mathscr{I}_K} (\mathbf{grad}\, v, \mathbf{grad}\, w)_{\mathbf{L}^2(K_\ell)}, \quad \|v\|_h^2 = \sum_{\ell\in\mathscr{I}_K} \|\mathbf{grad}\, v\|^2_{\mathbf{L}^2(K_\ell)}.$$

We set $\mathscr{P}_h\mathbf{H}^1 = \left(\mathscr{P}_h H^1\right)^d$, endowed with the scalar product:

$$(\mathbf{v},\mathbf{w})_h := \sum_{\ell\in\mathscr{I}_K} (\mathbf{Grad}\,\mathbf{v}, \mathbf{Grad}\,\mathbf{w})_{\mathbb{L}^2(K_\ell)}, \quad \|\mathbf{v}\|_h^2 = \sum_{\ell\in\mathscr{I}_K} \|\mathbf{Grad}\,\mathbf{v}\|^2_{\mathbb{L}^2(K_\ell)}.$$

Let $f \in \mathscr{I}_F^i$ such that $F_f = \partial K_L \cap \partial K_R$ and $\mathbf{n}_f$ is outward K_L oriented. The jump of $v \in \mathscr{P}_h H^1$ across the facet F_f is defined as follows: $[v]_{F_f} := v|_{K_L} - v|_{K_R}$. For $f \in \mathscr{I}_F^b$, we set: $[v]_{F_f} := v|_{F_f}$.

We set $\mathscr{P}_h\mathbf{H}(\mathrm{div}) = \{\mathbf{v} \in \mathbf{L}^2(\Omega) \,|\, \forall\, \ell \in \mathscr{I}_K, \mathbf{v}|_{K_\ell} \in \mathbf{H}(\mathrm{div}; K_\ell)\}$, and we define the operator div_h such that for all $\mathbf{v} \in \mathscr{P}_h\mathbf{H}(\mathrm{div})$, $\mathrm{div}_h\mathbf{v} \in L^2(\Omega)$ is such that:

$$\forall\, q \in L^2(\Omega),\ \left(\mathrm{div}_h\mathbf{v}, q\right)_{L^2(\Omega)} = \sum_{\ell\in\mathscr{I}_K} \left(\mathrm{div}\mathbf{v}, q\right)_{L^2(K_\ell)}.$$

We use the notation $A \lesssim B$ for $A \lesssim CB$ where A and B are scalar quantities and C is a generic positive constant which is independent of the sequence $(\mathscr{T}_h)_h$ and the quantities of interest.

We recall classical finite elements estimates [26]. Let $\widehat{K}$ be the reference simplex. For $\ell \in \mathscr{I}_K$, we denote by $T_\ell : \widehat{K} \to K_\ell$ an affine invertible mapping such that $T_\ell(\widehat{K}) = K_\ell$, $T_\ell(\partial\widehat{K}) = \partial K_\ell$. We set $T_\ell(\widehat{\mathbf{x}}) = \mathbb{B}_\ell\widehat{\mathbf{x}} + \mathbf{b}_\ell$, where $\mathbb{B}_\ell \in \mathbb{R}^{d\times d}$ and $\mathbf{b}_\ell \in \mathbb{R}^d$. Let $J_\ell = \det(\mathbb{B}_\ell)$. There holds:

$$|J_\ell| = d!|K_\ell|, \quad \|\mathbb{B}_\ell\| = \frac{h_\ell}{\rho_{\hat{K}}}, \quad \|\mathbb{B}_\ell{}^{-1}\| = \frac{h_{\hat{K}}}{\rho_\ell}. \tag{22}$$

Let $f \in \mathscr{I}_{F,\ell}$. According to [22, Equation (2.17)], we have:

$$|F_f||K_\ell|^{-1} \lesssim (\rho_\ell)^{-1}. \tag{23}$$

For $v \in L^2(K_\ell)$, we set $\widehat{v}_\ell = v \circ T_\ell$. Let $v \in \mathscr{P}_h H^1$. By changing the variable, $\mathbf{grad}\, v|_{K_\ell} = (\mathbb{B}_\ell{}^{-1})^T\, \mathbf{grad}_{\hat{\mathbf{x}}}\, \widehat{v}_\ell$, and it holds:

$$\|\mathbf{grad}\, v\|^2_{\mathbf{L}^2(K_\ell)} \lesssim \|\mathbb{B}_\ell{}^{-1}\|^2 |K_\ell| \|\mathbf{grad}_{\hat{\mathbf{x}}}\, \widehat{v}_\ell\|^2_{\mathbf{L}^2(\hat{K})}, \tag{24}$$

$$\|\mathbf{grad}_{\hat{\mathbf{x}}}\, \widehat{v}_\ell\|^2_{\mathbf{L}^2(\hat{K})} \lesssim \|\mathbb{B}_\ell\|^2\, |K_\ell|^{-1}\, \|\mathbf{grad}\, v\|^2_{\mathbf{L}^2(K_\ell)}. \tag{25}$$

We will use the following notations:

$$\forall\, \ell \in \mathscr{I}_K,\ \forall\, v \in L^2(K_\ell),\ \underline{v}_\ell = \int_{K_\ell} v/|K_\ell|, \tag{26}$$

$$\forall\, f \in \mathscr{I}_F,\ \forall\, v \in L^2(F_f),\ \underline{v}_f = \int_{F_f} v/|F_f|. \tag{27}$$

We recall the Poincaré–Steklov inequality in cells:

Proposition 8 ([26, Lemma 12.11]). *For all $\ell \in \mathscr{I}_K$ (K_ℓ is a convex set), for all $v \in H^1(K_\ell)$:*

$$\|v - \underline{v}_\ell\|_{L^2(K_\ell)} \leq \pi^{-1} h_\ell \|\mathbf{grad}\, v\|_{\mathbf{L}^2(K_\ell)}. \tag{28}$$

We will need the following Poincaré–Steklov inequality on faces:

Proposition 9 ([27, Lemma 36.8]). *For all $\ell \in \mathscr{I}_K$, for all $v \in H^1(K_\ell)$ and for all $f \in \mathscr{I}_{F,\ell}$, we have:*

$$\|v - \underline{v}_f\|_{L^2(F_f)} \lesssim \left(\frac{|F_f|}{|K_\ell|}\right)^{\frac{1}{2}} h_\ell \|\mathbf{grad}\, v\|_{\mathbf{L}^2(K_\ell)} \lesssim (\sigma_\ell)^{\frac{1}{2}} (h_\ell)^{\frac{1}{2}} \|\mathbf{grad}\, v\|_{\mathbf{L}^2(K_\ell)}. \tag{29}$$

Proof. We have: $v-\underline{v}_f=(v-\underline{v}_\ell)-|F|^{-1}\int_{F_f}(v-\underline{v}_\ell)$. Hence: $\|v-\underline{v}_f\|_{L^2(F_f)}\le 2\|v-\underline{v}_\ell\|_{L^2(F_f)}$. Changing the variable, using the continuity of the trace operator, we have: $\|v-\underline{v}_\ell\|_{L^2(F_f)}\lesssim |F_f|^{\frac12}\|\widehat{v}_\ell-\widehat{\underline{v}}_\ell\|_{H^1(\widehat{K})}$. Using (28) in $\widehat{K}$, we obtain $\|v-\underline{v}_\ell\|_{L^2(F_f)}\lesssim |F_f|^{\frac12}\|\mathbf{grad}_{\hat{\mathbf{x}}}\,\widehat{v}_\ell\|_{\mathbf{L}^2(\widehat{K})}$. Using (25) and applying (23), we get (29). □

For all $D\subset\mathbb{R}^d$, and $k\in\mathbb{N}$, we call $P^k(D)$ the set of order k polynomials on D, $\mathbf{P}^k(D)=\left(P^k(D)\right)^d$, and we consider the broken polynomial space:

$$P^k_{\mathrm{disc}}(\mathscr{T}_h)=\left\{q\in L^2(\Omega)\,\middle|\,\forall\,\ell\in\mathscr{I}_K,\ q|_{K_\ell}\in P^k(K_\ell)\right\},\quad \mathbf{P}^k_{\mathrm{disc}}(\mathscr{T}_h):=\left(P^k_{\mathrm{disc}}(\mathscr{T}_h)\right)^d.$$

2.5. *Nonconforming discretization and discrete well-posedness*

The nonconforming finite element method was introduced by Crouzeix and Raviart in [22] to solve Stokes problem (3). We approximate the vector space $\mathbf{H}^1(\Omega)$ component by component by piecewise polynomials of order $k\in\mathbb{N}^\star$. Let us consider X_h (resp. $X_{0,h}$), the space of nonconforming approximation of $H^1(\Omega)$ (resp. $H^1_0(\Omega)$) of order k:

$$\begin{aligned}X_h&=\left\{v_h\in P^k_{\mathrm{disc}}(\mathscr{T}_h)\,\middle|\,\forall\,f\in\mathscr{I}^i_F,\ \forall\,q_h\in P^{k-1}(F_f),\ \int_{F_f}[v_h]_{F_f}q_h=0\right\},\\ X_{0,h}&=\left\{v_h\in X_h\,\middle|\,\forall\,f\in\mathscr{I}^b_F,\ \forall\,q_h\in P^{k-1}(F_f),\ \int_{F_f}v_hq_h=0\right\}.\end{aligned}\tag{30}$$

The condition on the jumps of v_h on the inner facets is often called the patch-test condition. It allows to prove a discrete Poincaré–Steklov inequality, using Corollary 4, and the proof of [40, Theorem D.1]. The proof of [27, Lemma 36.6] is similar, but the vector $\mathbf{s}$ defined in Corollary 4 is constructed in [27, Lemma 36.6] as the gradient of a scalar function, so that it gives a lower estimate when Ω is nonconvex. Alternative proofs are given in [11,48].

Proposition 10. *The following discrete Poincaré–Steklov inequality holds:*

$$\forall\,v_h\in X_{0,h},\ \|v_h\|_{L^2(\Omega)}\lesssim\sigma C_\Omega h_\Omega\|v_h\|_h.\tag{31}$$

Proof. Let $v_h\in X_{0,h}$. According to Corollary 4, there exists $\mathbf{s}\in\mathbf{H}^1(\Omega)$ such that:

$$\operatorname{div}\mathbf{s}=v_h\qquad\text{and}\qquad\|\mathbf{s}\|_{\mathbf{L}^2(\Omega)}+h_\Omega\|\mathbf{Grad}\,\mathbf{s}\|_{\mathbb{L}^2(\Omega)}\le C_\Omega h_\Omega\|v_h\|_{L^2(\Omega)}\tag{32}$$

We have, by integration by parts:

$$\|v_h\|^2_{L^2(\Omega)}=\left(v_h,\operatorname{div}\mathbf{s}\right)_{L^2(\Omega)}=-\sum_{\ell\in\mathscr{I}_K}\left(\mathbf{grad}\,v_h,\mathbf{s}\right)_{\mathbf{L}^2(K_\ell)}+\sum_{\ell\in\mathscr{I}_K}\sum_{f\in\mathscr{I}_{F,\ell}}\left(v_h,\mathbf{s}\cdot\mathbf{n}_{f,\ell}\right)_{L^2(F_f)}.\tag{33}$$

The first term can be bounded as follows:

$$\left(\mathbf{grad}\,v_h,\mathbf{s}\right)_{\mathbf{L}^2(K_\ell)}\le\|\mathbf{grad}\,v_h\|_{\mathbf{L}^2(K_\ell)}\|\mathbf{s}\|_{\mathbf{L}^2(K_\ell)}.\tag{34}$$

Due to the patch-test, the second term reads:

$$\begin{aligned}\sum_{\ell\in\mathscr{I}_K}\sum_{f\in\mathscr{I}_{F,\ell}}\left(v_h,\mathbf{s}\cdot\mathbf{n}_{f,\ell}\right)_{L^2(F_f)}&=\sum_{\ell\in\mathscr{I}_K}\sum_{f\in\mathscr{I}_{F,\ell}}\left(v_h-\underline{v}_{h,f},(\mathbf{s}-\underline{\mathbf{s}}_f)\cdot\mathbf{n}_{f,\ell}\right)_{L^2(F_f)}\\&\le\sum_{\ell\in\mathscr{I}_K}\sum_{f\in\mathscr{I}_{F,\ell}}\left\|v_h-\underline{v}_{h,f}\right\|_{L^2(F_f)}\left\|(\mathbf{s}-\underline{\mathbf{s}}_f)\cdot\mathbf{n}_{f,\ell}\right\|_{L^2(F_f)}.\end{aligned}\tag{35}$$

Using inequality (29), we have:

$$\left\|v_h-\underline{v}_{h,f}\right\|_{L^2(F_f)}\left\|(\mathbf{s}-\underline{\mathbf{s}}_f)\cdot\mathbf{n}_{f,\ell}\right\|_{L^2(F_f)}\lesssim\sigma_\ell h_\ell\|\mathbf{grad}\,v_h\|_{\mathbf{L}^2(K_\ell)}\|\mathbf{Grad}\,\mathbf{s}\|_{\mathbb{L}^2(K_\ell)}.\tag{36}$$

Using (36) in (35), combining the result with (34), inequality (33) now reads:

$$\|v_h\|^2_{L^2(\Omega)} \lesssim \sum_{\ell\in\mathscr{I}_K} \|\mathbf{grad}\, v_h\|_{\mathbf{L}^2(K_\ell)}\Big(\|\mathbf{s}\|_{\mathbf{L}^2(K_\ell)} + \sigma_\ell h_\ell \|\mathbf{Grad}\,\mathbf{s}\|_{\mathbb{L}^2(K_\ell)}\Big)$$
$$\lesssim \sigma \sum_{\ell\in\mathscr{I}_K} \|\mathbf{grad}\, v_h\|_{\mathbf{L}^2(K_\ell)}\Big(\|\mathbf{s}\|_{\mathbf{L}^2(K_\ell)} + h_\Omega \|\mathbf{Grad}\,\mathbf{s}\|_{\mathbb{L}^2(K_\ell)}\Big).$$

We obtain (31) using the discrete Cauchy–Schwarz inequality and (32). □

As a consequence of Proposition 10, we have the following result.

Proposition 11. *The broken norm $v_h \to \|v_h\|_h$ is a norm over $X_{0,h}$.*

The space of nonconforming approximation of $\mathbf{H}^1(\Omega)$ (resp. $\mathbf{H}^1_0(\Omega)$) of order k is $\mathbf{X}_h = (X_h)^d$ (resp. $\mathbf{X}_{0,h} = (X_{0,h})^d$). We set $\mathscr{X}_h := \mathbf{X}_{0,h} \times Q_h$ where $Q_h = P^{k-1}_{\text{disc}}(\mathscr{T}_h) \cap L^2_{\text{zmv}}(\Omega)$. We deduce from Proposition 11 the following result.

Proposition 12. *The broken norm defined below is a norm on $\mathscr{X}_h$:*

$$\|(\cdot,\cdot)\|_{\mathscr{X}_h} : \begin{cases} \mathscr{X}_h \longrightarrow \mathbb{R} \\ (\mathbf{v}_h, q_h) \longmapsto \left(\|\mathbf{v}_h\|^2_h + \nu^{-2}\|q_h\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}} \end{cases}$$

Thus, the product space $\mathscr{X}_h$ endowed with the broken norm $\|(\cdot,\cdot)\|_{\mathscr{X}_h}$ is a Hilbert space. We consider the discrete continuous bilinear form $a_{S,h}(\cdot,\cdot)$ such that:

$$a_{S,h}: \begin{cases} \mathscr{X}_h \times \mathscr{X}_h \longrightarrow \mathbb{R} \\ (\mathbf{u}'_h, p'_h) \times (\mathbf{v}_h, q_h) \longmapsto \nu(\mathbf{u}'_h, \mathbf{v}_h)_h - \big(\text{div}_h\,\mathbf{v}_h, p'_h\big)_{L^2(\Omega)} - \big(\text{div}_h\,\mathbf{u}'_h, q_h\big)_{L^2(\Omega)} \end{cases}$$

Let us set $\mathbf{V}_h$ the discrete space of discrete divergence-free velocities :

$$\mathbf{V}_h := \big\{\mathbf{v}_h \in \mathbf{X}_{0,h} \,\big|\, \forall\, q_h \in Q_h, \big(\text{div}_h\,\mathbf{v}_h, q_h\big)_{L^2(\Omega)} = 0\big\}. \tag{37}$$

We recall that the velocities in $\mathbf{V}_h$ are piecewise divergence-free:

Proposition 13 ([13, Lemma 3.1]). *For all $\mathbf{v}_h \in \mathbf{V}_h$, for all $\ell \in \mathscr{I}_K$, $\text{div}\mathbf{v}_h\big|_{K_\ell} = 0$.*

Proof. Let $\mathbf{v}_h \in \mathbf{V}_h$. Integrating by parts and using the patch-test, we have:

$$(\text{div}_h\,\mathbf{v}_h, 1)_{L^2(\Omega)} = \sum_{\ell\in\mathscr{I}_K} \int_{K_\ell} \text{div}\mathbf{v}_h = \sum_{f\in\mathscr{I}_F} \int_{F_f} [\mathbf{v}_h]_{F_f} \cdot \mathbf{n}_f = 0.$$

Let $q_h \in P^{k-1}_{\text{disc}}(\mathscr{T}_h)$ and $\underline{q}_h = \int_\Omega q_h/|\Omega|$. Then $q_h - \underline{q}_h \in Q_h$ so that: $(\text{div}_h\,\mathbf{v}_h, q_h - \underline{q}_h)_{L^2(\Omega)} = 0$. Hence, we have: $(\text{div}_h\,\mathbf{v}_h, q_h)_{L^2(\Omega)} = (\text{div}_h\,\mathbf{v}_h, \underline{q}_h)_{L^2(\Omega)} = 0$. Let $\ell \in \mathscr{I}_K$. Let $q_h \in P^{k-1}_{\text{disc}}(\mathscr{T}_h)$ such that $q_h\big|_{K_\ell} = \text{div}\mathbf{v}_h\big|_{K_\ell}$ and for all $\ell' \in \mathscr{I}_K$, $\ell' \neq \ell$, $q_h\big|_{K_{\ell'}} = 0$. We have: $(\text{div}_h\,\mathbf{v}_h, q_h)_{L^2(\Omega)} = 0$ and $(\text{div}_h\,\mathbf{v}_h, q_h)_{L^2(\Omega)} = (\text{div}\mathbf{v}_h, q_h)_{L^2(K_\ell)} = \|\text{div}\mathbf{v}_h\|^2_{L^2(K_\ell)}$. Hence $\|\text{div}\mathbf{v}_h\|^2_{L^2(K_\ell)} = 0$. □

Let $\mathscr{J}_h : X_{0,h} \to Y_{0,h}$, with $Y_{0,h} = \big\{v_h \in H^1_0(\Omega) \,\big|\, \forall\, \ell \in \mathscr{I}_K, v_h\big|_{K_\ell} \in P^k(K_\ell)\big\}$ be the averaging operator described in [26, §22.4.1]. There exists a constant $C^{\text{nc}}_{\mathscr{J}_h} \approx \sigma$ and independent of h such that:

$$\forall\, v_h \in X_{0,h},\ \|\mathscr{J}_h v_h\|_{H^1_0(\Omega)} \le C^{\text{nc}}_{\mathscr{J}_h} \|v_h\|_h. \tag{38}$$

Let $\ell_{\mathbf{f}} \in \mathscr{L}(\mathscr{X}_h, \mathbb{R})$ be such that for all $(\mathbf{v}_h, q_h) \in \mathscr{X}_h$:

$$\ell_{\mathbf{f}}\big((\mathbf{v}_h, q_h)\big) = \begin{cases} (\mathbf{f}, \mathbf{v}_h)_{\mathbf{L}^2(\Omega)} & \text{if } \mathbf{f} \in \mathbf{L}^2(\Omega), \\ \langle \mathbf{f}, \mathscr{J}_h(\mathbf{v}_h)\rangle_{\mathbf{H}^1_0(\Omega)} & \text{if } \mathbf{f} \in \mathbf{H}^{-1}(\Omega). \end{cases}$$

The nonconforming discretization of Problem (12) reads:

$$\text{Find } (\mathbf{u}_h, p_h) \in \mathscr{X}_h \text{ such that } a_{S,h}\big((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)\big) = \ell_{\mathbf{f}}\big((\mathbf{v}_h, q_h)\big)\ \forall\, (\mathbf{v}_h, q_h) \in \mathscr{X}_h. \tag{39}$$

To prove that Problem (39) is well-posed, we will also use the T-coercivity theory. We do not need the well-posedness of the continuous problem, i.e. Proposition 6, but we will follow its proof, using a Fortin operator. This operator will be explained later, using the discrete basis functions. We will see that the discrete stability constant depends on this operator (hence polynomial order k).

Proposition 14. *Suppose that there exists a Fortin operator* $\Pi_{\rm nc} : \mathbf{H}^1(\Omega) \to \mathbf{X}_h$ *such that*

$$\exists\, C_{\rm nc} \text{ such that } \forall\, \mathbf{v} \in \mathbf{H}^1(\Omega),\ \|\Pi_{\rm nc}\mathbf{v}\|_h \le C_{\rm nc}\|\mathbf{Grad}\,\mathbf{v}\|_{\mathbb{L}^2(\Omega)}, \tag{40}$$

$$\forall\, \mathbf{v} \in \mathbf{H}^1(\Omega),\ (\mathrm{div}_h \Pi_{\rm nc}\mathbf{v}, q_h)_{L^2(\Omega)} = (\mathrm{div}\,\mathbf{v}, q_h)_{L^2(\Omega)},\ \forall\, q \in Q_h, \tag{41}$$

where the constant $C_{\rm nc}$ *does not depend on* h*. Then, the bilinear form* $a_{S,h}(\cdot,\cdot)$ *is* T*-coercive:*

$$\begin{aligned}&\exists\, T_h \in \mathscr{L}(\mathscr{X}_h),\ \text{bijective},\ \exists\, \alpha_{T_h} > 0, \forall\, (\mathbf{u}'_h, p'_h) \in \mathscr{X}_h,\\ &a_{S,h}\big((\mathbf{u}'_h, p'_h), T_h((\mathbf{u}'_h, p'_h))\big) \ge \alpha_{T_h}\big\|(\mathbf{u}'_h, p'_h)\big\|^2_{\mathscr{X}_h}.\end{aligned} \tag{42}$$

We will exhibit $C_{\rm nc}$ for $k = 1$, $d = 2,3$ in Section 3.1, obtaining $C_{\rm nc} = 1$; and then for $k = 2$, $d = 2$ in Section 3.3, obtaining $C_{\rm nc} = \sigma^2 + 1$. Let us prove Proposition 14.

Proof. We follow the proof of Proposition 6. Let us consider $(\mathbf{u}'_h, p'_h) \in \mathscr{X}_h$ and let us build $(\mathbf{v}^\star_h, q^\star_h) \in \mathscr{X}_h$ satisfying (2) (with $V = \mathscr{X}_h$). We need three main steps.

Step 1. According to Proposition 3, there exists $\widetilde{\mathbf{v}}_{p'_h} \in \mathbf{V}^\perp$ such that $\mathrm{div}\widetilde{\mathbf{v}}_{p'_h} = p'_h$ in Ω and $\|\widetilde{\mathbf{v}}_{p'_h}\|_{\mathbf{H}^1_0(\Omega)} \le C_{\rm div}\|p'_h\|_{L^2(\Omega)}$. Let us set $\mathbf{v}_{p'_h} = \nu^{-1}\widetilde{\mathbf{v}}_{p'_h}$. Consider $\mathbf{v}_{h,p'_h} = \Pi_{\rm nc}\mathbf{v}_{p'_h}$, for all $q_h \in Q_h$ we have:

$$\big(\mathrm{div}_h \mathbf{v}_{h,p'_h}, q_h\big)_{L^2(\Omega)} = \nu^{-1}(p'_h, q_h)_{L^2(\Omega)} \quad \text{and} \quad \|\mathbf{v}_{h,p'_h}\|_h \le \nu^{-1}C^{\rm nc}_{\rm div}\|p'_h\|_{L^2(\Omega)} \tag{43}$$

where $C^{\rm nc}_{\rm div} = C_{\rm nc}C_{\rm div}$. Let us set $(\mathbf{v}^\star_h, q^\star_h) := \big(\gamma_{\rm nc}\mathbf{u}'_h - \mathbf{v}_{h,p'_h}, -\gamma_{\rm nc}p'_h\big)$, with $\gamma_{\rm nc} > 0$. We obtain:

$$a_{S,h}\big((\mathbf{u}'_h, p'_h), (\mathbf{v}^\star_h, q^\star_h)\big) = \nu\gamma_{\rm nc}\|\mathbf{u}'_h\|^2_h + \nu^{-1}\|p'_h\|^2_{L^2(\Omega)} - \nu(\mathbf{u}'_h, \mathbf{v}_{h,p'_h})_h. \tag{44}$$

Step 2. In order to bound the last term of (44), we use Young inequality and then inequality (43) so that for all $\eta_{\rm nc} > 0$:

$$\big(\mathbf{u}'_h, \mathbf{v}_{h,p'_h}\big)_h \le \frac{\eta_{\rm nc}}{2}\|\mathbf{u}'_h\|^2_h + \frac{\eta^{-1}_{\rm nc}}{2}\left(\frac{C^{\rm nc}_{\rm div}}{\nu}\right)^2\|p'_h\|^2_{L^2(\Omega)}. \tag{45}$$

Step 3. Using the bound (45) in (44) and choosing $\eta_{\rm nc} = \gamma_{\rm nc}$, we get:

$$a_{S,h}\big((\mathbf{u}'_h, p'_h), (\mathbf{v}^\star_h, q^\star_h)\big) \ge \nu\left(\frac{\gamma_{\rm nc}}{2}\|\mathbf{u}'_h\|^2_h + \nu^{-2}\left(1 - \frac{(\gamma_{\rm nc})^{-1}}{2}(C^{\rm nc}_{\rm div})^2\right)\|p'_h\|^2_{L^2(\Omega)}\right).$$

Consider now $\gamma_{\rm nc} = (C^{\rm nc}_{\rm div})^2$. We obtain:

$$a_{S,h}\big((\mathbf{u}'_h, p'_h), (\mathbf{v}^\star_h, q^\star_h)\big) \ge \alpha_T C^{\rm nc}_{\rm min}\big\|(\mathbf{u}'_h, p'_h)\big\|^2_{\mathscr{X}_h}, \text{ where } C^{\rm nc}_{\rm min} = \min\big((C^{\rm nc}_{\rm div})^2, 1\big).$$

We obtain (42) with $\alpha_{T_h} = \alpha_T C^{\rm nc}_{\rm min}$. Suppose that $C_{\rm nc} \ge 1$. Then $\alpha_{T_h} = \alpha_T = \frac{\nu}{2}$.

The operator T_h such that $T_h(\mathbf{u}'_h, p'_h) = (\mathbf{v}^\star_h, p^\star_h)$ is linear and continuous. We have indeed:

$$\big\|T_h(\mathbf{u}'_h, p'_h)\big\|^2_{\mathscr{X}_h} = \|\mathbf{v}^\star_h\|^2_h + \nu^{-2}\|q^\star_h\|^2_{L^2(\Omega)} \le (C^{\rm nc}_{\rm max})^2\big\|(\mathbf{u}'_h, p'_h)\big\|^2_{\mathscr{X}_h}$$

where $C^{\rm nc}_{\rm max} = C^{\rm nc}_{\rm div}\big(\max\big(2 + (C^{\rm nc}_{\rm div})^2, 2(C^{\rm nc}_{\rm div})^2\big)\big)^{\frac{1}{2}}$.

Remark that, given $(\mathbf{v}^\star_h, q^\star_h) \in \mathscr{X}_h$, choosing $(\mathbf{u}'_h, p'_h) = \big(\gamma^{-1}_{\rm nc}\mathbf{v}^\star_h - \gamma^{-2}_{\rm nc}\mathbf{v}_{h,q^\star_h}, -\gamma^{-1}_{\rm nc}q^\star_h\big)$ yields $T_h\big((\mathbf{u}'_h, p'_h)\big) = (\mathbf{v}^\star_h, q^\star_h)$. Hence, the operator $T_h \in \mathscr{L}(\mathscr{X}_h)$ is bijective. □

Remark 15. We recover the first Banach–Nečas–Babuška condition [27, Theorem 25.9, (BNB1)]:

$$a_{S,h}\big((\mathbf{u}'_h, p'_h), (\mathbf{v}^\star_h, q^\star_h)\big) \ge \alpha_{T_h}(C^{\rm nc}_{\rm max})^{-1}\big\|(\mathbf{u}'_h, p'_h)\big\|_{\mathscr{X}_h}\big\|(\mathbf{v}^\star_h, q^\star_h)\big\|_{\mathscr{X}_h}.$$

We can now prove the discrete counterpart of Theorem 7.

Theorem 16. *Suppose that there exists a Fortin operator* $\Pi_{\rm nc}\colon \mathbf{H}^1(\Omega)\to\mathbf{X}_h$ *satifying* (40)-(41). *Then Problem* (39) *is well-posed. It admits one and only one solution* $(\mathbf{u}_h,p_h)$ *such that:*

$$\begin{aligned} &\text{if } \mathbf{f}\in\mathbf{L}^2(\Omega): \|\mathbf{u}_h\|_h \lesssim \nu^{-1}C_0^{\rm nc}\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)},\ \|p_h\|_{L^2(\Omega)} \lesssim 2C_0^{\rm nc}C_{\rm div}^{\rm nc}\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)},\\ &\text{if } \mathbf{f}\in\mathbf{H}^{-1}(\Omega): \|\mathbf{u}_h\|_h \lesssim \nu^{-1}C_{\mathscr{J}_h}^{\rm nc}\|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)},\ \|p_h\|_{L^2(\Omega)} \lesssim 2C_{\mathscr{J}_h}^{\rm nc}C_{\rm div}^{\rm nc}\|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}, \end{aligned} \tag{46}$$

where $C_0^{\rm nc}=\sigma C_\Omega h_\Omega$.

Proof. Consider $(\mathbf{u}_h,p_h)$ the unique solution of Problem (39). Choosing $\mathbf{v}_h=0$, we obtain that $\mathrm{div}_h\,\mathbf{u}_h=0$. Let $\mathbf{f}\in\mathbf{L}^2(\Omega)$. Now, choosing $\mathbf{v}_h=\mathbf{u}_h$ in (39), using Cauchy–Schwarz inequality, we get that: $\|\mathbf{u}_h\|_h\le\nu^{-1}\sigma C_\Omega h_\Omega\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}$ using inequality (31). Consider $(\mathbf{v}_h,q_h)=(\mathbf{v}_{h,p_h},0)$ in (39), where $\mathbf{v}_{h,p_h}=\Pi_{\rm nc}\mathbf{v}_{p_h}$ is built as $\mathbf{v}_{h,p_h'}$ in Step 1 of the proof of Proposition 14, setting $p_h'=p_h$. Suppose that $\mathbf{f}\in\mathbf{L}^2(\Omega)$. Notice that $\nu^{-1}\|p_h\|^2_{L^2(\Omega)}=\nu(\mathbf{u}_h,\mathbf{v}_{h,p_h})_h-(\mathbf{f},\mathbf{v}_{h,p_h})_{\mathbf{L}^2(\Omega)}$. Using Cauchy–Schwarz inequality, we have: $\nu^{-1}\|p_h\|^2_{L^2(\Omega)}\le\nu\|\mathbf{u}_h\|_h\|\mathbf{v}_{h,p_h}\|_h+\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}\|\mathbf{v}_{h,p_h}\|_{\mathbf{L}^2(\Omega)}$. Using Poincaré–Steklov inequality (31), hypothesis (40), and the previous estimate on $\|\mathbf{u}_h\|_h$, we have:

$$\|p_h\|^2_{L^2(\Omega)}\lesssim 2\sigma C_\Omega h_\Omega\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}\|\mathbf{v}_{h,p_h}\|_h\lesssim 2\sigma C_\Omega h_\Omega C_{\rm div}^{\rm nc}\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}\|p_h\|_{L^2(\Omega)}.$$

Let $\mathbf{f}\in\mathbf{H}^{-1}(\Omega)$. We apply the same reasoning, using inequality (38). □

As a corollary of Theorem 16, the following a priori error estimates follow:

Corollary 17 ([22, Theorems 3, 4, 6], [29, Equation (47)]). *Under the assumption of Theorem 16, suppose that* $(\mathbf{u},p)\in\left(\mathbf{H}^{1+k}(\Omega)\cap\mathbf{H}^1_0(\Omega)\right)\times\left(H^k(\Omega)\cap L^2_{\rm zmv}(\Omega)\right)$, *we have the estimates:*

$$\|\mathbf{u}-\mathbf{u}_h\|_h\lesssim\sigma^l h^k\left(|\mathbf{u}|_{\mathbf{H}^{k+1}(\Omega)}+\nu^{-1}|p|_{H^k(\Omega)}\right), \tag{47}$$

$$\nu^{-1}\|p-p_h\|_{L^2(\Omega)}\lesssim\sigma^l h^k\left(|\mathbf{u}|_{\mathbf{H}^{k+1}(\Omega)}+\nu^{-1}|p|_{H^k(\Omega)}\right). \tag{48}$$

Suppose moreover that the domain Ω *is convex. Then we have:*

$$\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{L}^2(\Omega)}\lesssim\sigma^{2l}h^{k+1}\left(|\mathbf{u}|_{\mathbf{H}^{k+1}(\Omega)}+\nu^{-1}|p|_{H^k(\Omega)}\right). \tag{49}$$

The hidden constants depend on k but they don't depend on the mesh. The parameter σ *is the shape regularity parameter and the exponent* $l\in\mathbb{N}^\star$ *depends on* k. *When* $k=1$, $d=2,3$, *we have* $l=1$, *and when* $k=2$, $d=2$, *we have* $l=2$.

When Ω is not convex, the exponent on h in Equation (49) is equal to $k+s$ where $s\in]0,1[$ depends on Ω (cf. [27, Theorem 31.33]).

The main issue with nonconforming mixed finite elements is the construction of the basis functions. In a recent paper, Sauter explains such a construction in two dimensions [41, Theorem 1.3], and gives a bound to the discrete counterpart $\beta_{\mathscr{T}}(\Omega)$ of $\beta(\Omega)$ defined in (7):

$$\beta_{\mathscr{T}}(\Omega)=\inf_{q_h\in Q_h\setminus\{0\}}\ \sup_{\mathbf{v}_h\in\mathbf{X}_{0,h}\setminus\{0\}}\frac{(\mathrm{div}_h\mathbf{v}_h,q_h)_{L^2(\Omega)}}{\|q_h\|_{L^2(\Omega)}\|\mathbf{v}_h\|_h}\ge c_{\mathscr{T}}\left(\log(k+1)\right)^{-\alpha}, \tag{50}$$

where the parameter α is explicit and depends on k and on the mesh topology; and the constant $c_{\mathscr{T}}$ depends only on the shape-regularity of the mesh.

3. Examples of nonconforming discretization for Stokes problem

3.1. *Nonconforming Crouzeix–Raviart mixed finite elements for $k = 1$*

We study the lowest order nonconforming Crouzeix–Raviart mixed finite elements [22]. Let us consider X_{CR} (resp. $X_{0,\text{CR}}$), the space of nonconforming approximation of $H^1(\Omega)$ (resp. $H^1_0(\Omega)$) of order 1:

$$\begin{aligned} X_{\text{CR}} &= \left\{ v_h \in P^1_{\text{disc}}(\mathcal{T}_h) \,\middle|\, \forall\, f \in \mathcal{I}^i_F,\ \int_{F_f} [v_h]_{F_f} = 0 \right\}, \\ X_{0,\text{CR}} &= \left\{ v_h \in X_{\text{CR}} \,\middle|\, \forall\, f \in \mathcal{I}^b_F,\ \int_{F_f} v_h = 0 \right\}. \end{aligned} \tag{51}$$

The space of nonconforming approximation of $\mathbf{H}^1(\Omega)$ (resp. $\mathbf{H}^1_0(\Omega)$) of order 1 is $\mathbf{X}_{\text{CR}} = (X_{\text{CR}})^d$ (resp. $\mathbf{X}_{0,\text{CR}} = (X_{0,\text{CR}})^d$). We set $\mathcal{X}_{\text{CR}} := \mathbf{X}_{0,\text{CR}} \times Q_{\text{CR}}$ where $Q_{\text{CR}} = P^0_{\text{disc}}(\mathcal{T}_h) \cap L^2_{\text{zmv}}(\Omega)$.

We can endow X_{CR} with the basis $(\psi_f)_{f \in \mathcal{I}_F}$ such that:

$$\forall\, \ell \in \mathcal{I}_K,\ \psi_f\big|_{K_\ell} = \begin{cases} 1 - d\lambda_{i,\ell} & \text{if } f \in \mathcal{I}_{F,\ell}, \\ 0 & \text{otherwise,} \end{cases}$$

where $i \in \mathcal{I}_S$ is such that S_i is the vertex opposite to F_f in K_ℓ. We then have $\psi_f\big|_{F_f} = 1$, so that $[\psi_f]_{F_f} = 0$ if $f \in \mathcal{I}^i_F$, and for all $f, f' \in \mathcal{I}_F$, $f' \neq f$, $\int_{F_{f'}} \psi_f = 0$.

We have: $X_{\text{CR}} = \text{vect}\big((\psi_f)_{f \in \mathcal{I}_F}\big)$ and $X_{0,\text{CR}} = \text{vect}\big((\psi_f)_{f \in \mathcal{I}^i_F}\big)$.

The Crouzeix–Raviart interpolation operator π_{CR} for scalar functions is defined by:

$$\pi_{\text{CR}} : \begin{cases} H^1(\Omega) \longrightarrow X_{\text{CR}} \\ \qquad v \longmapsto \displaystyle\sum_{f \in \mathcal{I}_F} \pi_f v \psi_f \end{cases} \qquad \text{where } \pi_f v = \frac{1}{|F_f|} \int_{F_f} v.$$

Notice that $\forall\, f \in \mathcal{I}_F$, $\int_{F_f} \pi_{\text{CR}} v = \int_{F_f} v$. Moreover, the Crouzeix–Raviart interpolation operator preserves the constants, so that $\pi_{\text{CR}} \underline{v}_\Omega = \underline{v}_\Omega$ where $\underline{v}_\Omega = \int_\Omega v/|\Omega|$. We recall that for $k = 1$, the coefficient C_{nc} in (40) is equal to 1:

Lemma 18 ([3, Lemma 2]). *The Crouzeix–Raviart interpolation operator π_{CR} is such that:*

$$\forall\, v \in H^1(\Omega),\ \|\pi_{\text{CR}} v\|_h \leq \|\mathbf{grad}\, v\|_{\mathbf{L}^2(\Omega)}. \tag{52}$$

Proof. We have, integrating by parts twice and using Cauchy–Schwarz inequality:

$$\begin{aligned} \mathbf{grad}\, \pi_{\text{CR}} v\big|_{K_\ell} &= |K_\ell|^{-1} \int_{K_\ell} \mathbf{grad}\, \pi_{\text{CR}} v \\ &= |K_\ell|^{-1} \sum_{f \in \mathcal{I}_{F,\ell}} \int_{F_f} \pi_{\text{CR}} v \mathbf{n}_f \\ &= |K_\ell|^{-1} \sum_{f \in \mathcal{I}_{F,\ell}} \int_{F_f} v \mathbf{n}_f \\ &= |K_\ell|^{-1} \int_{K_\ell} \mathbf{grad}\, v, \\ \left|\mathbf{grad}\, \pi_{\text{CR}} v\big|_{K_\ell}\right| &\leq |K_\ell|^{-\frac{1}{2}} \left\|\mathbf{grad}\, v\right\|_{\mathbf{L}^2(K_\ell)} \\ \Rightarrow \quad \left\|\mathbf{grad}\, \pi_{\text{CR}} v\right\|^2_{\mathbf{L}^2(K_\ell)} &\leq \left\|\mathbf{grad}\, v\right\|^2_{\mathbf{L}^2(K_\ell)}. \end{aligned}$$

Summing these local estimates over $\ell \in \mathcal{I}_K$, we obtain (52). □

For a vector $\mathbf{v} \in \mathbf{H}^1(\Omega)$ of components $(v_{d'})^d_{d'=1}$, the Crouzeix–Raviart interpolation operator is such that: $\Pi_{\text{CR}} \mathbf{v} = \big(\pi_{\text{CR}} v_{d'}\big)^d_{d'=1}$. Let us set $\Pi_f \mathbf{v} = \big(\pi_f v_{d'}\big)^d_{d'=1}$.

Lemma 19. *The Crouzeix–Raviart interpolation operator* $\Pi_{\rm CR}$ *can play the role of the Fortin operator:*

$$\forall\, \mathbf{v} \in \mathbf{H}^1(\Omega),\ \|\Pi_{\rm CR}\mathbf{v}\|_h \le \|\mathbf{Grad}\,\mathbf{v}\|_{\mathbb{L}^2(\Omega)}, \tag{53}$$

$$\forall\, \mathbf{v} \in \mathbf{H}^1(\Omega),\ \left(\mathrm{div}_h\,\Pi_{\rm CR}\mathbf{v}, q_h\right)_{L^2(\Omega)} = \left(\mathrm{div}\,\mathbf{v}, q_h\right)_{L^2(\Omega)},\ \forall\, q \in Q_h. \tag{54}$$

Moreover, for all $\mathbf{v} \in \mathbf{P}^1(\Omega)$, $\Pi_{\rm CR}\mathbf{v} = \mathbf{v}$.

Proof. We obtain (53) applying Lemma 18 component by component. By integrating by parts, we have $\forall\, \mathbf{v} \in \mathbf{H}^1(\Omega)$, $\forall\, \ell \in \mathscr{I}_K$:

$$\int_{K_\ell} \mathrm{div}\,\Pi_{\rm CR}\mathbf{v} = \sum_{f \in \mathscr{I}_{F,\ell}} \int_{F_f} \Pi_{\rm CR}\mathbf{v}\cdot\mathbf{n}_f = \sum_{f \in \mathscr{I}_{F,\ell}} \int_{F_f} \Pi_f\mathbf{v}\cdot\mathbf{n}_f = \sum_{f \in \mathscr{I}_{F,\ell}} \int_{F_f} \mathbf{v}\cdot\mathbf{n}_f = \int_{K_\ell} \mathrm{div}\,\mathbf{v},$$

so that (54) is satisfied. □

We can apply the T-coercivity theory to show the following result:

Theorem 20. *Let* $\mathscr{X}_h = \mathscr{X}_{\rm CR}$. *Then the continuous bilinear form* $a_{S,h}(\cdot,\cdot)$ *is* T_h*-coercive and Problem* (39) *is well-posed.*

Proof. Using estimates (53) and (31), we apply the proof of Theorem 16. □

Since the constant of the interpolation operator $\Pi_{\rm CR}$ is equal to 1, we have $C_{\min}^{\rm nc} = \min\left((C_{\rm div})^2, 1\right) = 1$ and $C_{\max}^{\rm nc} = C_{\max}$: the stability constant of the nonconforming Crouzeix–Raviart mixed finite elements is independent of the mesh. This is not the case for higher order (see [14, Theorem 2.2]).

3.2. *Comments on higher-order methods*

For higher order, we cannot build the interpolation operator component by component, since higher-order divergence moments must be preserved. Thus, for $k > 1$, we must build $\Pi_{\rm nc}$ so that for all $\mathbf{v} \in \mathbf{H}^1(\Omega)$, for all $\ell \in \mathscr{I}_K$, for all $q \in P^{k-1}(K_\ell)$:

$$\int_{K_\ell} q\,\mathrm{div}\,\Pi_{\rm nc}\mathbf{v} = \int_{K_\ell} q\,\mathrm{div}\,\mathbf{v}.$$

We recall that by integration by parts, we have:

$$\int_{K_\ell} q\,\mathrm{div}\,\Pi_{\rm nc}\mathbf{v} + \int_{K_\ell} \mathbf{grad}\,q\cdot\Pi_{\rm nc}\mathbf{v} = \int_{\partial K_\ell} q\Pi_{\rm nc}\mathbf{v}\cdot\mathbf{n}|_{\partial K_\ell}. \tag{55}$$

Hence, to obtain a local estimate of $\left\|\mathbf{Grad}\,\Pi_{\rm nc}\mathbf{v}\right\|_{\mathbb{L}^2(K_\ell)}$, we will need the following Lemma:

Lemma 21. *Let* $\mathbf{v} \in \mathbf{H}^1(K_\ell)$ *and* $q \in P^{k-1}(K_\ell) \cap L^2_{\rm zmv}(K_\ell)$. *We have:*

$$\left|\int_{\partial K_\ell} q(\mathbf{v}-\underline{\mathbf{v}}_\ell)\cdot\mathbf{n}|_{\partial K_\ell}\right| \le \left(\sqrt{d}+1\right)\pi^{-1}h_\ell\left\|\mathbf{grad}\,q\right\|_{\mathbf{L}^2(K_\ell)}\left\|\mathbf{Grad}\,\mathbf{v}\right\|_{\mathbb{L}^2(K_\ell)} \tag{56}$$

Proof. We have by integration by parts, and then using Cauchy–Schwarz inequality:

$$\begin{aligned}\left|\int_{\partial K_\ell} q(\mathbf{v}-\underline{\mathbf{v}}_\ell)\cdot\mathbf{n}|_{\partial K_\ell}\right| &\le \left|\int_{K_\ell} q\,\mathrm{div}(\mathbf{v}-\underline{\mathbf{v}}_\ell)\right| + \left|\int_{K_\ell} \mathbf{grad}\,q\cdot(\mathbf{v}-\underline{\mathbf{v}}_\ell)\right| \\ &\le \sqrt{d}\|q\|_{L^2(K_\ell)}\left\|\mathbf{Grad}(\mathbf{v}-\underline{\mathbf{v}}_\ell)\right\|_{\mathbb{L}^2(K_\ell)} + \left\|\mathbf{grad}\,q\right\|_{\mathbf{L}^2(K_\ell)}\left\|(\mathbf{v}-\underline{\mathbf{v}}_\ell)\right\|_{\mathbf{L}^2(K_\ell)} \\ &\le \left(\sqrt{d}+1\right)\pi^{-1}h_\ell\left\|\mathbf{grad}\,q\right\|_{\mathbf{L}^2(K_\ell)}\left\|\mathbf{Grad}\,\mathbf{v}\right\|_{\mathbb{L}^2(K_\ell)},\ \text{using (28) twice.}\end{aligned}$$ □

In the next section, we will see that for $k=2$, $d=2$, we will need Lemma 21. For $k\geq 3$, it could be necessary to bound the tangential components of $\mathbf{v}-\underline{\mathbf{v}}_\ell$. To do so, we would need to preserve curl integrals on K_ℓ. Indeed, by integration by parts, we have:

$$\text{For } d=2,\ \mathbf{v}\in\mathbf{H}^1(\Omega),\ q\in P^{k-1}(K_\ell): \int_{K_\ell} q(\mathbf{curl}\, q\cdot\mathbf{v}-\mathrm{curl}\,\mathbf{v}q) = \int_{\partial K_\ell} q\mathbf{v}\times\mathbf{n}|_{\partial K_\ell}.$$
$$\text{For } d=3, \mathbf{v}\in\mathbf{H}^1(\Omega), \mathbf{w}\in\mathbf{P}^{k-1}(K_\ell): \int_{K_\ell}(\mathbf{w}\cdot\mathbf{curl}\,\mathbf{v}-\mathbf{curl}\,\mathbf{w}\cdot\mathbf{v}) = \int_{\partial K_\ell}(\mathbf{n}|_{\partial K_\ell}\times\mathbf{v}\times\mathbf{n}|_{\partial K_\ell})\cdot(\mathbf{w}\times\mathbf{n}|_{\partial K_\ell}).$$

3.3. *Fortin–Soulie mixed finite elements*

We consider here the case $k=2$, $d=2$ and we study the so-called Fortin–Soulie mixed finite elements [29]. We consider a shape-regular triangulation sequence $(\mathcal{T}_h)_h$.

Let us consider X_{FS} (resp. $X_{0,\mathrm{FS}}$), the space of nonconforming approximation of $H^1(\Omega)$ (resp. $H^1_0(\Omega)$) of order 2:

$$\begin{aligned} X_{\mathrm{FS}} &= \left\{ v_h\in P^2_{\mathrm{disc}}(\mathcal{T}_h) \,\middle|\, \forall f\in\mathcal{I}^i_F,\ \forall q_h\in P^1(F_f),\ \int_{F_f}[v_h]_{F_f}q_h=0\right\}, \\ X_{0,\mathrm{FS}} &= \left\{ v_h\in X_{\mathrm{FS}} \,\middle|\, \forall f\in\mathcal{I}^b_F,\ \forall q_h\in P^1(F_f),\ \int_{F_f} v_h q_h=0\right\}. \end{aligned} \tag{57}$$

The space of nonconforming approximation of $\mathbf{H}^1(\Omega)$ (resp. $\mathbf{H}^1_0(\Omega)$) of order 2 is $\mathbf{X}_{\mathrm{FS}}=(X_{\mathrm{FS}})^2$ (resp. $\mathbf{X}_{0,\mathrm{FS}}=(X_{0,\mathrm{FS}})^2$). We set $\mathcal{X}_{\mathrm{FS}}=\mathbf{X}_{0,\mathrm{FS}}\times Q_{\mathrm{FS}}$ where $Q_{\mathrm{FS}}:=P^1_{\mathrm{disc}}(\mathcal{T}_h)\cap L^2_{\mathrm{zmv}}(\Omega)$.

The building of a basis for $X_{0,\mathrm{FS}}$ is more involved than for $X_{0,\mathrm{CR}}$ since we cannot use two points per facet as degrees of freedom. Indeed, for all $\ell\in\mathcal{I}_K$, there exists a polynomial of order 2 vanishing on the Gauss–Legendre points of the facets of the boundary ∂K_ℓ. Let $f\in\mathcal{I}_F$. The barycentric coordinates of the two Gauss–Legendre points $(p_{+,f},p_{-,f})$ on F_f are such that:

$$p_{+,f}=(c_+,c_-),\ p_{-,f}=(c_-,c_+),\ \text{ where } c_\pm=\left(1\pm 1/\sqrt{3}\right)/2.$$

These points can be used to integrate exactly order three polynomials:

$$\forall g\in P^3(F_f),\ \int_{F_f} g=\frac{|F_f|}{2}\left(g(p_{+,f})+g(p_{-,f})\right).$$

For all $\ell\in\mathcal{I}_K$, we define the quadratic function ϕ_{K_ℓ} that vanishes on the six Gauss–Legendre points of the facets of K_ℓ (see Figure 1):

$$\phi_{K_\ell}:=2-3\sum_{i\in\mathcal{I}_{S,\ell}}\lambda^2_{i,\ell} \text{ such that } \forall f\in\mathcal{I}_{F,\ell},\forall q\in P^1(F_f),\ \int_{F_f}\phi_{K_\ell}q=0. \tag{58}$$

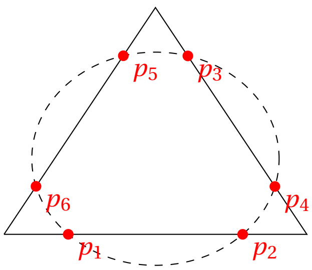

Figure 1. The six Gauss–Legendre points of an element K_ℓ and the elliptic function ϕ_{K_ℓ}.

We consider the set of the elliptic functions ϕ_{K_ℓ}:

$$\Phi_h := \{\phi_h \in L^2(\Omega) \mid \forall\, \ell \in \mathscr{I}_K,\ \phi_h|_{K_\ell} = v_{K_\ell}\phi_{K_\ell},\ v_{K_\ell} \in \mathbb{R}\}. \tag{59}$$

We also define the spaces of P^2-Lagrange functions:

$$X_{\mathrm{LG}} := \{v_h \in H^1(\Omega) \mid \forall\, \ell \in \mathscr{I}_K,\ v_h|_{K_\ell} \in P^2(K_\ell)\}, \quad X_{0,\mathrm{LG}} := \{v_h \in X_{\mathrm{LG}} \mid v_h|_{\partial\Omega} = 0\}.$$

The proposition below allows to build a basis for $X_{0,\mathrm{FS}}$:

Proposition 22 ([29, Proposition 1]). *We have the following decomposition: $X_{\mathrm{FS}} = X_{\mathrm{LG}} + \Phi_h$ with $\dim(X_{\mathrm{LG}} \cap \Phi_h) = 1$. Any function of X_{FS} can be written as the sum of a function of X_{LG} and a function of Φ_h. This representation can be made unique by specifying one degree of freedom.*

Notice that $\Phi_h \cap X_{\mathrm{LG}} = \mathrm{vect}(v_\Phi)$, where for all $\ell \in \mathscr{I}_K$, $v_\Phi|_{K_\ell} = \phi_{K_\ell}$. Then, counting the degrees of freedom, one can show that $\dim(X_{\mathrm{FS}}) = \dim(X_{\mathrm{LG}}) + \dim(\Phi_h) + 1$. For problems involving Dirichlet boundary conditions we can thus prove that for $X_{0,\mathrm{FS}}$ the representation is unique and $X_{0,\mathrm{FS}} = X_{0,\mathrm{LG}} \oplus \Phi_h$. We have $X_{\mathrm{LG}} = \mathrm{vect}\big((\phi_{S_i})_{i\in\mathscr{I}_S}, (\phi_{F_f})_{f\in\mathscr{I}_F}\big)$ where the basis functions are such that:

$$\forall\, i,j \in \mathscr{I}_S, \forall\, f,f' \in \mathscr{I}_F: \quad \phi_{S_i}(S_j) = \delta_{i,j}, \phi_{S_i}(M_f) = 0, \quad \phi_{M_f}(M_{f'}) = \delta_{f,f'}, \phi_{M_f}(S_i) = 0.$$

For all $\ell \in \mathscr{I}_K$, we will denote by $\big(\phi_{\ell,j}\big)_{j=1}^6$ the local nodal basis such that:

$$\big(\phi_{\ell,j}\big)_{j=1}^3 = \big(\phi_{S_i}|_{K_\ell}\big)_{i\in\mathscr{I}_{S,\ell}} \qquad \text{and} \qquad \big(\phi_{\ell,j}\big)_{j=4}^6 = \big(\phi_{F_f}|_{K_\ell}\big)_{f\in\mathscr{I}_{F,\ell}}.$$

The spaces X_{FS} and $X_{0,\mathrm{FS}}$ are such that:

$$\begin{aligned} X_{\mathrm{FS}} &= \mathrm{vect}\big((\phi_{S_i})_{i\in\mathscr{I}_S}, (\phi_{F_f})_{f\in\mathscr{I}_F}, (\phi_{K_\ell})_{\ell\in\mathscr{I}_K}\big), \\ X_{0,\mathrm{FS}} &= \mathrm{vect}\big((\phi_{S_i})_{i\in\mathscr{I}_S^i}, (\phi_{F_f})_{f\in\mathscr{I}_F^i}, (\phi_{K_\ell})_{\ell\in\mathscr{I}_K}\big). \end{aligned} \tag{60}$$

We propose here an alternative definition of the Fortin interpolation operator proposed in [29]. Let us first recall the Scott–Zhang interpolation operator [18,44]. For all $i \in \mathscr{I}_S$, we choose some $\ell_i \in \mathscr{I}_{K,i}$, and we build the $L^2(K_{\ell_i})$-dual basis $\big(\widetilde{\phi}_{\ell_i,j}\big)_{j=1}^6$ of the local nodal basis such that:

$$\forall\, j,j' \in \{1,\ldots,6\},\ \int_{K_{\ell_i}} \widetilde{\phi}_{\ell_i,j}\phi_{\ell_i,j'} = \delta_{j,j'}.$$

Let us define the Fortin–Soulie interpolation operator for scalar functions by:

$$\pi_{\mathrm{FS}}: \left\{ \begin{array}{l} H^1(\Omega) \longrightarrow X_{\mathrm{FS}} \\ \quad v \longmapsto \widetilde{\pi} v + \displaystyle\sum_{\ell\in\mathscr{I}_K} v_{K_\ell}\phi_{K_\ell} \end{array} \right. \quad \text{with} \quad \widetilde{\pi} v = \sum_{i\in\mathscr{I}_S} v_{S_i}\phi_{S_i} + \sum_{f\in\mathscr{I}_F} v_{F_f}\phi_{F_f}. \tag{61}$$

- The coefficients $(v_{S_i})_{i\in\mathscr{I}_S}$ are fixed so that: $\forall\, i \in \mathscr{I}_S$, $v_{S_i} = \int_{K_{\ell,i}} v\widetilde{\phi}_{\ell_i,j_i}$, where j_i is the index such that $\int_{K_{\ell_i}} \widetilde{\phi}_{\ell_i,j_i}\phi_{S_i}|_{K_{\ell_i}} = 1$.
- The coefficients $(v_{F_f})_{f\in\mathscr{I}_F}$ are fixed so that: $\forall\, f \in \mathscr{I}_F$, $\int_{F_f} \widetilde{\pi} v = \int_{F_f} v$.
- The coefficients v_{K_ℓ} are fixed so that: $\int_{K_\ell} \pi_{\mathrm{FS}} v = \int_{K_\ell} v$.

The definition (61) is more general than the one given in [29], which holds for $v \in H^2(\Omega)$.

We set $\mathbf{v}_{S_i} := \big(\widetilde{\pi} v_1(S_i), \widetilde{\pi} v_2(S_i)\big)^T$ and $\mathbf{v}_{F_f} := \big(\widetilde{\pi} v_1(F_f), \widetilde{\pi} v_2(F_f)\big)^T$.

We can define two different Fortin–Soulie interpolation operators for vector functions. First, let

$$\widetilde{\Pi}_{\mathrm{FS}}: \left\{ \begin{array}{l} \mathbf{H}^1(\Omega) \longrightarrow \mathbf{X}_{\mathrm{FS}} \\ \quad \mathbf{v} \longmapsto \displaystyle\sum_{i\in\mathscr{I}_S} \mathbf{v}_{S_i}\phi_{S_i} + \sum_{f\in\mathscr{I}_F} \mathbf{v}_{F_f}\phi_{F_f} + \sum_{\ell\in\mathscr{I}_K} \widetilde{\mathbf{v}}_{K_\ell}\phi_{K_\ell}, \end{array} \right.$$

where the coefficients $(\widetilde{\mathbf{v}}_{K_\ell})_{\ell\in\mathscr{I}_K}$ are such that:

$$\forall\, \ell \in \mathscr{I}_K,\ \int_{K_\ell} \widetilde{\Pi}_{\mathrm{FS}}\mathbf{v} = \int_{K_\ell} \mathbf{v}. \tag{62}$$

The interpolation operator $\widetilde{\Pi}_{\rm FS}$ preserves the local averages, but it doesn't preserve the divergence. We then define a second interpolation operator which preserves the divergence in a weak sense:

$$\Pi_{\rm FS}: \begin{cases} \mathbf{H}^1(\Omega) \longrightarrow \mathbf{X}_{\rm FS} \\ \quad \mathbf{v} \longmapsto \sum_{i\in\mathcal{I}_S} \mathbf{v}_{S_i}\phi_{S_i} + \sum_{f\in\mathcal{I}_F} \mathbf{v}_{F_f}\phi_{F_f} + \sum_{\ell\in\mathcal{I}_K} \mathbf{v}_{K_\ell}\phi_{K_\ell}. \end{cases}$$

For all $\ell \in \mathcal{I}_K$, the vector coefficient $\mathbf{v}_{K_\ell} \in \mathbb{R}^2$ is now fixed so that condition (41) is satisfied. We can impose for example that the projection $\Pi_{\rm FS}\mathbf{v}$ satisfies:

$$\int_{K_\ell} T_\ell^{-1}(\mathbf{x})\,\mathrm{div}\,\Pi_{\rm FS}\mathbf{v} = \int_{K_\ell} T_\ell^{-1}(\mathbf{x})\,\mathrm{div}\,\mathbf{v}. \tag{63}$$

Notice that due to (58), the patch-test condition is still satisfied.

Proposition 23. *The Fortin–Soulie interpolation operator* $\Pi_{\rm FS}$ *is such for all* $0 \le s \le 1$, *for all* $\mathbf{v} \in \mathbf{H}^{1+s}(\Omega)$, *we have:*

$$\forall\, \ell \in \mathcal{I}_K, \quad \left\|\mathbf{Grad}\big(\Pi_{\rm FS}\mathbf{v} - \mathbf{v}\big)\right\|_{\mathbb{L}^2(K_\ell)} \lesssim (\sigma_\ell)^2 (h_\ell)^s |\mathbf{v}|_{\mathbf{H}^{1+s}(K_\ell)}, \tag{64}$$

$$\left|\Pi_{\rm FS}\mathbf{v} - \mathbf{v}\right|_h \lesssim \sigma^2 h^s |\mathbf{v}|_{\mathbf{H}^{1+s}(\Omega)}. \tag{65}$$

Remark 24. Albeit we are inspired by the proof of [22, Lemma 4], we changed the transition from Equation (4.27) to (4.29) there by using only the properties related to the normal component of the velocity, cf. (56). In the original proof, one needs a stronger assumption on the regularity of $\mathbf{v}$ (namely, $\mathbf{v} \in \bigcap_{0<s<s_\Omega} \mathbf{H}^{1+s}(\Omega)$ with $s_\Omega > \frac{1}{2}$). Finally, because we do not split the integral over the boundaries of elements into the sum of $d+1$ integrals over the facets, we obtain purely local estimates, which appear to be new for the Fortin–Soulie element in the case of low-regularity fields $\mathbf{v}$.

Proof of Proposition 23. Let $\mathbf{v} \in \mathbf{H}^1(\Omega)$. We have:

$$\left\|\mathbf{Grad}\big(\Pi_{\rm FS}\mathbf{v} - \mathbf{v}\big)\right\|_{\mathbb{L}^2(K_\ell)} \le \left\|\mathbf{Grad}\big(\Pi_{\rm FS}\mathbf{v} - \widetilde{\Pi}_{\rm FS}\mathbf{v}\big)\right\|_{\mathbb{L}^2(K_\ell)} + \left\|\mathbf{Grad}\big(\widetilde{\Pi}_{\rm FS}\mathbf{v} - \mathbf{v}\big)\right\|_{\mathbb{L}^2(K_\ell)}. \tag{66}$$

Notice that for all $\ell \in \mathcal{I}_K$, $\big(\Pi_{\rm FS}\mathbf{v} - \widetilde{\Pi}_{\rm FS}\mathbf{v}\big)\big|_{K_\ell} = (\mathbf{v}_{K_\ell} - \widetilde{\mathbf{v}}_{K_\ell})\phi_{K_\ell}$. Using (24), we obtain that:

$$\left\|\mathbf{Grad}\big(\Pi_{\rm FS}\mathbf{v} - \widetilde{\Pi}_{\rm FS}\mathbf{v}\big)\right\|_{\mathbb{L}^2(K_\ell)} \lesssim |\mathbf{v}_{K_\ell} - \widetilde{\mathbf{v}}_{K_\ell}| \left\|\mathbf{grad}\,\phi_{K_\ell}\right\|_{\mathbf{L}^2(K_\ell)} \lesssim \|\mathbb{B}_\ell^{-1}\| |K_\ell|^{\frac{1}{2}} |\mathbf{v}_{K_\ell} - \widetilde{\mathbf{v}}_{K_\ell}|. \tag{67}$$

Let us estimate $|\mathbf{v}_{K_\ell} - \widetilde{\mathbf{v}}_{K_\ell}|$. On the one hand, we have[2]:

$$\begin{aligned} \int_{K_\ell} \big(\Pi_{\rm FS}\mathbf{v} - \widetilde{\Pi}_{\rm FS}\mathbf{v}\big) &= \int_{K_\ell} \big(\Pi_{\rm FS}\mathbf{v} - \mathbf{v}\big) && \text{from (62)}, \\ &= \int_{\partial K_\ell} \mathbf{x}\big(\Pi_{\rm FS}\mathbf{v} - \mathbf{v}\big)\cdot\mathbf{n}\big|_{\partial K_\ell} && \text{by IBP and using (63)}, \\ &= \int_{\partial K_\ell} (\mathbf{x} - \underline{\mathbf{x}})\big(\Pi_{\rm FS}\mathbf{v} - \mathbf{v}\big)\cdot\mathbf{n}\big|_{\partial K_\ell} && \text{since } \int_{\partial K_\ell} \big(\Pi_{\rm FS}\mathbf{v} - \mathbf{v}\big)\cdot\mathbf{n}\big|_{\partial K_\ell} = 0, \\ &= \int_{\partial K_\ell} (\mathbf{x} - \underline{\mathbf{x}})\big(\widetilde{\Pi}_{\rm FS}\mathbf{v} - \mathbf{v}\big)\cdot\mathbf{n}\big|_{\partial K_\ell} && \text{from (58)}. \end{aligned}$$

On the other hand, it holds: $\int_{K_\ell} \big(\Pi_{\rm FS}\mathbf{v} - \widetilde{\Pi}_{\rm FS}\mathbf{v}\big) = (\mathbf{v}_{K_\ell} - \widetilde{\mathbf{v}}_{K_\ell}) \int_{K_\ell} \phi_{K_\ell} = \frac{|K_\ell|}{4}(\mathbf{v}_{K_\ell} - \widetilde{\mathbf{v}}_{K_\ell})$.

Hence we have:

$$|\mathbf{v}_{K_\ell} - \widetilde{\mathbf{v}}_{K_\ell}| \le 4|K_\ell|^{-1} \left| \int_{\partial K_\ell} (\mathbf{x} - \underline{\mathbf{x}})\big(\widetilde{\Pi}_{\rm FS}\mathbf{v} - \mathbf{v}\big)\cdot\mathbf{n}\big|_{\partial K_\ell} \right|. \tag{68}$$

[2]Let $\mathbf{w} = \Pi_{\rm FS}\mathbf{v} - \mathbf{v} = (w_1, w_2)^T$ and $\mathbf{x} := (x_1, x_2)^T$. By integration by parts, we have for $d' = 1,2$: $\int_{\partial K_\ell} x_{d'}\mathbf{w}\cdot\mathbf{n}|_{\partial K_\ell} = \int_{K_\ell} x_{d'}\,\mathrm{div}\,\mathbf{w} + \int_{K_\ell} \mathbf{w}\cdot\mathbf{e}_{d'}$. Due to (63), $\int_{K_\ell} x_{d'}\,\mathrm{div}\,\mathbf{w} = 0$, so that for $d' = 1,2$: $\int_{K_\ell} w_{d'} = \int_{\partial K_\ell} x_{d'}\mathbf{w}\cdot\mathbf{n}|_{\partial K_\ell}$.

In order to bound the right-hand side of (68) component by component, we can use Lemma 21, with $q = x_{d'} - \int_{K_\ell} x_{d'}/|K_\ell|$ ($d' = 1,2$), so that $\|\mathbf{grad}\, q\|_{\mathbf{L}^2(K_\ell)} = |K_\ell|^{\frac{1}{2}}$. We obtain:

$$|\mathbf{v}_{K_\ell} - \widetilde{\mathbf{v}}_{K_\ell}| \le 4d \times \left(\sqrt{d}+1\right)|K_\ell|^{-\frac{1}{2}}\pi^{-1}h_\ell \left\|\mathbf{Grad}\left(\widetilde{\Pi}_{\mathrm{FS}}\mathbf{v}-\mathbf{v}\right)\right\|_{\mathbb{L}^2(K_\ell)}. \tag{69}$$

Combining (67) and (69), we have:

$$\left\|\mathbf{Grad}\left(\Pi_{\mathrm{FS}}\mathbf{v} - \widetilde{\Pi}_{\mathrm{FS}}\mathbf{v}\right)\right\|_{\mathbb{L}^2(K_\ell)} \lesssim \|\mathbb{B}_\ell{}^{-1}\| h_\ell \left\|\mathbf{Grad}\left(\widetilde{\Pi}_{\mathrm{FS}}\mathbf{v}-\mathbf{v}\right)\right\|_{\mathbb{L}^2(K_\ell)} \lesssim \sigma_\ell \left\|\mathbf{Grad}\left(\widetilde{\Pi}_{\mathrm{FS}}\mathbf{v}-\mathbf{v}\right)\right\|_{\mathbb{L}^2(K_\ell)}.$$

For all $\mathbf{v} \in \mathbf{P}^2(K_\ell)$ we have $\widetilde{\Pi}_{\mathrm{FS}}(\mathbf{v}) = \mathbf{v}$ and $\widehat{\widetilde{\Pi}}_{\mathrm{FS}}\widehat{\mathbf{v}}_\ell = \widehat{\mathbf{v}}_\ell = \widehat{\mathbf{v}}_\ell$. Hence, using Bramble–Hilbert/Deny–Lions Lemma [26, Lemma 11.9], we have for $m = 0,1$, for all $\mathbf{v} \in \mathbf{H}^{1+m}(\Omega)$:

$$\forall\, \ell \in \mathscr{I}_K, \quad \left\|\mathbf{Grad}\left(\widetilde{\Pi}_{\mathrm{FS}}\mathbf{v}-\mathbf{v}\right)\right\|_{\mathbb{L}^2(K_\ell)} \lesssim \sigma_\ell (h_\ell)^m |\mathbf{v}|_{\mathbf{H}^{1+m}(K_\ell)}.$$

We deduce that for $m = 0,1$, for all $\mathbf{v} \in \mathbf{H}^{1+m}(\Omega)$:

$$\begin{aligned} \forall\, \ell \in \mathscr{I}_K, \quad \left\|\mathbf{Grad}\left(\Pi_{\mathrm{FS}}\mathbf{v}-\mathbf{v}\right)\right\|_{\mathbb{L}^2(K_\ell)} &\lesssim (\sigma_\ell)^2 (h_\ell)^m |\mathbf{v}|_{\mathbf{H}^{1+m}(K_\ell)}, \\ \left\|\mathbf{Grad}\left(\Pi_{\mathrm{FS}}\mathbf{v}-\mathbf{v}\right)\right\|_{\mathbb{L}^2(\Omega)} &\lesssim \sigma^2 h^m |\mathbf{v}|_{\mathbf{H}^{1+m}(\Omega)} \text{ by summation.} \end{aligned} \tag{70}$$

Using interpolation property [46, Lemma 22.2], we obtain (64) and (65). □

Hence, using the triangular inequality, we have:

$$\left\|\mathbf{Grad}\,\Pi_{\mathrm{FS}}\mathbf{v}\right\|_{\mathbb{L}^2(K_\ell)} \le \left\|\mathbf{Grad}\left(\Pi_{\mathrm{FS}}\mathbf{v}-\mathbf{v}\right)\right\|_{\mathbb{L}^2(K_\ell)} + \left\|\mathbf{Grad}\,\mathbf{v}\right\|_{\mathbb{L}^2(K_\ell)} \lesssim \left((\sigma_\ell)^2+1\right)\|\mathbf{v}\|_{\mathbf{H}^1(K_\ell)}.$$

By summation over ℓ, we deduce that the coefficient C_{nc} in (40) is here equal to σ^2+1. We recall that the discrete Poincaré–Steklov inequality (31) holds.

Theorem 25. *Let $\mathscr{X}_h = \mathscr{X}_{\mathrm{FS}}$. Then the continuous bilinear form $a_{S,h}(\cdot,\cdot)$ is T_h-coercive and Problem* (39) *is well-posed.*

Proof. According to Proposition 23, the Fortin–Soulie interpolation operator Π_{FS} satisfies (40)-(41), so that we can apply the proof of Theorem 16. □

Notice that in the recent paper [42], the inf-sup condition of the mixed Fortin–Soulie finite element is proved directly on a triangle and then using the macro-element technique [45], but it seems difficult to use this technique to build a Fortin operator, which is needed to compute error estimates.

The study can be extended to higher orders for $d = 2$ using the following papers: [4] for $k \ge 4$, k even, [15] for $k = 3$ and [14] for $k \ge 5$, k odd. In [24], the authors propose a local Fortin operator for the lowest order Taylor–Hood finite element [47] for $d = 3$.

4. Numerical results improving consistency

4.1. $\mathbf{H}$(div)*-conforming velocity reconstruction*

Consider Problem (3) with data $\mathbf{f} = -\mathbf{grad}\,\phi$, where $\phi \in H^1(\Omega) \cap L^2_{\mathrm{zmv}}(\Omega)$. The unique solution is then $(\mathbf{u}, p) := (0, \phi)$. By integrating by parts, the source term in (9) reads:

$$\forall\, \mathbf{v} \in \mathbf{H}^1_0(\Omega), \int_\Omega \mathbf{f}\cdot\mathbf{v} = \int_\Omega \phi\, \mathrm{div}\,\mathbf{v}. \tag{71}$$

Recall that the nonconforming space $\mathbf{X}_h$ defined in (30) is a subset of $\mathscr{P}_h\mathbf{H}^1$: using a nonconforming finite element method, the integration by parts must be done on each element of the triangulation, and we have:

$$\forall\, \mathbf{v} \in \mathscr{P}_h\mathbf{H}^1, \int_\Omega \mathbf{f}\cdot\mathbf{v} = \left(\mathrm{div}_h\,\mathbf{v}, \phi\right)_{L^2(\Omega)} + \sum_{f\in\mathscr{I}_F}\int_{F_f} [\mathbf{v}]_{F_f}\cdot\mathbf{n}_f\,\phi. \tag{72}$$

Using Lemma 21, we have: $\sum_{f\in\mathcal{I}_F}\int_{F_f}[\mathbf{v}]_{F_f}\cdot\mathbf{n}_f\phi \lesssim h\|\mathbf{v}_h\|_h\|\mathbf{grad}\,\phi\|_{\mathbf{L}^2(\Omega)}$. Applying (72) to the right-hand side of (39) and choosing $\mathbf{v}_h=\mathbf{u}_h$, it holds: $\nu\|\mathbf{u}_h\|_h \lesssim h\|\mathbf{grad}\,\phi\|_{\mathbf{L}^2(\Omega)}$ (as expected by (47)). Hence, the term with the jumps acts as a numerical source for the discrete velocity, whose numerical influence is proportional to h/ν. Thus, we cannot obtain exactly $\mathbf{u}_h=0$. Linke proposed in [38] to project the test function $\mathbf{v}_h\in\mathbf{X}_h$ on a discrete subspace of $\mathbf{H}(\mathrm{div};\Omega)$, like Raviart–Thomas or Brezzi–Douglas–Marini finite elements (see [12,39], or the monograph [8]). Let $\Pi_{\mathrm{div}}:\mathbf{X}_{0,h}\to\mathbf{P}^k_{\mathrm{disc}}(\mathcal{T}_h)\cap\mathbf{H}_0(\mathrm{div};\Omega)$ be some interpolation operator built so that for all $\mathbf{v}_h\in\mathbf{X}_{0,h}$, for all $\ell\in\mathcal{I}_K$, $\left(\mathrm{div}\,\Pi_{\mathrm{div}}\mathbf{v}_h\right)\big|_{K_\ell}=\mathrm{div}\,\mathbf{v}_h\big|_{K_\ell}$. Integrating by parts, we have for all $\mathbf{v}_h\in\mathbf{X}_{0,h}$:

$$\int_\Omega \mathbf{f}\cdot\Pi_{\mathrm{div}}\mathbf{v}_h=\int_\Omega\phi\,\mathrm{div}\,\Pi_{\mathrm{div}}\mathbf{v}_h=\sum_{\ell\in K_\ell}\int_{K_\ell}\phi\,\mathrm{div}\,\Pi_{\mathrm{div}}\mathbf{v}_h=\sum_{\ell\in K_\ell}\int_{K_\ell}\phi\,\mathrm{div}\,\mathbf{v}_h=\left(\mathrm{div}_h\,\mathbf{v}_h,\phi\right)_{L^2(\Omega)}.$$

The projection Π_{div} allows to eliminate the terms of the integrals of the jumps in (72).

Let us write Problem (39) as:

$$\text{Find } (\mathbf{u}_h,p_h)\in\mathcal{X}_h \text{ such that } a_{S,h}\big((\mathbf{u}_h,p_h),(\mathbf{v}_h,q_h)\big)=\ell_{\mathbf{f}}\big((\Pi_{\mathrm{div}}\mathbf{v}_h,q_h)\big)\ \ \forall\,(\mathbf{v}_h,q_h)\in\mathcal{X}_h. \tag{73}$$

In the case of $\mathcal{X}_h=\mathcal{X}_{\mathrm{CR}}$ and a projection on Brezzi–Douglas–Marini finite elements, the following error estimate holds if $(\mathbf{u},p)\in\mathbf{H}^2(\Omega)\times H^1(\Omega)$:

$$\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{L}^2(\Omega)}\le\widetilde{C}h^2|\mathbf{u}|_{\mathbf{H}^2(\Omega)}, \tag{74}$$

where the constant $\widetilde{C}$ is independent of h. The proof is detailed in [10] for shape-regular meshes and [2] for anisotropic meshes. We remark that the error doesn't depend on the norm of the pressure nor on the ν parameter. We will provide some numerical results to illustrate the effectiveness of this formulation, even with a projection on the Raviart–Thomas finite elements, which, for a fixed polynomial order, are less precise than the Brezzi–Douglas–Marini finite elements.

For $k\in\mathbb{N}^\star$, the space of Raviart–Thomas finite elements can be defined as:

$$\mathbf{X}_{\mathrm{RT}_k}:=\left\{\mathbf{v}\in\mathbf{H}(\mathrm{div};\Omega)\ \Big|\ \forall\,\ell\in\mathcal{I}_k,\ \mathbf{v}|_{K_\ell}=\mathbf{a}_\ell+b_\ell\mathbf{x},\ (\mathbf{a}_\ell,b_\ell)\in P^k(K_\ell)^d\times P^k(K_\ell)\right\}.$$

Let $k\le1$. The Raviart–Thomas interpolation operator $\Pi_{\mathrm{RT}_k}:\mathbf{H}^1(\Omega)\cup\mathbf{X}_h\to\mathbf{X}_{\mathrm{RT}_k}$ is defined by:

$$\forall\,\mathbf{v}\in\mathbf{H}^1(\Omega)\cup\mathbf{X}_h,\ \begin{cases}\forall\,f\in\mathcal{I}_F,\ \displaystyle\int_{F_f}\Pi_{\mathrm{RT}_k}\mathbf{v}\cdot\mathbf{n}_f q=\int_{F_f}\mathbf{v}\cdot\mathbf{n}_f q,\ \forall\,q\in P^k(F_f),\\ \text{for } k=1,\ \forall\,\ell\in\mathcal{I}_K,\ \displaystyle\int_{K_\ell}\Pi_{\mathrm{RT}_1}\mathbf{v}=\int_{K_\ell}\mathbf{v}.\end{cases} \tag{75}$$

Note that the Raviart–Thomas interpolation operator preserves the constants. Let $\mathbf{v}_h\in\mathbf{X}_h$. In order to compute the left-hand side of (72), we must evaluate $(\Pi_{\mathrm{RT}_k}\mathbf{v}_h)\big|_{K_\ell}$ for all $\ell\in\mathcal{I}_K$. Calculations are performed using the proposition below:

Proposition 26 ([31, Lemma 3.11]). *Let $k\le1$. Let $\widehat{\Pi}_{\mathrm{RT}_k}:\mathbf{H}^1(\widehat{K})\to\mathbf{P}^k(\widehat{K})$ be the Raviart–Thomas interpolation operator restricted to the reference element, so that:*

$$\forall\,\widehat{\mathbf{v}}\in\mathbf{H}^1(\widehat{K}),\ \begin{cases}\forall\,\widehat{F}\in\partial\widehat{K},\ \displaystyle\int_{\widehat{F}}\widehat{\Pi}_{\mathrm{RT}_k}\widehat{\mathbf{v}}\cdot\mathbf{n}_{\widehat{F}}\widehat{q}=\int_{\widehat{F}}\widehat{\mathbf{v}}\cdot\mathbf{n}_{\widehat{F}}\widehat{q},\ \forall\,\widehat{q}\in P^k(\widehat{F}),\\ \textit{for } k=1,\ \displaystyle\int_{\widehat{K}}\widehat{\Pi}_{\mathrm{RT}_k}\widehat{\mathbf{v}}=\int_{\widehat{K}}\widehat{\mathbf{v}}.\end{cases} \tag{76}$$

We then have: $\forall\,\ell\in\mathcal{I}_K$,

$$\left(\Pi_{\mathrm{RT}_k}\mathbf{v}\right)\big|_{K_\ell}(\mathbf{x})=\mathbb{B}_\ell\big(\widehat{\Pi}_{\mathrm{RT}_k}\mathbb{B}_\ell^{-1}\widehat{\mathbf{v}}_\ell\big)\circ T_\ell^{-1}(\mathbf{x})\ \ \textit{where}\ \widehat{\mathbf{v}}_\ell=\mathbf{v}\circ T_\ell(\widehat{\mathbf{x}}). \tag{77}$$

The proof is based on the equality of the $\widehat{F}$ and $\widehat{K}$-moments of $\left(\Pi_{\mathrm{RT}_k}\mathbf{v}\right)\big|_{K_\ell}\circ T_\ell(\widehat{\mathbf{x}})$ and $\mathbb{B}_\ell\big(\widehat{\Pi}_{\mathrm{RT}_k}\mathbb{B}_\ell^{-1}\widehat{\mathbf{v}}_\ell\big)(\widehat{\mathbf{x}})$. For $k=0$, setting for $d'\in\{1,\dots,d\}$: $\boldsymbol{\psi}_{f,d'}:=\psi_f\mathbf{e}_{d'}$, we obtain that:

$$\forall\,\ell\in\mathcal{I}_K,\forall\,f\in\mathcal{I}_{F,\ell},\ \left(\Pi_{\mathrm{RT}_0}\boldsymbol{\psi}_{f,d'}\right)\Big|_{K_\ell}=\left(d|K_\ell|\right)^{-1}\big(\mathbf{x}-\vec{O}S_{f,\ell}\big)\mathcal{S}_{f,\ell}\cdot\mathbf{e}_{d'}, \tag{78}$$

where $S_{f,\ell}$ is the vertex opposite to F_f in K_ℓ.

For $k=1$, the vector $\left(\Pi_{\mathrm{RT}_1}\mathbf{v}_h\right)\big|_{K_\ell}$ is described by eight unknowns:

$$\left(\Pi_{\mathrm{RT}_1}\mathbf{v}_h\right)\big|_{K_\ell}=\mathbb{A}_\ell\mathbf{x}+(\mathbf{b}_\ell\cdot\mathbf{x})\mathbf{x}+\mathbf{d}_\ell,\ \text{ where }\ \mathbb{A}_\ell\in\mathbb{R}^{2\times2},\mathbf{b}_\ell\in\mathbb{R}^2,\mathbf{d}_\ell\in\mathbb{R}^2.$$

We compute only once the inverse of the matrix of the linear system (76), in $\mathbb{R}^{8\times8}$.

4.2. *Application with manufactured solutions*

In the Tables 1, 2 and 3, we call $\varepsilon_0^\nu(\mathbf{u})=\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{L}^2(\Omega)}/\|(\mathbf{u},p)\|_{\mathscr{X}}$ the velocity error in $\mathbf{L}^2(\Omega)$-norm, where $\mathbf{u}_h$ is the solution to Problem (39) (columns $\mathbf{X}_{\mathrm{CR}}$ and $\mathbf{X}_{\mathrm{FS}}$) or (73) (columns $\mathbf{X}_{\mathrm{CR}}+\Pi_{\mathrm{RT}_0}$ and $\mathbf{X}_{\mathrm{FS}}+\Pi_{\mathrm{RT}_1}$) and h is the mesh size.

We first consider Stokes problem (3) in $\Omega=(0,1)^2$ with $\mathbf{u}=0$, $p=(x_1)^3+(x_2)^3-0.5$, $\mathbf{f}=\mathbf{grad}\,p=3\left((x_1)^2,(x_2)^2\right)^T$. We report in Table 1 the velocity error $\varepsilon_0^\nu(\mathbf{u}):=\nu\|\mathbf{u}_h\|_{\mathbf{L}^2(\Omega)}/\|p\|_{L^2(\Omega)}$, for $h=5.00\times10^{-2}$ and for different values of ν.

Table 1. Values of $\varepsilon_0^\nu(\mathbf{u})$ for $h=5.00\times10^{-2}$.

ν	$\mathbf{X}_{\mathrm{CR}}$	$\mathbf{X}_{\mathrm{CR}}+\Pi_{\mathrm{RT}_0}$	$\mathbf{X}_{\mathrm{FS}}$	$\mathbf{X}_{\mathrm{FS}}+\Pi_{\mathrm{RT}_1}$
1.00×10^{-4}	7.96×10^{-4}	4.59×10^{-17}	8.81×10^{-7}	1.54×10^{-16}
1.00×10^{-5}	7.96×10^{-4}	4.59×10^{-17}	8.81×10^{-7}	1.54×10^{-16}
1.00×10^{-6}	7.96×10^{-4}	4.59×10^{-17}	8.81×10^{-7}	1.54×10^{-16}

The $\mathbf{L}^2(\Omega)$-norm of the discrete velocity $\|\mathbf{u}_h\|_{\mathbf{L}^2(\Omega)}$ is proportional to ν^{-1}. Using the projection, we obtain $\varepsilon_0^\nu(\mathbf{u})=0$ close to machine precision.

We now consider Stokes problem (3) in $\Omega=(0,1)^2$ with:

$$\mathbf{u}=\begin{pmatrix}\left(1-\cos(2\pi x_1)\right)\sin(2\pi x_2)\\ \left(\cos(2\pi x_2)-1\right)\sin(2\pi x_1)\end{pmatrix},\ p=\sin(2\pi x_1)\ \text{ and }\ \mathbf{f}=-\nu\Delta\mathbf{u}+\mathbf{grad}\,p.$$

We report in Table 2 (resp. 3) the values of $\varepsilon_0^\nu(\mathbf{u})$ in the case $\nu=1.00\times10^{-3}$ (resp. $\nu=1.00\times10^{-4}$) for mesh sizes. We observe that when there is no projection, $\varepsilon_0^\nu(\mathbf{u})$ is independent of ν, whereas using the projection, $\varepsilon_0^\nu(\mathbf{u})$ is proportional to ν.

Table 2. Values of $\varepsilon_0^\nu(\mathbf{u})$ for $\nu=1.00\times10^{-3}$.

h	$\mathbf{X}_{\mathrm{CR}}$	$\mathbf{X}_{\mathrm{CR}}+\Pi_{\mathrm{RT}_0}$	$\mathbf{X}_{\mathrm{FS}}$	$\mathbf{X}_{\mathrm{FS}}+\Pi_{\mathrm{RT}_1}$
5.00×10^{-2}	1.32×10^{-3}	2.74×10^{-5}	4.73×10^{-6}	5.05×10^{-7}
2.50×10^{-2}	3.30×10^{-4}	6.93×10^{-6}	5.06×10^{-7}	6.42×10^{-8}
1.25×10^{-2}	8.25×10^{-5}	1.74×10^{-6}	6.31×10^{-8}	8.10×10^{-9}
6.25×10^{-3}	2.04×10^{-5}	4.35×10^{-7}	7.44×10^{-9}	1.03×10^{-9}
Rate	$h^{2.00}$	$h^{1.99}$	$h^{3.08}$	$h^{2.97}$

Let us consider Stokes problem (3) with a low-regular velocity. Let $\Omega=(0,1)^2$, $S_0=(0.5,0.5)$, and (r,θ) be the polar coordinates centered on S_0. We set:

$$\mathbf{u}=r^\alpha\mathbf{e}_\theta,\ p=r-|\Omega|^{-1}\int_\Omega r\ \text{ so that }\ \mathbf{f}:=-\nu\Delta\mathbf{u}+\mathbf{grad}\,p=\nu(1-\alpha^2)r^{\alpha-2}\mathbf{e}_\theta+\mathbf{e}_r.$$

We report in Table 4 the values of $\varepsilon_0^\nu(\mathbf{u})$ for $\nu=1.00\times10^{-4}$, and for different mesh sizes, with $\alpha=1$ and $\alpha=0.45$. For $\alpha=1$, $\mathbf{u}=(-y,x)^T\in\mathbf{H}^2(\Omega)$. For $\alpha=0.45$, $\mathbf{u}\in\bigcap_{0<s<\alpha}\mathbf{H}^{1+s}(\Omega)$, hence $\mathbf{u}\notin\mathbf{H}^2(\Omega)$. It seems that the Raviart–Thomas projection is less efficient in that last case.

Table 3. Values of $\varepsilon_0^\nu(\mathbf{u})$ for $\nu = 1.00 \times 10^{-4}$.

h	$\mathbf{X}_{\mathrm{CR}}$	$\mathbf{X}_{\mathrm{CR}} + \Pi_{\mathrm{RT}_0}$	$\mathbf{X}_{\mathrm{FS}}$	$\mathbf{X}_{\mathrm{FS}} + \Pi_{\mathrm{RT}_1}$
5.00×10^{-2}	1.32×10^{-3}	2.74×10^{-6}	4.70×10^{-6}	5.05×10^{-8}
2.50×10^{-2}	3.30×10^{-4}	6.93×10^{-7}	5.10×10^{-7}	6.43×10^{-9}
1.25×10^{-2}	8.25×10^{-5}	1.74×10^{-7}	6.37×10^{-8}	8.11×10^{-10}
6.25×10^{-3}	2.04×10^{-5}	4.36×10^{-8}	7.51×10^{-9}	9.77×10^{-11}
Rate	$h^{2.00}$	$h^{1.99}$	$h^{3.08}$	$h^{2.99}$

Table 4. Values of $\varepsilon_0^\nu(\mathbf{u})$, regular and low-regular velocity, $\nu = 1.00 \times 10^{-4}$.

	$\alpha = 1$		$\alpha = 0.45$	
h	$\mathbf{X}_{\mathrm{FS}}$	$\mathbf{X}_{\mathrm{FS}} + \Pi_{\mathrm{RT}_1}$	$\mathbf{X}_{\mathrm{FS}}$	$\mathbf{X}_{\mathrm{FS}} + \Pi_{\mathrm{RT}_1}$
1.00×10^{-1}	3.03×10^{-5}	2.81×10^{-6}	3.05×10^{-5}	3.94×10^{-6}
5.00×10^{-2}	4.34×10^{-6}	1.54×10^{-6}	4.57×10^{-6}	2.15×10^{-6}
2.50×10^{-2}	4.72×10^{-7}	2.41×10^{-8}	9.70×10^{-7}	8.52×10^{-7}
Rate	$h^{3.00}$	$h^{3.43}$	$h^{2.48}$	$h^{1.11}$

In order to enhance the numerical results, one could also use a posteriori error estimators to adapt the mesh near point S_0 (see [23,25] for $k = 1$ and [1] for $k = 2$).

Alternatively, using the nonconforming Crouzeix–Raviart mixed finite element method, one can build a divergence-free basis, as described in [34] for $k = 1$. When $k = 1$, following the initial work of [7], one can also add P^1-Lagrange basis functions to the space of the discrete pressures as explained in [37]. The consistency of the discrete velocity is then improved. Notice that using conforming finite elements, the Scott–Vogelius finite elements [28,43,49] produce velocity approximations that are exactly divergence free.

The code used to get the numerical results can be downloaded on GitHub [35].

5. Conclusion

We analysed the discretization of Stokes problem with nonconforming finite elements in light of the T-coercivity theory. Furthermore, we obtained local stability estimates for order 1 in 2 or 3 dimensions without mesh regularity assumption; and for order 2 in 2 dimensions in the case of a shape-regular triangulation sequence. This local approach, splitting the normal and the tangential components could help to generalize our results to order $k \geq 3$ (using maybe also other internal moment conservation). This is ongoing work. We then provided numerical results to illustrate the importance of using $\mathbf{H}(\mathrm{div})$-conforming projection. Further, we intend to extend the study to other mixed finite element methods.

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Declaration of interests

The author does not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and has declared no affiliation other than her research organization.

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