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The isospectral problem for p-widths: an application of Zoll metrics

Le problème isospectral pour les p-largeurs : une application des métriques de Zoll

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Abstract. We pose the isospectral problem for the p-widths: is a Riemannian manifold (M^{n+1}, g) uniquely determined by its p-widths, $\{\omega_p(M,g)\}_{p=1}^{\infty}$? We construct many counterexamples on S^2 using Zoll metrics and the fact that geodesic p-widths are given by unions of immersed geodesics.

Résumé. Nous posons le problème isospectral pour les p-largeurs : une variété Riemannienne (M^{n+1}, g) est-elle uniquement déterminée par ses p-largeurs, $\{\omega_p(M,g)\}_{p=1}^{\infty}$? Nous construisons de nombreux contre-exemples sur S^2 en utilisant les métriques de Zoll et le fait que les p-largeurs géodésiques sont données par des unions de géodésiques immergées.

Keywords. Min-max, p-widths, Zoll, geodesics.

Mots-clés. Min-max, p-largeurs, Zoll, géodesiques.

2020 Mathematics Subject Classification. 53C22.

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1. Introduction

In [15], Gromov introduced the *p-widths*, $\{\omega_p\}$, of a Riemannian manifold, (M^{n+1}, g), as an analogue of the spectrum of the Laplacian, $\{\lambda_k\}$, the discrete collection of eigenvalues. Intuitively, one replaces the Rayleigh quotient in the definition of eigenvalues with the area functional in the definition of ω_p – see Section 2, as well as [10,16,17,19] for more background. The p-widths have proven to be extremely useful due to the Almgren–Pitts/Marques–Neves Morse theory program for the area functional (see [2,3,25–27,30,34,38]). In [24], Liokumovich–Marques–Neves demonstrated that $\{\omega_p\}$ satisfy a Weyl law, which later led to a resolution of Yau's conjecture [37]: any closed three-dimensional manifold must contain an infinite number of immersed minimal surfaces. In some sense, the resolution was stronger than expected:

Theorem 1 ([8,21,28,31,36]). *On any closed manifold (M^{n+1}, g) with $3 \leq n+1 \leq 7$, there exist infinitely many embedded minimal hypersurfaces.*

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We emphasize the result of *embedded* surfaces over *immersed* ones, and also note that embedded minimal hypersurfaces are automatically smooth when $3 \le n+1 \le 7$. We recall the following timeline: Marques–Neves [28] first proved the result in the $\mathrm{Ric}_g > 0$ setting, Irie–Marques–Neves [21] proved the result for generic metrics, Chodosh–Mantoulidis [8] proved the result using the Allen–Cahn equation for a generic metric when $n+1=3$, and Song [36] proved the result for non-generic metrics. We also note the work of Li [23] who showed the existence of infinitely many immersed minimal surfaces with optimal regularity when $n+1 \ge 8$. We refer the reader to the introduction of [9] for a more thorough overview of the history and applications of the p-widths.

Recall the isospectral problem for the Laplacian:

Problem 2. *Is a closed manifold, (M^{n+1}, g), uniquely determined (up to isometry) by its discrete spectrum, $\{\lambda_k\}$?*

Informally, this is known as "can you hear the shape of a drum?" and was made famous by Kac [22]. In [33], Milnor constructed two 16-dimensional tori which are isospectral but not isometric, providing the first resolution of Problem 2. Gordon–Webb–Wolpert [12] constructed simpler examples within the class of polygons in $\mathbb{R}^2$. By contrast, Berger [6] showed that S^2 is spectrally rigid (see also Hersch [20] who deduced the same from only the first 3 eigenvalues).

Given the analogy between $\{\omega_p\}$ and $\{\lambda_p\}$, it is natural to pose the *p-width isospectral problem*:

Problem 3. *Is a closed manifold, (M^{n+1}, g), uniquely determined (up to isometry) by the values $\omega_p(g)$?*

We show that the answer is "no":

Theorem 4. *For any $p > 0$, ω_p is constant on any fixed connected component of the space of Zoll metrics on S^2.*

As a corollary, we can compute the p-widths explicitly for new metrics.

Corollary 5. *Let g be any Zoll metric on S^2 which lies in the connected component of g_{round} in the space of Zoll metrics on S^2. Then $\omega_p(g) = \omega_p(g_{\text{round}}) = 2\pi\lfloor\sqrt{p}\rfloor$ for all p.*

The above gives rise to a continuous deformation of p-width isospectral metrics (compare [13]). In particular, any odd function $f\colon S^2 \to \mathbb{R}$ gives rise to a 1-parameter family of geometrically distinct Zoll metrics near the round metric (see [18]). The p-widths of (S^2, g_{round}) were computed explicitly in [9], and this was the first example of the p-width spectrum being known for all p. Corollary 5 now gives uncountably more manifolds where the full p-width spectrum is known.

The proof of Theorem 4 uses the defining properties of Zoll metrics on S^2, which are metrics for which all geodesics (with multiplicity one) are simple and of length 2π (note this choice of length, as opposed to any constant c, agrees with [18]). These were first constructed by Zoll [39], and we refer the reader to [7,11,14,18] for a by no means complete list of background sources. The proof utilizes a recent result of Chodosh–Mantoulidis [9], which states that each $\omega_p(M^2, g)$ is achieved by a union of closed, potentially immersed geodesics.

This paper is organized as follows: in Section 2, we briefly define the p-widths, and recall the results of Chodosh–Mantoulidis [9]. We also define Zoll metrics and list some relevant properties. In Section 3 we prove the main theorem, and in Section 4 we pose some open questions.

2. Background

We recall the essential notation from the min-max theory of Almgren–Pitts [2] and Marques–Neves [29]. Let M be an $(n+1)$-dimensional manifold, let $X \subseteq [0,1]^k$ denote a cubical subcomplex of the k-dimensional unit cube (cf. [29, Section 2.2]), and let $\mathcal{Z}_n(M;\mathbb{Z}_2)$ denote the set of n-cycles mod $\mathbb{Z}_2$ – informally, a cycle is the boundary of a nice (Caccioppoli) open set. In [2], Almgren showed that $\mathcal{Z}_n(M;\mathbb{Z}_2)$ is weakly homotopic to $\mathbb{RP}^\infty$. We let $\overline{\lambda} \in H^1\big(\mathcal{Z}_n(M;\mathbb{Z}_2),\mathbb{Z}_2\big) \cong \mathbb{Z}_2$ denote the generator of this group, so that $\overline{\lambda}^p$ generates $H^p\big(\mathcal{Z}_n(M;\mathbb{Z}_2),\mathbb{Z}_2\big) \cong \mathbb{Z}_2$. For $T \in \mathcal{Z}_n(M;\mathbb{Z}_2)$, there is a natural associated varifold to T and we let $\|T\|$ denote the corresponding Radon measure, with $\|T\|(U)$ denoting the integral of the measure over U. We further let $\mathbf{M}(T) = \|T\|(M)$.

Definition 6 ([28, Definition 4.1]). *A map $\Phi\colon X \to \mathcal{Z}_n(M;\mathbb{Z}_2)$ is a p-*sweepout *if it is continuous (with the standard flat norm topology on $\mathcal{Z}_n(M;\mathbb{Z}_2)$) and $\Phi^*(\overline{\lambda}^p) \neq 0 \in H^p(X)$.*

Definition 7 ([28, Section 3.3]). *A map $\Phi\colon X \to \mathcal{Z}_n(M;\mathbb{Z}_2)$ is said to have* no concentration of mass *if*

$$\lim_{r\to 0}\sup\Big\{\|\Phi(x)\|\big(B_r(p)\big) \,:\, x \in X,\ p \in M\Big\} = 0.$$

Definition 8 ([16,19]). *We define $\mathcal{P}_p = \mathcal{P}_p(M)$ to be the set of all p-sweepouts, out of any cubical subcomplex X, with no concentration of mass. The p-*width *of (M,g) is*

$$\omega_p(M,g) := \inf_{\Phi\in\mathcal{P}_p}\sup\Big\{\mathbf{M}\big(\Phi(x)\big) \,:\, x \in \mathrm{domain}(\Phi)\Big\}.$$

To see the analogy between ω_p and λ_p, consider the following characterization of $\lambda_p(M)$ as

$$\lambda_p(M) := \inf_{\substack{V\subseteq C^\infty(M)\\ \dim V = p+1}}\ \sup_{f\in V\setminus\{0\}} \frac{\int_M |\nabla f|^2}{\int_M f^2}.$$

Note the parallels between the min-max definitions of ω_p, λ_p, with $\Phi^*(\overline{\lambda}^p) \neq 0$ being analogous to $\dim(V) = p+1$.

Having defined the p-widths, one expects from the min-max theory of Almgren–Pitts [3,34] that $\omega_p = \sum_i \mathbf{M}(V_i)$ where V_i are stationary varifolds for $\mathbf{M}$. When $3 \le n+1 \le 7$, the regularity theory of Schoen–Simon [35] shows that such varifolds are induced by smooth minimal hypersurfaces, potentially with multiplicity (see also [27, Section 3]). However, when $n+1 = 2$, the V_i may correspond to geodesic networks, as opposed to unions of closed geodesics (cf. [27, Remark 1.1] and [9, Section 5], as well as [32, Figure 1] for a visual). In [9], Chodosh–Mantoulidis used the Allen–Cahn equation to prove that the p-widths can always be achieved by a union of closed geodesics:

Theorem 9 ([9, Theorem 1.2]). *Let (M^2, g) be a closed Riemannian manifold. For every $p \in \mathbb{Z}^+$, there exists a collection of primitive, closed geodesics $\{\sigma_{p,j}\}$ such that*

$$\omega_p(M,g) = \sum_{j=1}^{N(p)} m_j \cdot L_g(\sigma_{p,j}), \quad m_j \in \mathbb{Z}^+.$$

Here, “primitive” means that the geodesic is connected and transversed with multiplicity one. They also showed:

Theorem 10 ([9, Theorem 1.4]). *For g_0 the unit round metric on S^2*

$$\forall\, p \in \mathbb{N}, \quad \omega_p(S^2, g_0) = 2\pi\lfloor\sqrt{p}\rfloor.$$

We now collect relevant facts about Zoll metrics on S^2.

Definition 11. *A metric, g, on S^2 is called a* Zoll metric *if all of the primitive geodesics are closed and of length 2π.*

Zoll [39], Funk [11], Guillemin [18], and Gromoll–Grove [14] each contributed significantly to the theory of Zoll metrics. In particular, Gromoll–Grove showed that all of the primitive geodesics on a Zoll surface are simple. Furthermore, the combined work of Guillemin and Funk demonstrated that the tangent space of Zoll metrics at the round metric is given by odd functions $f\colon S^2 \to \mathbb{R}$. We also mention a higher dimensional analogue of these metrics constructed by Ambrozio–Marques–Neves [4].

Let $\mathcal{Z}(S^2)$ denote the space of Zoll metrics on S^2, $\mathcal{Z}_\alpha$, an arbitrary connected component of $\mathcal{Z}(S^2)$, and $\mathcal{Z}_0$, the connected component containing g_0, the unit round metric.

3. Proof of main theorem

Let $\mathcal{M}_\infty(S^2)$ denote the set of all smooth metrics on S^2, and let g_0 denote the round metric. For fixed p, $\omega_p\colon \mathcal{M}_\infty(S^2) \to \mathbb{R}$ is continuous due to the following lemma of Marques–Neves–Song:

Lemma 12 ([31, Lemma 1]). *Let $C_1 < C_2$. There exists a $K(C_1, C_2)$, such that*

$$|\omega_p(g) - \omega_p(g')| \leq K(C_1, C_2)\, p^{1/2} \|g - g'\|_{C^0(g_0)}$$

for any g, g' such that $C_1 g_0 \leq g \leq C_2 g_0$ and $C_1 g_0 \leq g' \leq C_2 g_0$.

Take any Zoll metric, g, on S^2. Applying Theorem 9, we have

$$\begin{aligned}\omega_p(S^2, g) &= \sum_{j=1}^{N(p)} m_j \cdot L_g(\sigma_{p,j}) \\ &= 2\pi \sum_{j=1}^{N(p)} m_j \in 2\pi\mathbb{Z}^+\end{aligned}$$

since each $\sigma_{p,j}$ is primitive and $L_g(\sigma_{p,j}) = 2\pi$ by definition of Zoll metric. Let $\mathcal{Z}_\alpha$ denote a fixed, connected component of $\mathcal{Z}$. Consider the restriction of $\omega_p\colon \mathcal{Z}_\alpha \to \mathbb{R}$. Its image is discrete, lying inside $2\pi\mathbb{N}$, but ω_p is continuous in each component due to Lemma 12, thus the map is constant on connected components, proving Theorem 4.

4. Open questions

Having shown a counterexample to p-width spectral rigidity, it would be interesting to know any of the following:

- Do there exist counterexamples to the isospectral problem in dimensions $3 \leq n+1 \leq 7$? The minimal hypersurfaces which achieve the p-widths behave differently (e.g. are embedded and disjoint) in these dimensions compared to $n+1 = 2$, where self-intersecting geodesics, or two geodesics intersecting may occur. See [1,32] which highlight the differences between multiplicity, immersion vs. embeddedness, and index bounds of the underlying minimal objects in dimensions $n+1 = 2$ vs. $3 \leq n+1 \leq 7$.
- Are there other surfaces for which (Σ^2, g) is p-width spectrally rigid up to isometry? We note that shortly after the release of this work, Ambrozio–Marques–Neves [5] demonstrated that $\mathbb{RP}^2$ with the round metric is p-width spectrally rigid.
- Do there exist other reasonable geometric conditions that one can impose on a manifold so that isospectral rigidity holds?

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Declaration of interests

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