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Liouville theorems for p -Laplace equations with critical Hardy–Hénon exponents

Théorèmes de Liouville pour les équations de p -Laplace avec des exposants critiques de Hardy–Hénon

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Abstract. This paper is concerned with possibly sign-changing solutions to the critical p -Laplace equation $-\Delta_p u = |x|^\alpha |u|^{p_\alpha^* - 2} u$ in Ω , $u = 0$ on $\partial\Omega$, where $1 < p < N$, $\alpha > -p$, $p_\alpha^* = \frac{(N+\alpha)p}{N-p}$, and Ω is a bounded or unbounded domain of \mathbb{R}^N . We first derive a Liouville type theorem on half-spaces. Then we classify solutions via the radial symmetry and Morse index. Moreover, we characterize the compactness of radial Palais–Smale sequences on radial domains.

Résumé. Cet article s'intéresse aux solutions de l'équation critique de p -Laplace qui peuvent changer de signe. $-\Delta_p u = |x|^\alpha |u|^{p_\alpha^* - 2} u$ dans Ω , $u = 0$ sur $\partial\Omega$, où $1 < p < N$, $\alpha > -p$, $p_\alpha^* = \frac{(N+\alpha)p}{N-p}$, et Ω est un domaine borné ou non de \mathbb{R}^N . Nous dérivons d'abord un théorème de type Liouville sur les demi-espaces. Ensuite, nous classons les solutions en fonction de la symétrie radiale et de l'indice de Morse. De plus, nous caractérisons la compacité des suites de Palais–Smale radiales sur les domaines radiaux.

Keywords. p -Laplace equations, Hardy–Hénon exponents, Liouville theorems, Palais–Smale sequences.

Mots-clés. Équations de p -Laplace, exposants de Hardy–Hénon, théorèmes de Liouville, suites de Palais–Smale.

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1. Introduction

Unless specified, in this paper we always assume that $N \geq 2$, $1 < p < N$, $-p < \alpha < +\infty$ and Ω is a smooth possibly unbounded domain of \mathbb{R}^N . We are concerned with the critical Hardy–Hénon problem

$$\begin{cases} -\Delta_p u = |x|^\alpha |u|^{p_\alpha^* - 2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p -Laplacian and $p_\alpha^* = \frac{(N+\alpha)p}{N-p}$ is the critical Hardy–Hénon–Sobolev exponent. To introduce a functional setting for (1), we recall the homogeneous Sobolev space

$$\mathcal{D}^{1,p}(\mathbb{R}^N) = \left\{ u \in L^{p_\alpha^*}(\mathbb{R}^N) : \nabla u \in L^p(\mathbb{R}^N; \mathbb{R}^N) \right\},$$

which is endowed with the norm

$$\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u|^p dx \right)^{\frac{1}{p}}.$$

When $-p < \alpha \leq 0$, problem (1) is related to Hardy–Sobolev’s inequality

$$S_\alpha \left(\int_{\mathbb{R}^N} |x|^\alpha |u|^{p_\alpha^*} dx \right)^{\frac{p}{p_\alpha^*}} \leq \int_{\mathbb{R}^N} |\nabla u|^p dx \tag{2}$$

which holds for all $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$. The best Hardy–Sobolev constant $S_\alpha > 0$ is defined as

$$S_\alpha = \inf \left\{ \int_{\mathbb{R}^N} |\nabla u|^p dx : \int_{\mathbb{R}^N} |x|^\alpha |u|^{p_\alpha^*} dx = 1 \right\}$$

(see [1,2]). In the case $\alpha > 0$, inequality (2) holds for all $u \in \mathcal{D}_{\text{rad}}^{1,p}(\mathbb{R}^N)$, where we denote by $\mathcal{D}_{\text{rad}}^{1,p}(\mathbb{R}^N)$ the subspace of radial functions in $\mathcal{D}^{1,p}(\mathbb{R}^N)$, i.e.,

$$\mathcal{D}_{\text{rad}}^{1,p}(\mathbb{R}^N) = \left\{ u \in \mathcal{D}^{1,p}(\mathbb{R}^N) : u(x) = u(|x|) \right\}$$

(see [22]).

We denote by $\mathcal{D}_0^{1,p}(\Omega)$ the closure of $\mathcal{D}(\Omega)$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$. Notice that $\mathcal{D}_0^{1,p}(\Omega) = W_0^{1,p}(\Omega)$ when Ω is bounded.

The corresponding energy functional of (1) is

$$E(u) = \int_{\Omega} \left(\frac{|\nabla u|^p}{p} - \frac{|x|^\alpha |u|^{p_\alpha^*}}{p_\alpha^*} \right) dx.$$

Taking into account (2), we see that E is well-defined on the space

$$\mathcal{W}(\Omega) = \begin{cases} \mathcal{D}_0^{1,p}(\Omega) & \text{if } -p < \alpha \leq 0, \\ \mathcal{D}_0^{1,p}(\Omega) \cap L^{p_\alpha^*}(\Omega, |x|^\alpha) & \text{if } \alpha > 0, \end{cases}$$

where the space

$$L^{p_\alpha^*}(\Omega, |x|^\alpha) = \left\{ u : \Omega \rightarrow \mathbb{R} \text{ is measurable such that } \int_{\mathbb{R}^N} |x|^\alpha |u|^{p_\alpha^*} dx < +\infty \right\}$$

is endowed with the norm $\|u\|_{L^{p_\alpha^*}(\omega, |x|^\alpha)} = \left(\int_{\mathbb{R}^N} |x|^\alpha |u|^{p_\alpha^*} dx \right)^{\frac{1}{p_\alpha^*}}$.

In the scope of this paper, we are only interested in finite energy solutions to (1), which are defined as critical points of E in $\mathcal{W}(\Omega)$. More precisely, we call $u \in \mathcal{W}(\Omega)$ a weak solution to (1) if

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx = \int_{\Omega} |x|^\alpha |u|^{p_\alpha^* - 2} u \varphi dx \quad \text{for all } \varphi \in C_c^1(\Omega). \tag{3}$$

If Ω is radially symmetric, we denote by $\mathcal{D}_{0,\text{rad}}^{1,p}(\Omega)$ (resp. $W_{0,\text{rad}}^{1,p}(\Omega)$) the space of radial functions in $\mathcal{D}_0^{1,p}(\Omega)$ (resp. $W_0^{1,p}(\Omega)$). Then $\mathcal{D}_{0,\text{rad}}^{1,p}(\Omega) \subset \mathcal{W}(\Omega)$.

By exploiting Moser’s iteration technique and the standard elliptic regularity (see [23] and also [8,17,25]), for solution u to (1), we have

$$\begin{aligned} u &\in C^1(\overline{\Omega}) \cap L^\infty(\Omega) \quad \text{if } \alpha \geq 0, \\ u &\in C^1(\overline{\Omega} \setminus \{0\}) \cap L^\infty(\Omega) \quad \text{if } -p < \alpha < 0. \end{aligned}$$

When $\Omega = \mathbb{R}^N$, there is no boundary condition and hence problem (1) reads as

$$-\Delta_p u = |x|^\alpha |u|^{p_\alpha^* - 2} u \quad \text{in } \mathbb{R}^N. \tag{4}$$

Positive $\mathcal{D}^{1,p}(\mathbb{R}^N)$ solutions to (4) with $-p < \alpha < 0$ have been classified recently in [15] (see also [3]). More precisely, every positive $\mathcal{D}^{1,p}(\mathbb{R}^N)$ solution to (4) with $-p < \alpha < 0$ must be of the form

$$U_\lambda(x) = \left(\frac{(N + \alpha)^{\frac{1}{p}} \left(\frac{N-p}{p-1} \right)^{\frac{p-1}{p}} \lambda}{\lambda^p + |x|^{\frac{p+\alpha}{p-1}}} \right)^{\frac{N-p}{p+\alpha}} \tag{5}$$

for some $\lambda > 0$. Moreover, in the case $\alpha > 0$, all radial positive solutions to (4) are given by formula (5) (see [13]).

By Pohozaev’s identity [13,14], it is also well-known that problem (1) has no positive solutions if Ω is a bounded star-shaped domain about the origin. However, (1) may have positive solutions if Ω is not star-shaped. Indeed, it was proved in [18] that problem (1) with $\alpha = 0$ admits a positive solution on a bounded non-contractible domain with a small hole. This result extends the classical one of Coron [6].

In this paper, we study possibly sign-changing solutions to problem (1) and (4). Since the class of sign-changing solutions may have a very complex structure, a phenomenon that has been seen in the case $\alpha = 0$ (see [4,9,21] for $p = 2$ and [5] for $p \neq 2$), there is little hope to classify all possibly sign-changing solutions to these critical problems in general domains. Nevertheless, we have the following Liouville type theorem for quasilinear Hardy–Hénon equations on the halfspace $\mathbb{R}_+^N = \{x \in \mathbb{R}^N : x_N > 0\}$.

Theorem 1. *Assume $\Omega = \mathbb{R}_+^N$ and $\alpha > 0$. Let $u \in \mathcal{W}(\mathbb{R}_+^N)$ be a possibly sign-changing weak solution of problem (1). Then $u \equiv 0$.*

The same conclusion holds for the case $-p < \alpha < 0$ if we further assume that $|\nabla u(x)| = o(|x|^{\frac{1-N}{p}})$ as $|x| \rightarrow 0$.

To prove Theorem (1) we derive a local Pohozaev type identity by following the ideas in [10,12,19]. However, we use a regular form of $\partial_1 u$ instead of $\partial_N u$ as a test function. This new choice of test function allows us to exploit the assumption $\alpha \neq 0$ and get the conclusion. Notice that the case $\alpha = 0$, mentioned in [12], is still an open question. To the best of our knowledge, the best Liouville type theorem for possibly sign-changing solutions in the case $\alpha = 0$ is [12, Theorem 1.5], where the authors proved the result under the additional assumption that the Morse index of the solution is not greater than 1.

Next, we focus on radial solutions and try to derive some classification results. Problem of this type has been studied by Ni in [20] in the case $p = 2$. The following results extend the corresponding ones in [12] for $\alpha = 0$ to the case $\alpha > -p$.

Theorem 2. *Assume Ω is a ball centered at the origin. Let $u \in W_{0,\text{rad}}^{1,p}(\Omega)$ be a possibly sign-changing radial weak solution to problem (1). Then $u \equiv 0$.*

Theorem 3. *Let $u \in \mathcal{D}_{\text{rad}}^{1,p}(\mathbb{R}^N) \setminus \{0\}$ be a possibly sign-changing radial weak solution to problem (4). Then, either $u \equiv U_\lambda$ or $u \equiv -U_\lambda$ for some $\lambda > 0$.*

Theorem 3 allows us to derive the full description of the Palais–Smale sequences to the functional

$$J(u) = \int_{\Omega} \left(\frac{|\nabla u|^p}{p} + a(x) \frac{|u|^p}{p} - \frac{|x|^\alpha |u|^{p_\alpha^*}}{p_\alpha^*} \right) dx$$

in the radial setting (see Section 6 for more details).

In a non-radial setting, we prove some classification results via the Morse index. For $p \geq 2$, we can compute the second variation of E as

$$E''(u)[v, v] = \int_{\Omega} [|\nabla u|^{p-2} |\nabla v|^2 + (p-2) |\nabla u|^{p-4} (\nabla u \cdot \nabla v)^2 - (p_\alpha^* - 1) |x|^\alpha |u|^{p_\alpha^*-2} |v|^2] dx$$

for all $v \in C_c^1(\Omega)$. As in [7,11], we say that u has *Morse index* $i(u)$ if $i(u)$ is the maximal dimension of the subspaces V of $C_c^1(\Omega)$ such that

$$E''(u)[v, v] < 0 \quad \text{for all } v \in V \setminus \{0\}.$$

We have the following classification result for solutions with Morse index up to 1.

Theorem 4. *Assume $2 \leq p < N$ and $-p < a < 0$. Let $u \in \mathcal{D}^{1,p}(\mathbb{R}^N) \setminus \{0\}$ be a weak solution of problem (4) such that $i(u) \leq 1$. Then, either $u \equiv U_\lambda$ or $u \equiv -U_\lambda$ for some $\lambda > 0$.*

We do not know, however, if solutions U_λ have an index exactly equal to 1. This is still an open question even in the case $\alpha = 0$.

For bounded domains, we have the following Liouville type result.

Theorem 5. *Assume $2 \leq p < N$ and Ω is a smooth bounded domain of \mathbb{R}^N , star-shaped about the origin, namely $x \cdot \nu \geq 0$ on $\partial\Omega$, where ν is the exterior normal unit vector. Let $u \in \mathcal{W}(\Omega)$ be a weak solution to problem (1) such that $i(u) \leq 1$. Then $u \equiv 0$.*

The remainder of this paper is organized as follows. In Section 2 we prove Liouville type theorems on half space. In Section 3 we prove some claims on the nodal regions of solutions. In Section 4 and Section 5 we classify solutions via radial symmetry and Morse index, respectively. Section 6 is devoted to the descriptions for the radial Palais–Smale sequences.

2. Liouville type theorem on a halfspace

In this section, we prove Theorem 1. To this end, we always assume that $u \in \mathcal{W}(\mathbb{R}_+^N)$ is a weak solution of problem (1). Since $u \in C^1(\overline{\mathbb{R}_+^N}) \cap L^\infty(\mathbb{R}_+^N)$ if $\alpha \geq 0$ and $u \in C^1(\overline{\mathbb{R}_+^N} \setminus \{0\}) \cap L^\infty(\mathbb{R}_+^N)$ if $-p < \alpha < 0$, we may extend u by odd reflection, i.e., $u(x', x_N) = -u(x', -x_N)$ for $x_N < 0$. Therefore, in this section we can assume that $u \in \mathcal{W}(\mathbb{R}^N)$ is a solution to (4) that satisfies the condition $u = 0$ on $\partial\mathbb{R}_+^N$.

Hereafter we set $B_R = \{x \in \mathbb{R}^N : |x| < R\}$ and denote by $\nu(\cdot)$ the exterior unit normal to $\partial(B_R \cap \mathbb{R}_+^N)$ whose the first component is $\nu_1(\cdot)$. The partial derivative in x_1 will be denoted by ∂_1 .

Proof of Theorem 1. Following [8, p. 833], we take $R > \delta > 0$ and let u_ε be the unique solution to the nondegenerate boundary value problem

$$\begin{cases} -\operatorname{div}\left(\left(\varepsilon + |\nabla u_\varepsilon|^2\right)^{\frac{p-2}{2}} \nabla u_\varepsilon\right) = |x|^\alpha |u|^{p_\alpha^*-2} u & \text{in } B_{2R} \setminus B_{\delta/2}, \\ u_\varepsilon = u & \text{on } \partial(B_{2R} \setminus B_{\delta/2}). \end{cases} \tag{6}$$

Then $u_\varepsilon \in C^2(\overline{B_R} \setminus B_\delta)$ and uniformly bounded for $\varepsilon \in (0, 1]$ in $C^{1,\gamma}(\overline{B_R} \setminus B_\delta)$ for some $0 < \gamma < 1$. The Ascoli–Arzelà theorem implies the existence of a suitable sequence $\varepsilon \rightarrow 0^+$ such that $u_\varepsilon \rightarrow u$ and $\nabla u_\varepsilon \rightarrow \nabla u$ uniformly on $\overline{B_R} \setminus B_\delta$.

Let us define

$$v_\varepsilon = \partial_1 u_\varepsilon \left(\varepsilon + |\nabla u_\varepsilon|^2 \right)^{\frac{p-2}{2}} \nabla u_\varepsilon.$$

Since

$$\begin{aligned} \operatorname{div}(v_\varepsilon) &= \partial_1 u_\varepsilon \operatorname{div}\left((\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-2}{2}} \nabla u_\varepsilon\right) + (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-2}{2}} \nabla u_\varepsilon \cdot \nabla \partial_1 u_\varepsilon \\ &= \partial_1 u_\varepsilon \operatorname{div}\left((\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-2}{2}} \nabla u_\varepsilon\right) + \partial_1 \left(\frac{(\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p}{2}}}{p}\right), \end{aligned}$$

by the divergence theorem, we have

$$\begin{aligned} \int_{(B_R \setminus B_\delta) \cap \mathbb{R}_+^N} \partial_1 u_\varepsilon \operatorname{div}\left((\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-2}{2}} \nabla u_\varepsilon\right) dx \\ = \int_{\partial((B_R \setminus B_\delta) \cap \mathbb{R}_+^N)} v_\varepsilon \cdot \nu(\sigma) d\sigma - \int_{\partial((B_R \setminus B_\delta) \cap \mathbb{R}_+^N)} \frac{(\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p}{2}}}{p} \nu_1(\sigma) d\sigma. \end{aligned} \tag{7}$$

On the other hand,

$$\begin{aligned} \int_{(B_R \setminus B_\delta) \cap \mathbb{R}_+^N} \partial_1 u |x|^\alpha |u|^{p_\alpha^* - 2} u dx \\ = \int_{(B_R \setminus B_\delta) \cap \mathbb{R}_+^N} \partial_1 \left(\frac{|x|^\alpha |u|^{p_\alpha^*}}{p_\alpha^*}\right) dx - \frac{\alpha}{p_\alpha^*} \int_{(B_R \setminus B_\delta) \cap \mathbb{R}_+^N} |x|^{\alpha-2} x_1 |u|^{p_\alpha^*} dx \\ = \int_{\partial((B_R \setminus B_\delta) \cap \mathbb{R}_+^N)} \frac{|x|^\alpha |u|^{p_\alpha^*}}{p_\alpha^*} \nu_1(\sigma) d\sigma - \frac{\alpha}{p_\alpha^*} \int_{(B_R \setminus B_\delta) \cap \mathbb{R}_+^N} |x|^{\alpha-2} x_1 |u|^{p_\alpha^*} dx. \end{aligned} \tag{8}$$

From (6), we deduce that

$$\int_{(B_R \setminus B_\delta) \cap \mathbb{R}_+^N} \partial_1 u_\varepsilon \operatorname{div}\left((\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-2}{2}} \nabla u_\varepsilon\right) dx = - \int_{(B_R \setminus B_\delta) \cap \mathbb{R}_+^N} \partial_1 u |x|^\alpha |u|^{p_\alpha^* - 2} u dx + o(\varepsilon). \tag{9}$$

Substituting (7) and (8) into (9), we get

$$\begin{aligned} \int_{\partial((B_R \setminus B_\delta) \cap \mathbb{R}_+^N)} v_\varepsilon \cdot \nu(\sigma) d\sigma - \int_{\partial((B_R \setminus B_\delta) \cap \mathbb{R}_+^N)} \frac{(\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p}{2}}}{p} \nu_1(\sigma) d\sigma \\ = - \int_{\partial((B_R \setminus B_\delta) \cap \mathbb{R}_+^N)} \frac{|x|^\alpha |u|^{p_\alpha^*}}{p_\alpha^*} \nu_1(\sigma) d\sigma + \frac{\alpha}{p_\alpha^*} \int_{(B_R \setminus B_\delta) \cap \mathbb{R}_+^N} |x|^{\alpha-2} x_1 |u|^{p_\alpha^*} dx + o(\varepsilon). \end{aligned}$$

Let $\varepsilon \rightarrow 0^+$, we derive

$$\begin{aligned} \int_{\partial((B_R \setminus B_\delta) \cap \mathbb{R}_+^N)} \partial_1 u |\nabla u|^{p-2} \nabla u \cdot \nu(\sigma) d\sigma - \int_{\partial((B_R \setminus B_\delta) \cap \mathbb{R}_+^N)} \frac{|\nabla u|^p}{p} \nu_1(\sigma) d\sigma \\ = - \int_{\partial((B_R \setminus B_\delta) \cap \mathbb{R}_+^N)} \frac{|x|^\alpha |u|^{p_\alpha^*}}{p_\alpha^*} \nu_1(\sigma) d\sigma + \frac{\alpha}{p_\alpha^*} \int_{(B_R \setminus B_\delta) \cap \mathbb{R}_+^N} |x|^{\alpha-2} x_1 |u|^{p_\alpha^*} dx. \end{aligned}$$

This implies

$$\begin{aligned} \frac{\alpha}{p_\alpha^*} \int_{(B_R \setminus B_\delta) \cap \mathbb{R}_+^N} |x|^{\alpha-2} x_1 |u|^{p_\alpha^*} dx \\ = \int_{\mathbb{R}_+^N \cap \partial(B_R \setminus B_\delta)} \left[\partial_1 u |\nabla u|^{p-2} \nabla u \cdot \nu(\sigma) - \frac{|\nabla u|^p}{p} \nu_1(\sigma) + \frac{|x|^\alpha |u|^{p_\alpha^*}}{p_\alpha^*} \nu_1(\sigma) \right] d\sigma \\ \leq C \int_{\mathbb{R}_+^N \cap \partial B_\delta} (|\nabla u|^p + |x|^\alpha |u|^{p_\alpha^*}) d\sigma + C \int_{\mathbb{R}_+^N \cap \partial B_R} (|\nabla u|^p + |x|^\alpha |u|^{p_\alpha^*}) d\sigma \end{aligned} \tag{10}$$

for some constant $C > 0$.

If $\alpha > 0$, then u and $|\nabla u|$ are bounded, hence

$$\int_{\mathbb{R}_+^N \cap \partial B_\delta} (|\nabla u|^p + |x|^\alpha |u|^{p_\alpha^*}) d\sigma \rightarrow 0 \quad \text{as } \delta \rightarrow 0. \tag{11}$$

If $-p < \alpha < 0$, by exploiting the assumption $|\nabla u(x)| = o(|x|^{\frac{1-N}{p}})$ as $|x| \rightarrow 0$, we derive $|u(x)| = o(|x|^{\frac{1-N+p}{p}})$ and

$$\int_{\mathbb{R}_+^N \cap \partial B_\delta} (|\nabla u|^p + |x|^\alpha |u|^{p_\alpha^*}) d\sigma \leq C \left(\delta^{N-1} o(\delta^{1-N}) + \delta^{N-1+\alpha} o\left(s^{\frac{(1-N+p)(N+\alpha)}{N-p}}\right) \right).$$

Since

$$N - 1 + \alpha + \frac{(1 - N + p)(N + \alpha)}{N - p} = \frac{\alpha + p}{N - p} > 0,$$

we deduce that (11) also holds in this case.

On the other hand, since $|\nabla u| \in L^p(\mathbb{R}_+^N)$ and $u \in L^{p_\alpha^*}(\mathbb{R}_+^N, |x|^\alpha)$, there exists a sequence of radii $R_k \rightarrow \infty$ such that

$$R_k \int_{\mathbb{R}_+^N \cap \partial B_{R_k}} (|\nabla u|^p + |x|^\alpha |u|^{p_\alpha^*}) d\sigma \rightarrow 0 \quad \text{as } k \rightarrow \infty. \tag{12}$$

Hence letting $\delta \rightarrow 0^+$ and $R = R_k \rightarrow \infty$ in (10) and using (11) and (12), we deduce

$$\frac{\alpha}{p_\alpha^*} \int_{\mathbb{R}_+^N} |x|^{\alpha-2} x_1 |u|^{p_\alpha^*} dx = 0.$$

Since $\alpha \neq 0$, this implies $u \equiv 0$. □

3. On nodal regions of solutions

We call ω a nodal domain of a continuous function $u: \Omega \rightarrow \mathbb{R}$ if ω is a connected component of the set $\{x \in \Omega : u(x) \neq 0\}$. In order to use $u|_\omega$ as a test function in (3), we need the following approximation result.

Lemma 6. *Assume $u \in C_{loc}^{0,1}(\mathbb{R}^N \setminus \{0\}) \cap \mathcal{W}(\mathbb{R}^N)$. Let ω be a nodal domain of u , and $u|_\omega$ its restriction to ω . Then there exists a sequence $\{u_n\}_n \subset C_c^{0,1}(\omega)$ such that:*

- (i) $u_n \rightarrow u|_\omega$ in $L^{p_\alpha^*}(\omega, |x|^\alpha)$ and everywhere in ω ;
- (ii) $\nabla u_n \rightarrow \nabla u|_\omega$ in $L^p(\omega; \mathbb{R}^N)$ and almost everywhere in ω .

The proof of Lemma 6 is similar to that of [12, Lemma 3.1] and hence will be omitted. Motivated by [12], the main result of this section is the following claim on nodal regions of solutions.

Theorem 7. *Let $u \in \mathcal{W}(\Omega)$ be a nontrivial solution to (1). Then:*

- (i) *For every nodal domain ω of u it holds that*

$$\int_\omega |\nabla u|^p dx = \int_\omega |x|^\alpha |u|^{p_\alpha^*} dx.$$

- (ii) *Let \mathcal{N}_u be the set of nodal domains of u and $|\mathcal{N}_u|$ its cardinality. If $-p < \alpha < 0$, then*

$$|\mathcal{N}_u| \leq S_\alpha^{-\frac{N+\alpha}{p+\alpha}} \int_\Omega |x|^\alpha |u|^{p_\alpha^*} dx.$$

In particular, u has at most a finite number of nodal domains. The same conclusions hold for nontrivial solutions $u \in \mathcal{D}_{0,rad}^{1,p}(\Omega)$ to (1) in the case that $\alpha > 0$ and Ω is radially symmetric.

Proof of Theorem 7. Since $u \in C^1(\overline{\Omega} \setminus \{0\})$, we may extend u by zero outside Ω to obtain $u \in C_{loc}^{0,1}(\mathbb{R}^N \setminus \{0\}) \cap \mathcal{W}(\Omega)$.

Now we take a sequence (u_n) given by Lemma 6 and extend u_n by zero outside ω . By a standard density argument for every $n \in \mathbb{N}$ we can use u_n as a test function in (3) to deduce

$$\int_\omega |\nabla u|^{p-2} \nabla u \cdot \nabla u_n dx = \int_\omega |x|^\alpha |u|^{p_\alpha^*-2} u u_n dx.$$

Letting $n \rightarrow \infty$ and using Lemma (6), we obtain (i).

Now we prove (ii). Using inequality (2) with $v = u|_{\omega}$ extended by zero outside ω from (i) we have

$$\int_{\omega} |\nabla u|^p dx = \int_{\mathbb{R}^N} |\nabla v|^p dx \geq S_{\alpha} \left(\int_{\omega} |x|^{\alpha} |u|^{p_{\alpha}^*} dx \right)^{\frac{p}{p_{\alpha}^*}} = S_{\alpha} \left(\int_{\omega} |\nabla u|^p dx \right)^{\frac{p}{p_{\alpha}^*}}.$$

Hence

$$\int_{\omega} |\nabla u|^p dx \geq S_{\alpha}^{\frac{N+\alpha}{p+\alpha}}.$$

It follows that

$$\int_{\Omega} |x|^{\alpha} |u|^{p_{\alpha}^*} dx = \int_{\Omega} |\nabla u|^p dx = \sum_{\omega \in \mathcal{N}_u} \int_{\omega} |\nabla u|^p dx \geq \sum_{\omega \in \mathcal{N}_u} S_{\alpha}^{\frac{N+\alpha}{p+\alpha}} = |\mathcal{N}_u| S_{\alpha}^{\frac{N+\alpha}{p+\alpha}}.$$

This proves (ii). □

4. Classification of radial solutions

We prove Theorems 2 and 3 in this section. First, we recall that if u is a solution to

$$\begin{cases} -\Delta_p u = |x|^{\alpha} |u|^{q-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $1 < q \leq p_{\alpha}^*$, then it is standard to derive the Pohozaev's identity

$$\frac{p-1}{p} \int_{\partial\Omega} |\nabla u|^p x \cdot \nu(\sigma) d\sigma = \left(\frac{N+\alpha}{q} - \frac{N-p}{p} \right) \int_{\Omega} |x|^{\alpha} |u|^q dx,$$

see [13, Theorem 2.1] for instance. In particular, if u is a solution to (1), then

$$\int_{\partial\Omega} |\nabla u|^p x \cdot \nu(\sigma) d\sigma = 0. \tag{13}$$

Proof of Theorem 2. Suppose $u \neq 0$. We recall that $u \in C^1(\overline{\Omega} \setminus \{0\}) \cap L^{\infty}(\Omega)$.

If u does not change sign, then by the strong maximum principle we may assume $u > 0$ in $\Omega \setminus \{0\}$. Since Ω is a ball, identity (13) implies $|\nabla u| = 0$ on $\partial\Omega$. However, this contradicts Hopf's boundary point lemma.

Therefore, u must change the sign. Since u is radial, the nodal regions of u are spherically symmetric. By Theorem 7, the number of such regions is finite. We can take an annular nodal region $A = \{x \in \Omega : R_1 < |x| < R_2\}$ with $R_2 > R_1 > 0$ and assume $u > 0$ in A . Then u solves

$$\begin{cases} -\Delta_p u = |x|^{\alpha} |u|^{p_{\alpha}^*-2} u & \text{in } B_{R_2}, \\ u > 0 & \text{in } A, \\ u = 0 & \text{on } \partial A. \end{cases} \tag{14}$$

By Pohozaev's identity (13) again, we have $|\nabla u| = 0$ on ∂B_{R_2} . This again contradicts Hopf's boundary point lemma. The proof is concluded. □

Proof of Theorem 3. If u does not change the sign, then $|u| > 0$ by the strong maximum principle. Hence [13, Theorem 3.1] can be applied to yield $u \equiv U_{\lambda}$ or $u \equiv -U_{\lambda}$ for some $\lambda > 0$.

It remains to consider the case that u changes sign. By radial symmetry and Theorem 7, the nodal regions are spherically symmetric and their number is finite. Hence, by replacing u by $-u$ if necessary, we find $R > 0$ large enough such that

$$\begin{aligned} u &= 0 && \text{on } \partial B_R, \\ u &> 0 && \text{in } \mathbb{R}^N \setminus \overline{B_R}, \end{aligned}$$

and so, by Theorem 2,

$$\begin{aligned} u &= 0 && \text{in } \overline{B_R}, \\ u &> 0 && \text{in } \mathbb{R}^N \setminus \overline{B_R}. \end{aligned}$$

On the other hand, by continuity of ∇u , we have $|\nabla u| = 0$ on ∂B_R . However, this is in contradiction with Hopf’s boundary point lemma on $\mathbb{R}^N \setminus \overline{B_R}$. This finishes the proof. \square

5. Classification of solutions with Morse index at most 1

Motivated by [12], we show that the number of nodal regions of a solution is bounded by its own Morse index.

Theorem 8. *Assume $2 \leq p < N$ and Ω is a smooth domain of \mathbb{R}^N . Let $u \in \mathcal{W}(\Omega)$ be a weak solution to problem (1). Then*

$$|\mathcal{N}_u| \leq i(u),$$

where $i(u)$ is the Morse index of u .

Proof. Consider a nodal region ω . Let (u_n) be given by Lemma 6 and extend u_n by zero outside ω . By Lemma 6 we have

$$\begin{aligned} E''(u)[u_n, u_n] &= \int_{\omega} [|\nabla u|^{p-2} |\nabla u_n|^2 + (p-2) |\nabla u|^{p-4} (\nabla u \cdot \nabla u_n)^2 - (p_{\alpha}^* - 1) |x|^{\alpha} |u|^{p_{\alpha}^* - 2} |u_n|^2] dx \\ &\longrightarrow \int_{\omega} [(p-1) |\nabla u|^p - (p_{\alpha}^* - 1) |x|^{\alpha} |u|^{p_{\alpha}^*}] dx \quad \text{as } n \rightarrow \infty. \end{aligned}$$

This fact and Theorem 7 (i) yield

$$E''(u)[u_n, u_n] \longrightarrow (p - p_{\alpha}^*) \int_{\omega} |x|^{\alpha} |u|^{p_{\alpha}^*} dx < 0.$$

Therefore, for every nodal region ω there exists $v \in C_c^{0,1}(\omega)$ (and by density in $C_c^1(\omega)$) such that

$$E''(u)[v, v] < 0.$$

By the definition of Morse index, this implies $|\mathcal{N}_u| \leq i(u)$. \square

Now we can prove our classification results for solutions u with Morse index $i(u) \leq 1$.

Proof of Theorem 4. By Theorem 8, u has exactly one nodal region, which is denoted by A .

If A is a proper subset and $A \neq \mathbb{R}^N \setminus \{0\}$, we can assume, up to consider $-u$ instead of u , that $A = \{u > 0\}$. Since $u \in C^1(\mathbb{R}^N \setminus \{0\})$, we have $|\nabla u| = 0$ on $\partial A \setminus \{0\}$. We can pick an interior point $p \in A$ and a radius $R > 0$ such that the ball $B_R(p) \subset A$ touches ∂A at some point $p' \neq 0$. Now we can apply Hopf’s boundary point lemma at boundary point p' for equation on $B_R(p)$ to get $|\nabla u(p')| \neq 0$. This is a contradiction.

Therefore, either $A = \mathbb{R}^N$ or $A = \mathbb{R}^N \setminus \{0\}$. Then the conclusion follows by a recent classification result in [3,15]. This concludes the proof. \square

Proof of Theorem 5. Since Ω is star-shaped, we may use (13) to deduce that the normal derivative $\partial_{\nu} u$ vanishes at some point $x_0 \in \partial\Omega$. On the other hand, Theorem 8 implies that u has at most one nodal region. If u were nontrivial, we get a contradiction with Hopf’s boundary point lemma. Hence it must happen that $u \equiv 0$. \square

6. Descriptions for the Palais–Smale sequences in the radial setting

In this section, we consider the problem

$$\begin{cases} -\Delta_p u + a(x)|u|^{p-2}u = |x|^\alpha |u|^{p_\alpha^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{15}$$

where $-p < \alpha < 0$, Ω is bounded and $a \in L^{\frac{N}{p}}(\Omega)$. Solutions to (15) coincides with the critical points in $W_0^{1,p}(\Omega)$ of the functional

$$J(u) = \int_{\Omega} \left(\frac{|\nabla u|^p}{p} + a(x) \frac{|u|^p}{p} - \frac{|x|^\alpha |u|^{p_\alpha^*}}{p_\alpha^*} \right) dx.$$

We also define on $\mathcal{D}^{1,p}(\mathbb{R}^N)$ the limit functional

$$J_\infty(u) = \int_{\mathbb{R}^N} \left(\frac{|\nabla u|^p}{p} - \frac{|x|^\alpha |u|^{p_\alpha^*}}{p_\alpha^*} \right) dx.$$

We recall that $\{u_n\}_n \subset W_0^{1,p}(\Omega)$ is called a Palais–Smale sequence of J at energy level $c \in \mathbb{R}$ if

$$J(u_n) \rightarrow c \quad \text{and} \quad J'(u_n) \rightarrow 0 \quad \text{in } W^{-1,p'}(\Omega).$$

The compactness of Palais–Smale sequences is a vital condition in the theory of critical points. Using the global compactness results in [16] and an idea in [12] we can give a precise description of Palais–Smale sequences for functional J on the unit ball as follows.

Proposition 9. *Assume $-p < \alpha < 0$, Ω is the unit ball centered at the origin in \mathbb{R}^N and $a \in L^{\frac{N}{\text{rad}}}_p(\Omega)$ such that*

$$\inf_{\substack{u \in W_0^{1,p}(\Omega) \\ \|u\|=1}} \int_{\Omega} [|\nabla u|^p + a(x)|u|^p] dx > 0.$$

Let $\{u_n\}_n$ be a sequence in $W_{0,\text{rad}}^{1,p}(\Omega)$ such that

$$J(u_n) \rightarrow c \quad \text{and} \quad J'(u_n) \rightarrow 0 \quad \text{in } (W_{0,\text{rad}}^{1,p}(\Omega))'.$$

Then, passing if necessary to a subsequence, there exists a possibly trivial solution $v_0 \in W_{0,\text{rad}}^{1,p}(\Omega)$ to (15) and k sequences $\{\lambda_n^i\}_n \subset \mathbb{R}_+$, $i = 1, \dots, k$, such that $\lambda_n^i \rightarrow 0$ as $n \rightarrow \infty$ and

$$\left\| u_n - v_0 - \sum_{i=1}^k (\lambda_n^i)^{\frac{p-N}{p}} v_i \left(\frac{\cdot}{\lambda_n^i} \right) \right\| \rightarrow 0, \quad n \rightarrow \infty, \tag{16}$$

$$\|u_n\|^p \rightarrow \sum_{i=0}^k \|v_i\|^p, \quad n \rightarrow \infty, \tag{17}$$

$$J(v_0) + \sum_{i=1}^k J_\infty(v_i) = c, \tag{18}$$

where for each $i = 1, \dots, k$, either $v_i \equiv V_{\mu_i}$ or $v_i \equiv -V_{\mu_i}$ for some $\mu_i > 0$ and

$$J_\infty(v_i) = \frac{p + \alpha}{(N + \alpha)p} S_\alpha^{\frac{N+\alpha}{p+\alpha}}.$$

Moreover if $a \equiv 0$, then all weakly convergent subsequences of $\{u_n\}_n$ are weakly convergent to zero in $W_{0,\text{rad}}^{1,p}(\Omega)$. In particular, $v_0 \equiv 0$ and hence $J(v_0) = 0$.

Proof. By [16, Theorem 3.1], we know that up to a subsequence, there exists a possibly trivial solution $v_0 \in W_{0,\text{rad}}^{1,p}(\Omega)$ to (15), k nontrivial solutions $\{v_1, \dots, v_k\} \subset \mathcal{D}^{1,p}_{\text{rad}}(\mathbb{R}^N)$ to (4) and k sequences $\{\lambda_n^i\}_n \subset \mathbb{R}_+$, such that $\lambda_n^i \rightarrow 0$ as $n \rightarrow \infty$ and (16)–(18) is satisfied. By Theorem 3, we have either $v_i \equiv V_{\mu_i}$ or $v_i \equiv -V_{\mu_i}$ for some $\mu_i > 0$. Hence $J_\infty(v_i)$ can be computed explicitly as in [24].

Assume now $a \equiv 0$. Then v_0 solves (1) in the unit ball. By Theorem 2, we deduce $v_0 \equiv 0$. Since v_0 is the weak limit of $\{u_n\}_n$, the conclusion follows. \square

We can state a similar result for the problem

$$-\Delta_p u + a(x)|u|^{p-2}u = |x|^\alpha |u|^{p_\alpha^* - 2}u \quad \text{in } \mathbb{R}^N. \tag{19}$$

Solutions in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ to (19) coincide with the critical points in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ of the functional

$$J(u) = \int_{\mathbb{R}^N} \left(\frac{|\nabla u|^p}{p} + a(x) \frac{|u|^p}{p} - \frac{|x|^\alpha |u|^{p_\alpha^*}}{p_\alpha^*} \right) dx.$$

As before, we also define on $\mathcal{D}^{1,p}(\mathbb{R}^N)$ the functional

$$J_\infty(u) = \int_{\mathbb{R}^N} \left(\frac{|\nabla u|^p}{p} - \frac{|x|^\alpha |u|^{p_\alpha^*}}{p_\alpha^*} \right) dx.$$

We have the following result.

Proposition 10. *Assume $-p < \alpha < 0$ and $a \in L^{\frac{N}{\text{rad}}}(\mathbb{R}^N)$ such that*

$$\inf_{\substack{u \in \mathcal{D}_{\text{rad}}^{1,p}(\mathbb{R}^N) \\ \|u\|=1}} \int_{\mathbb{R}^N} [|\nabla u|^p + a(x)|u|^p] dx > 0.$$

Let $\{u_n\}_n$ be a sequence in $\mathcal{D}_{\text{rad}}^{1,p}(\mathbb{R}^N)$ such that

$$J(u_n) \rightarrow c \quad \text{and} \quad J'(u_n) \rightarrow 0 \quad \text{in } (\mathcal{D}_{\text{rad}}^{1,p}(\mathbb{R}^N))'.$$

Then, passing if necessary to a subsequence, there exists a possibly trivial solution $v_0 \in \mathcal{D}_{\text{rad}}^{1,p}(\mathbb{R}^N)$ to (19) and k sequences $\{\lambda_n^i\}_n \subset \mathbb{R}_+$, $i = 1, \dots, k$, such that $\lambda_n^i \rightarrow 0$ as $n \rightarrow \infty$ and

$$\left\| u_n - v_0 - \sum_{i=1}^k (\lambda_n^i)^{\frac{p-N}{p}} v_i \left(\frac{\cdot}{\lambda_n^i} \right) \right\| \rightarrow 0, \quad n \rightarrow \infty,$$

$$\|u_n\|^p \rightarrow \sum_{i=0}^k \|v_i\|^p, \quad n \rightarrow \infty,$$

$$J(v_0) + \sum_{i=1}^k J_\infty(v_i) = c,$$

where for each $i = 1, \dots, k$, either $v_i \equiv V_{\mu_i}$ or $v_i \equiv -V_{\mu_i}$ for some $\mu_i > 0$ and

$$J_\infty(v_i) = \frac{p + \alpha}{(N + \alpha)p} S_\alpha^{\frac{N+\alpha}{p+\alpha}}.$$

Moreover if $a \equiv 0$, then either v_0 is identically zero, or it holds that for some $\lambda > 0$ and up to the sign $v_0 \equiv V_\lambda$ and

$$J(v_0) = \frac{p + \alpha}{(N + \alpha)p} S_\alpha^{\frac{N+\alpha}{p+\alpha}}.$$

Proof. It follows from [16, Theorem 4.1] and Theorem 3. Note that $J_\infty(v_i)$ can be computed explicitly by [24], using Theorem 3. \square

Declaration of interests

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