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A simple construction of the Anderson operator via its quadratic form in dimensions two and three

Une construction simple de l'opérateur d'Anderson par sa forme quadratique en dimensions deux et trois

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Abstract. We provide a simple construction of the Anderson operator in dimensions two and three. This is done through its quadratic form. We rely on an exponential transform instead of the regularity structures or paracontrolled calculus which are usually used for the construction of the operator. The knowledge of the form is robust enough to deduce important properties such as positivity and irreducibility of the corresponding semigroup. The latter property gives existence of a spectral gap.

Résumé. Nous fournissons une construction simple de l'opérateur d'Anderson en dimensions deux et trois. Cela est réalisé à travers sa forme quadratique. Nous nous appuyons sur une transformation exponentielle au lieu des structures de régularité ou du calcul paracontrôlé, qui sont généralement utilisés pour la construction de l'opérateur. La connaissance de la forme est suffisamment robuste pour déduire des propriétés importantes telles que la positivité et l'irréductibilité du semi-groupe correspondant. Cette dernière propriété permet de démontrer l'existence d'un trou spectral.

Keywords. Anderson form, singular stochastic operator, Schrödinger operator, renormalization, positivity, spectral gap.

Mots-clés. Forme d'Anderson, opérateurs stochastiques singuliers, opérateur de Schrödinger, renormalisation, positivité, trou spectral.

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Introduction

Over the last decade, the study of singular stochastic PDEs has grown to an important field with the introduction of regularity structures by Hairer [16] and paracontrolled calculus by Gubinelli, Imkeller and Perkowski [13]. The theory first aimed at the resolution of parabolic equations such as the Parabolic Anderson Model (PAM) equation or the Kardar–Parisi–Zhang (KPZ) equation, it then led to the construction of the Anderson Hamiltonian

$$H = -\Delta + \xi$$

with ξ the spatial white noise, see [1,9,15,19,23] in dimension 2 and 3, on a finite box with periodic or Dirichlet boundary conditions or even compact Riemannian manifolds.

In this note we provide a simple construction of this operator via its quadratic form without using regularity structures or paracontrolled calculus. We rely on an exponential transform first used by Hairer and Labbé [17] for the continuum parabolic Anderson model on $\mathbb{R}^2$ and then used in different context, see for example [3,8,10,18,28]. In particular, this was already used by Matsuda and van Zuijlen [20] to construct the rough form in the full subcritical regime using also regularity structures. Our work can be seen as an unsophisticated approach since no regularity structures or paracontrolled calculus appear here. See also [21,22] for other singular stochastic operators.

The Anderson Hamiltonian is the Schrödinger operator $H = -\Delta + \xi$ with ξ the space white noise which is a random distribution of negative Hölder regularity $-\frac{d}{2} - \kappa$ for any $\kappa > 0$. In one dimension, it is the derivative of the Brownian motion and the associated form

$$\mathfrak{a}(u, v) = \int_0^1 \nabla u(x) \cdot \nabla v(x) \mathrm{d}x + \int_0^1 u(x) v(x) \xi(\mathrm{d}x)$$

was constructed by Fukushima and Nakao [12] with domain the usual Sobolev space $\mathscr{H}^1$. The idea is that one can multiply two distributions if the sum of their regularity is positive, hence $uv\xi$ is well-defined as a distribution for $u, v \in \mathscr{H}^1$ since $\xi \in \mathscr{C}^{-\frac{1}{2}-\kappa}$. In two dimensions, $\xi \in \mathscr{C}^{-1-\kappa}$ and this construction is not possible anymore. Following recent progress on singular stochastic PDEs, the operator can be constructed with a random domain $\mathscr{D}_\xi^2$ depending on the noise ξ such that $H \colon \mathscr{D}_\xi^2 \subset L^2 \to L^2$ is an unbounded closed operator. Taking $u \in L^2$ and assuming that Hu is an element of L^2, one obtains the relation

$$\Delta u = u\xi - Hu$$

which induces an expansion of u with respect to the noise using regularity structures or paracontrolled calculus. In particular, [15] and [23] also identify a form domain, that is a random subspace $\mathscr{D}_\xi^1 \subset L^2$ such that

$$\forall\, u \in \mathscr{D}_\xi^1, \quad \big|\langle Hu, u\rangle\big| < \infty.$$

We emphasize in the notation that the domains are random and consist of random functions depending on the noise. In each case, the operator is a singular stochastic operator and a renormalization procedure is involved in its construction. For a regularization ξ_ε of the noise, the operator is constructed as a limit in the resolvent sense, that is

$$H = \lim_{\varepsilon \to 0} \big(-\Delta + \xi_\varepsilon - c_\varepsilon\big)$$

with a constant c_ε which explodes when $\varepsilon \to 0$. It is related to the definition of the product $\Delta^{-1}\xi \cdot \xi$ and the divergence of the Green function of the Laplacian. In two dimensions, one has $c_\varepsilon \sim \log(\varepsilon)$ while $c_\varepsilon \sim \varepsilon^{-1}$ in three dimensions. In one dimension, this product is well-defined and one can take $c_\varepsilon = 0$ which is coherent with [12] and $\mathscr{D}_\xi^1 = \mathscr{H}^1$ does not depend on the noise. However the domain of the operator is random and this method was recently used by Dumaz and Labbé to provide a precise study of the operator, see [11] and references therein. The Anderson form also appears as the energy for dispersive PDEs such as the nonlinear Schrödinger equation

$$i\partial_t u = \Delta u + u\xi + |u|^2 u$$

and was used to obtain solutions, see for example [8,10,15,23,24,28] and references therein. In particular, uniform bounds in energy is the crucial property of such singular stochastic PDEs where one does not have the regularizing properties of the parabolic equation. In this context, one has to work with random initial data depending on the noise and the conservation of energy makes the form domain of the Anderson operator a natural space to get a global solution.

In this work, we consider a new variable $u = e^X v$ for a suitable random field X. In this case, we have

$$\Delta u = e^X \Delta v + 2e^X \nabla X \cdot \nabla v + e^X \left(|\nabla X|^2 + \Delta X\right) v$$

and if X is a solution to $\Delta X = \xi$, the Anderson operator is formaly given by

$$Hu = -e^X \Delta v - 2e^X \nabla X \cdot \nabla v - e^X |\nabla X|^2 v.$$

In two dimensions, we have

$$\xi \in \mathscr{C}^{-1-\kappa} \quad \Longrightarrow \quad \nabla X \in \mathscr{C}^{-\kappa},$$

hence the square $|\nabla X|^2$ is singular and has to be defined with a renormalization procedure as a Wick product $|\nabla X|^{2\diamond} \in \mathscr{C}^{-2\kappa}$. In this case, $v \in \mathscr{H}^1$ is regular enough for the associated form to make sense and one can construct the Anderson form with domain $\mathscr{D}(\mathfrak{a}) = e^X \mathscr{H}^1$. In three dimensions, we have

$$\xi \in \mathscr{C}^{-\frac{3}{2}-\kappa} \quad \Longrightarrow \quad \nabla X \in \mathscr{C}^{-\frac{1}{2}-\kappa}$$

and the Wick product $|\nabla X|^{2\diamond} \in \mathscr{C}^{-1-2\kappa}$ is too rough to be multiplied by $v \in \mathscr{H}^1$. One can apply the same method and construct the Anderson form with domain $e^{X+Y}\mathscr{H}^1$ with a suitable second random field Y.

This exponential transform allows us to construct a symmetric form $\mathfrak{a}$ whose associated operator H is the Anderson Hamiltonian. Once $\mathfrak{a}$ is defined, the associated operator is given by

$$\mathscr{D}(H) = \left\{u \in L^2(\mathbb{T}^d)\ ;\ \exists\, v \in L^2(\mathbb{T}^d), \forall\, \varphi \in \mathscr{D}(\mathfrak{a}), \mathfrak{a}(u,\varphi) = \langle v, \varphi\rangle\right\}, \quad Hu = v.$$

This operator is well defined with dense domain $\mathscr{D}(H)$ in $L^2(\mathbb{T}^d)$. This weak formulation does not however give a precise description of $\mathscr{D}(H)$. Also, the approach using the exponential transform does not allow to construct the explicit domain of the operator. For this a more involved theory such as regularity structures or paracontrolled calculus seems to be necessary. See Section 4 for some additional details. Nevertheless, the knowledge of H through its form is enough to deduce that H is self-adjoint, it has a discrete spectrum and an L^2 orthonormal basis given by eigenfunctions. In addition, relying on a criterion from [25], we prove that the associated semigroup is irreducible. In particular, this implies the existence of a spectral gap $\lambda_1 < \lambda_2$ with a positive ground state $\Psi \in \mathscr{D}(H)$. This result was already proved in [4] by relying on a quantitative estimate for the linear Parabolic Anderson Model equation. Our work provides a pedestrian approach to this result even in three dimensions which usually relies on involved computations with expansion of order 5 using regularity structures or paracontrolled calculus.

In order to keep the ideas and tools simple we restrict ourselves to the case of the torus $\mathbb{T}^d$ for $d \in \{2,3\}$ (endowed with the Lebesgue measure $\mathrm{d}x$). Our construction works on any compact manifold without boundary even though we rely on Fourier series here. On a manifold, one can use Calderón–Zygmund formula as in [23, Section 2.1] with tools based on the heat semigroup of the Laplace–Beltrami operator on the manifold. Similar renormalization problems were also considered in [4] with microlocal and harmonic analysis on local charts.

In the work [20], Matsuda and van Zuijlen use the exponential transform to construct the Dirichlet form associated to a Schrödinger operator for a general class of random potentials in the full subcritical regime in the sense of singular SPDEs using regularity structures, that is dimension $d \in \{2,3\}$ as far as the Anderson hamiltonian is concerned. To do so, they use higher order iteration of the argument we use in this note and for which an arbitrary large number of singular stochastic terms need to be renormalized. They rely on the renormalization algorithm from the very involved works [6,7] which form two important part of the full theory of regularity structures with [5,16]. In particular, they show that the exponential transform can be used in the full subcritical regime. The present work aims at making accessible such interesting ideas without the knowledge of regularity structures, we only rely on a direct renormalization and Fourier series while still covering the case of the Anderson Hamiltonian in two and three dimensions.

In Section 1, we provide several bounds on stochastic functions and distributions that we need to construct the form. In particular, this is where the renormalization of the singular products is done. In Sections 2 and 3, we respectively construct the Anderson form in two and three dimensions using the first and second order exponential transform. In Section 4, we prove irreducibility and existence of a spectral gap.

1. Stochastic bounds and renormalization

On the torus $\mathbb{T}^d$, the white noise is given by

$$\xi(x) = \sum_{k\in\mathbb{Z}^d} \xi_k e^{ik\cdot x}$$

with $(\xi_k)_{k\in\mathbb{Z}^d}$ a family of independent and identically distributed random variables of centered standard complex Gaussian with $\xi_{-k} = \overline{\xi_k}$. This gives a centered real Gaussian field with covariance function

$$\mathbb{E}\big[\xi(x)\xi(y)\big] = \delta_0(x-y),$$

that is a random distribution $(\langle\xi,\varphi\rangle)_{\varphi\in L^2(\mathbb{T}^d)}$ such that

$$\mathbb{E}\big[\langle\xi,\varphi\rangle\langle\xi,\psi\rangle\big] = \langle\varphi,\psi\rangle_{L^2(\mathbb{T}^d)}$$

which indeed gives the Fourier coefficient $(\xi_k)_{k\in\mathbb{Z}^d}$. Because of the lack of decay at infinity of the Fourier coefficients, this sum has to be interpreted in a weak sense. Its first construction is due to Paley and Zygmund [26,27] and is actually the first random distribution ever considered. A natural and convenient setting is given by the Besov space $\mathscr{B}^\alpha_{p,q}$ which can be defined using the Littlewood–Paley decomposition, see for example [2]. This decomposition can be stated as follows

$$u = \sum_{n\ge0}\Delta_n u$$

with $\Delta_n u = \big(\mathscr{F}^{-1}\mathbb{1}_{|\cdot|\simeq 2^n}\mathscr{F}\big)u$, that is the projection of u in frequencies on an annulus of size 2^n. It is defined by

$$\big(\Delta_n u\big)(x) := 2^{d(n-1)}\int_{\mathbb{R}^d}\chi\big(2^{n-1}(x-y)\big)u(y)\mathrm{d}y$$

with $\chi\in\mathscr{S}(\mathbb{R}^d)$ and $\operatorname{supp}\widehat{\chi}\subset\big\{\frac12\le|z|\le2\big\}$ for $n\ge1$ and

$$\big(\Delta_0 u\big)(x) := \int_{\mathbb{R}^d}\chi_0(x-y)u(y)\mathrm{d}y$$

with $\chi_0\in\mathscr{S}(\mathbb{R}^d)$ and $\operatorname{supp}\widehat{\chi}_0\subset\big\{|z|\le1\big\}$. We also denote $K=\widehat{\chi}$ such that $\Delta_n u = \big(\mathscr{F}^{-1}K(2^n\cdot)\mathscr{F}\big)u$. Then the Besov space $\mathscr{B}^\alpha_{p,q}$ are distributions such that

$$\|u\|_{\mathscr{B}^\alpha_{p,q}} := \Big(\sum_{n\ge0}2^{\alpha pn}\|\Delta_n u\|^p_{L^q(\mathbb{T}^d)}\Big)^{\frac1p} < \infty.$$

The particular case $p=q=2$ corresponds to the Sobolev space $\mathscr{B}^\alpha_{2,2}=\mathscr{H}^\alpha$ and for $p=q=\infty$ with $\alpha\in\mathbb{R}_+\backslash\mathbb{N}$, one gets the usual Hölder spaces $\mathscr{B}^\alpha_{\infty,\infty}=\mathscr{C}^\alpha$. One also has the continuous Besov embedding

$$\mathscr{B}^\alpha_{p_1,q_1}\hookrightarrow\mathscr{B}^{\alpha-d\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}_{p_2,q_2}$$

for $p_1\le p_2$, $q_1\le q_2$ and $\alpha\in\mathbb{R}$. While one can a priori only multiply a distribution by a smooth function, one has the following product rule in the case of Besov spaces which corresponds to Young condition.

Proposition 1. *For $\alpha,\beta\in\mathbb{R}$ such that $\alpha+\beta>0$ and $p,q,r\in[1,\infty]$ such that $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$, there exists a constant $C>0$ such that*

$$\|uv\|_{\mathscr{B}^{\alpha\wedge\beta}_{r,r}}\le C\|u\|_{\mathscr{B}^{\alpha}_{p,p}}\|v\|_{\mathscr{B}^{\beta}_{q,q}}.$$

The following proposition gives a similar result at the level of the duality bracket.

Proposition 2. *For $\alpha\in\mathbb{R}$ and $p,p',q,q'\in[1,\infty]$ such that $1=\frac{1}{p}+\frac{1}{p'}=\frac{1}{q}+\frac{1}{q'}$, there exists a constant $C>0$ such that*

$$\big|\langle u,v\rangle\big|\le C\|u\|_{\mathscr{B}^{\alpha}_{p,q}}\|v\|_{\mathscr{B}^{-\alpha}_{p',q'}}.$$

For later use, we introduce a new random field X defined by

$$X(x)=\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\frac{1}{|k|^2}\xi_k e^{ik\cdot x}.$$

It satisfies the equation

$$\Delta X=\xi-\xi_0.$$

The following proposition gives Hölder regularity of ξ and X. This result is well-known but we give a proof for self-completeness and also to give a flavor of the arguments for the stochastic renormalization.

Proposition 3. *For any $\kappa>0$, one has almost surely*

$$\xi\in\mathscr{C}^{-\frac{d}{2}-\kappa}(\mathbb{T}^d)\quad \text{and}\quad X\in\mathscr{C}^{2-\frac{d}{2}-\kappa}(\mathbb{T}^d).$$

Proof. Since the noise is Gaussian, we have

$$\mathbb{E}\big[\langle\xi,\varphi\rangle^p\big]\le(p-1)^{\frac{p}{2}}\mathbb{E}\big[\langle\xi,\varphi\rangle^2\big]^{\frac{p}{2}}$$

for any test function φ. This is usually referred to as Gaussian hypercontractivity. In order to use this, we estimate the Besov norm $\mathscr{B}^{\gamma}_{p,p}$ for p large and use the embedding

$$\mathscr{B}^{\gamma}_{p,p}(\mathbb{T}^d)\hookrightarrow\mathscr{B}^{\gamma-\frac{d}{p}}_{\infty,\infty}(\mathbb{T}^d).$$

We have

$$\begin{aligned}\mathbb{E}\Big[\|\Delta_n\xi\|^p_{L^p(\mathbb{T}^d)}\Big]&=\int_{\mathbb{T}^d}\mathbb{E}\big[\big\langle\xi,\chi_n(x-\cdot)\big\rangle^p\big]\mathrm{d}x\\&\le(p-1)^{\frac{p}{2}}\int_{\mathbb{T}^d}\mathbb{E}\big[\big\langle\xi,\chi_n(x-\cdot)\big\rangle^2\big]^{\frac{p}{2}}\mathrm{d}x\\&\le(p-1)^{\frac{p}{2}}\|\chi_n\|^p_{L^2(\mathbb{T}^d)}|\mathbb{T}^d|\end{aligned}$$

with $\chi_n(\cdot)=2^{dn}\chi(2^n\cdot)$ and using that ξ is an isometry from $L^2(\mathbb{T}^d)$ to $L^2(\Omega)$. We have

$$\|\chi_n\|^2_{L^2(\mathbb{T}^d)}=2^{2dn}\|\chi(2^n\cdot)\|^2_{L^2(\mathbb{T}^d)}=2^{dn}\|\chi\|^2_{L^2(\mathbb{T}^d)}$$

hence

$$\mathbb{E}\Big[\|\Delta_n\xi\|^p_{L^p(\mathbb{T}^d)}\Big]\le(p-1)^{\frac{p}{2}}2^{pn\frac{d}{2}}\|\chi\|^p_{L^2(\mathbb{T}^d)}|\mathbb{T}^d|.$$

This gives

$$\mathbb{E}\Big[\|\xi\|_{\mathscr{B}^{-\frac{d}{2}}_{p,p}}\Big]<\infty,$$

and hence $\xi\in\mathscr{C}^{-\frac{d}{2}-\frac{d}{p}}(\mathbb{T}^d)$ for any $p\ge1$ which completes the proof for the regularity of ξ while the regularity of X follows from a standard regularity estimate. □

In two dimensions, one has $X \in \mathscr{C}^{1-\kappa}$ hence $\nabla X \in \mathscr{C}^{-\kappa}$ and the square $|\nabla X|^2$ is ill-defined since $-2\kappa < 0$. Consider a regularization of the noise $\xi_\varepsilon = \xi * \rho_\varepsilon$ with ρ_ε a mollifier, radial to simplify the computations. Then ξ_ε converges to ξ as ε goes to 0 in $\mathscr{C}^{-1-\kappa}$ and one can consider X_ε the solution with null mean to

$$\Delta X_\varepsilon = \xi_\varepsilon - \langle \xi_\varepsilon, 1\rangle$$

which converges to X in $\mathscr{C}^{1-\kappa}$ as ε goes to 0. Since the square $|\nabla X|^2$ is ill-defined, the quantity $|\nabla X_\varepsilon|^2$ diverges and this is described by the Wick square as proved in the following proposition.

Proposition 4. *There exists a distribution* $|\nabla X|^{2\diamond} \in \mathscr{C}^{-2\kappa}(\mathbb{T}^2)$ *such that*

$$|\nabla X|^{2\diamond} = \lim_{\varepsilon\to 0}\Big(|\nabla X_\varepsilon|^2 - \mathbb{E}\big[|\nabla X_\varepsilon|^2\big]\Big)$$

in $\mathscr{C}^{-2\kappa}(\mathbb{T}^2)$ *in probability. Moreover, one has*

$$\mathbb{E}\big[|\nabla X_\varepsilon|^2\big] \underset{\varepsilon\to 0}{\sim} -\frac{1}{(2\pi)^2}\log(\varepsilon).$$

Proof. Since $\xi_\varepsilon = \xi * \rho_\varepsilon$, we have

$$X_\varepsilon(x) = \sum_{k\in\mathbb{Z}^2\setminus\{0\}} \frac{\widehat{\rho_\varepsilon}(k)}{|k|^2}\xi_k e^{ikx}$$

thus

$$\big|\nabla X_\varepsilon(x)\big|^2 = \sum_{k,k'\in\mathbb{Z}^2\setminus\{0\}} \theta\big(\varepsilon|k|\big)\theta\big(\varepsilon|k'|\big)\frac{k\cdot k'}{|k|^2|k'|^2}\xi_k\overline{\xi_{k'}}e^{i(k-k')\cdot x}$$

with $\theta\big(\varepsilon|k|\big) := \widehat{\rho_\varepsilon}(k)$ since ρ is radial. Using $\mathbb{E}\big[\xi_k\overline{\xi_{k'}}\ \big] = \delta_0(k-k')$, we have

$$\mathbb{E}\Big[\big|\nabla X_\varepsilon(x)\big|^2\Big] = \sum_{k\in\mathbb{Z}^2\setminus\{0\}} \frac{\theta\big(\varepsilon|k|\big)^2}{|k|^2} \underset{\varepsilon\to 0}{\sim} \frac{1}{(2\pi)^2}\log(\varepsilon)$$

which gives the second part of the statement. For $n \geq 1$, we have

$$\Delta_n\Big(|\nabla X_\varepsilon|^2 - \mathbb{E}\big[|\nabla X_\varepsilon|^2\big]\Big)(x) = \sum_{k\neq k'} K_n(k-k')\theta\big(\varepsilon|k|\big)\theta\big(\varepsilon|k'|\big)\frac{k\cdot k'}{|k|^2|k'|^2}\xi_k\overline{\xi_{k'}}e^{i(k-k')\cdot x}$$

with $K_n(\cdot) = K(2^{-n}\cdot)$ since $\Delta_n\mathbb{E}\big[|\nabla X_\varepsilon|^2\big] = 0$ for $n\geq 1$, the expectation of $|\nabla X_\varepsilon|^2$ being a constant function. Then the expectation of $\Big|\Delta_n\Big(|\nabla X_\varepsilon|^2 - \mathbb{E}\big[|\nabla X_\varepsilon|^2\big]\Big)\Big|^2(x)$ is given by

$$\sum_{\substack{k_1\neq k_1'\\ k_2\neq k_2'}} K_n(k_1-k_1')\overline{K_n(k_2-k_2')}\theta\big(\varepsilon|k_1|\big)\theta\big(\varepsilon|k_1'|\big)\theta\big(\varepsilon|k_2|\big)\theta\big(\varepsilon|k_2'|\big)\frac{k_1\cdot k_1'}{|k_1|^2|k_1'|^2}\frac{k_2\cdot k_2'}{|k_2|^2|k_2'|^2}\mathbb{E}\big[\xi_{k_1}\overline{\xi_{k_1'}}\xi_{k_2}\overline{\xi_{k_2'}}\big] \times e^{i(k_1-k_1')\cdot x}e^{-i(k_2-k_2')\cdot x}$$

and we have

$$\begin{aligned}\mathbb{E}\big[\xi_{k_1}\overline{\xi_{k_1'}}\xi_{k_2}\overline{\xi_{k_2'}}\big] &= \mathbb{E}\big[\xi_{k_1}\overline{\xi_{k_1'}}\big]\mathbb{E}\big[\xi_{k_2}\overline{\xi_{k_2'}}\big] + \mathbb{E}\big[\xi_{k_1}\xi_{k_2}\big]\mathbb{E}\big[\overline{\xi_{k_1'}\xi_{k_2'}}\big] + \mathbb{E}\big[\xi_{k_1}\overline{\xi_{k_2'}}\big]\mathbb{E}\big[\overline{\xi_{k_1'}}\xi_{k_2}\big]\\ &= \delta_0(k_1-k_1')\delta_0(k_2-k_2') + \delta_0(k_1+k_2)\delta_0(k_1'+k_2') + \delta_0(k_1-k_2')\delta_0(k_1'-k_2)\end{aligned}$$

for any $k_1, k_1', k_2, k_2' \in \mathbb{Z}^2$. It follows that

$$\begin{aligned}\mathbb{E}\Big|\Delta_n\Big(|\nabla X_\varepsilon|^2 - \mathbb{E}\big[|\nabla X_\varepsilon|^2\big]\Big)\Big|^2 &= 2\sum_{k_1,k_2}\big|K_n(k_1-k_2)\big|^2\frac{\theta\big(\varepsilon|k_1|\big)^2\theta\big(\varepsilon|k_2|\big)^2}{|k_1|^2|k_2|^2}\\ &= 2\sum_{k_1,k_2}\big|K\big(2^{-n}(k_1-k_2)\big)\big|^2\frac{\theta\big(\varepsilon|k_1|\big)^2\theta\big(\varepsilon|k_2|\big)^2}{|k_1|^2|k_2|^2}\\ &= 2\sum_k\big|K(2^{-n}k)\big|^2\sum_{k_1-k_2=k}\frac{\theta\big(\varepsilon|k_1|\big)^2\theta\big(\varepsilon|k_2|\big)^2}{|k_1|^2|k_2|^2}\\ &= 2\sum_k\big|K(2^{-n}k)\big|^2\sum_{k_2}\frac{\theta(\varepsilon|k+k_2|)^2\theta\big(\varepsilon|k_2|\big)^2}{|k+k_2|^2|k_2|^2}\\ &\le C2^{2n}2^{-(2-2\kappa)n}\sum_{k_2}\frac{\theta\big(\varepsilon|k_2|\big)^2}{|k_2|^{2+2\kappa}}\end{aligned}$$

for any $\kappa > 0$ and a constant $C > 0$ using the support of K. The Gaussian hypercontractivity yields

$$\begin{aligned}\mathbb{E}\Big|\Delta_n\Big(|\nabla X_\varepsilon|^2 - \mathbb{E}\big[|\nabla X_\varepsilon|^2\big]\Big)\Big|^p &\le (p-1)^p\bigg(\mathbb{E}\Big|\Delta_n\Big(\big|\nabla X_\varepsilon(x)\big|^2 - \mathbb{E}\Big[\big|\nabla X_\varepsilon(x)\big|^2\Big]\Big)\Big|^2\bigg)^{\frac{p}{2}}\\ &\le C2^{\kappa np}.\end{aligned}$$

Thus, $|\nabla X_\varepsilon|^2 - \mathbb{E}\big[|\nabla X_\varepsilon|^2\big]$ is bounded in $\mathscr{B}^{-\kappa}_{p,p}$ for any $\kappa > 0$ and $p \ge 1$. Using the embedding $\mathscr{B}^{-\kappa}_{p,p} \hookrightarrow \mathscr{C}^{-\kappa-\frac{d}{p}}$ and a similar bound, one proves that $\big(|\nabla X_\varepsilon|^2 - \mathbb{E}\big[|\nabla X_\varepsilon|^2\big]\big)_{\varepsilon>0}$ is a Cauchy family in $\mathscr{C}^{-\kappa}$ for any $\kappa > 0$ which completes the proof. □

We define the two dimensional enhanced noise

$$\Xi = \big(\xi, |\nabla X|^{2\diamond}\big)$$

which belongs to

$$\mathscr{X}^\kappa(\mathbb{T}^2) = \mathscr{C}^{-1-\kappa}(\mathbb{T}^2) \times \mathscr{C}^{-2\kappa}(\mathbb{T}^2)$$

for any $\kappa > 0$. We also have that

$$\Xi_\varepsilon = \big(\xi_\varepsilon, |\nabla X_\varepsilon|^2 - (2\pi)^{-2}\log(\varepsilon)\big)$$

converges to Ξ in $\mathscr{X}^\kappa(\mathbb{T}^2)$ for any $\kappa > 0$. In three dimensions, one has $X \in \mathscr{C}^{\frac{1}{2}-\kappa}$ hence this term is even more singular with $-1-2\kappa < 0$. The analog of the previous renormalization is the following proposition with a larger divergence. Its proof follows the same path as the previous one.

Proposition 5. *There exists a distribution* $|\nabla X|^{2\diamond} \in \mathscr{C}^{-1-2\kappa}(\mathbb{T}^3)$ *such that*

$$|\nabla X|^{2\diamond} = \lim_{\varepsilon\to 0}\Big(|\nabla X_\varepsilon|^2 - \mathbb{E}\big[|\nabla X_\varepsilon|^2\big]\Big)$$

in $\mathscr{C}^{-1-2\kappa}(\mathbb{T}^3)$ *in probability. Moreover, one has*

$$\mathbb{E}\big[|\nabla X_\varepsilon|^2\big] \underset{\varepsilon\to 0}{\sim} -\frac{1}{(2\pi)^2}\frac{1}{\varepsilon}.$$

Since the noise is more irregular, $|\nabla X|^{2\diamond}$ is too rough to make sense of its bracket with $\mathscr{H}^1$ hence we will also need Y the solution to

$$\Delta Y = |\nabla X|^{2\diamond} - \big\langle |\nabla X|^{2\diamond}, 1\big\rangle$$

which belongs to $\mathscr{C}^{1-2\kappa}(\mathbb{T}^3)$. The square of its gradient is also singular and can be defined as a Wick product, as well as the product $\nabla X \cdot \nabla Y$. Note that for this product, there is no renormalization constant due to an algebraic cancellation.

Proposition 6. *There exists a distribution* $|\nabla Y|^{2\diamond} \in \mathscr{C}^{-4\kappa}(\mathbb{T}^3)$ *such that*

$$|\nabla Y|^{2\diamond} = \lim_{\varepsilon\to 0}\Big(|\nabla Y_\varepsilon|^2 - \mathbb{E}\big[|\nabla Y_\varepsilon|^2\big]\Big)$$

in $\mathscr{C}^{-4\kappa}(\mathbb{T}^3)$ *in probability. Moreover, one has*

$$\mathbb{E}\big[|\nabla Y_\varepsilon|^2\big] \underset{\varepsilon\to 0}{\sim} -C\log(\varepsilon)$$

with $C>0$ *a constant. There also exists a distribution* $\nabla X \diamond \nabla Y \in \mathscr{C}^{-\frac{1}{2}-3\kappa}(\mathbb{T}^3)$ *such that*

$$\nabla X \diamond \nabla Y = \lim_{\varepsilon\to 0}\big(\nabla X_\varepsilon \cdot \nabla Y_\varepsilon\big).$$

Proof. We have

$$Y_\varepsilon(x) = \sum_{\substack{k,k'\neq 0\\ k\neq k'}} \theta\big(\varepsilon|k|\big)\theta\big(\varepsilon|k'|\big)\frac{k\cdot k'}{|k|^2|k'|^2}\xi_k\overline{\xi_{k'}}\frac{e^{i(k-k')\cdot x}}{|k-k'|^2}$$

hence

$$\begin{aligned}\big|\nabla Y_\varepsilon(x)\big|^2 = &\sum_{\substack{k_1\neq k_1'\\ k_2\neq k_2'\\ k_1,k_1',k_2,k_2'\neq 0}} \theta\big(\varepsilon|k_1|\big)\theta\big(\varepsilon|k_1'|\big)\theta\big(\varepsilon|k_2|\big)\theta\big(\varepsilon|k_2'|\big)\frac{k_1\cdot k_1'}{|k_1|^2|k_1'|^2}\frac{k_2\cdot k_2'}{|k_2|^2|k_2'|^2}\xi_{k_1}\overline{\xi_{k_1'}}\xi_{k_2'}\overline{\xi_{k_2}}\\ &\frac{(k_1-k_1')\cdot(k_2-k_2')}{|k_1-k_1'|^2|k_2-k_2'|^2}e^{i(k_1-k_1')\cdot x}e^{i(k_2'-k_2)\cdot x}\\ =&\sum_{n\in\mathbb{Z}^3} e^{in\cdot x}\widehat{|\nabla Y_\varepsilon|^2}(n)\end{aligned}$$

with $\widehat{|\nabla Y_\varepsilon|^2}(n)$ given by

$$\sum_{\substack{k_1-k_1'+k_2'-k_2=n\\ k_1\neq k_1',k_2\neq k_2'\\ k_1,k_1',k_2,k_2'\neq 0}} \theta\big(\varepsilon|k_1|\big)\theta\big(\varepsilon|k_1'|\big)\theta\big(\varepsilon|k_2|\big)\theta\big(\varepsilon|k_2'|\big)\xi_{k_1}\overline{\xi_{k_1'}}\xi_{k_2'}\overline{\xi_{k_2}}\frac{k_1\cdot k_1'}{|k_1|^2|k_1'|^2}\frac{k_2\cdot k_2'}{|k_2|^2|k_2'|^2}\frac{(k_1-k_1')\cdot(k_2'-k_2)}{|k_1-k_1'|^2|k_2-k_2'|^2}.$$

Using again

$$\mathbb{E}\big[\xi_{k_1}\overline{\xi_{k_1'}}\xi_{k_2}\overline{\xi_{k_2'}}\big] = \delta_0(k_1-k_1')\delta_0(k_2-k_2') + \delta_0(k_1+k_2)\delta_0(k_1'+k_2') + \delta_0(k_1-k_2')\delta_0(k_1'-k_2),$$

we get

$$\begin{aligned}\mathbb{E}\big[\big|\nabla Y_\varepsilon(x)\big|^2\big] &= \sum_{\substack{k\neq k'\\ k,k'\neq 0}}\theta\big(\varepsilon|k|\big)^2\theta\big(\varepsilon|k'|\big)^2\frac{(k\cdot k')^2}{|k|^4|k'|^4|k-k'|^2} + \sum_{\substack{k_1\neq k_2\\ k_1,k_2\neq 0}}\theta\big(\varepsilon|k_1|\big)^2\theta\big(\varepsilon|k_2|\big)^2\frac{(k_1\cdot k_2)^2}{|k_1|^4|k_2|^4|k_1-k_2|^2}\\ &= 2\sum_{\substack{k\neq k'\\ k,k'\neq 0}}\theta\big(\varepsilon|k|\big)^2\theta\big(\varepsilon|k'|\big)^2\frac{(k\cdot k')^2}{|k|^4|k'|^4|k-k'|^2}\end{aligned}$$

which diverges logarithmically in ε. As the process is invariant in law by translation again, the expectation is a constant function hence

$$\Delta_n\Big(|\nabla Y_\varepsilon|^2 - \mathbb{E}\big[|\nabla Y_\varepsilon|^2\big]\Big)(x) = \sum_{m\geq 1} K_n(m)\widehat{|\nabla Y_\varepsilon|^2}(m)e^{im\cdot x}$$

for $n \geq 1$. We obtain that $\left|\Delta_n\Big(|\nabla Y_\varepsilon|^2 - \mathbb{E}\big[|\nabla Y_\varepsilon|^2\big]\Big)\right|^2(x)$ is of order

$$
\begin{aligned}
&2^{3n}\left|\widehat{|\nabla Y_\varepsilon|^2}(2^n)\right|^2\\
&= 2^{3n}\sum_{\substack{k_1-k_1'+k_2-k_2'=2^n\\ |k_i|,|k_i'|\leq\varepsilon^{-1}}}\ \sum_{\substack{\ell_1-\ell_1'+\ell_2-\ell_2'=2^n\\ |\ell_i|,|\ell_i'|\leq\varepsilon^{-1}}} \frac{k_1\cdot k_1'}{|k_1|^2|k_1'|^2}\frac{k_2\cdot k_2'}{|k_2|^2|k_2'|^2}\frac{(k_1-k_1')\cdot(k_2'-k_2)}{|k_1-k_1'|^2|k_2-k_2'|^2}\xi_{k_1}\overline{\xi_{k_1'}}\xi_{k_2'}\overline{\xi_{k_2}}\\
&\qquad\qquad \frac{\ell_1\cdot\ell_1'}{|\ell_1|^2|\ell_1'|^2}\frac{\ell_2\cdot\ell_2'}{|\ell_2|^2|\ell_2'|^2}\frac{(\ell_1-\ell_1')\cdot(\ell_2'-\ell_2)}{|\ell_1-\ell_1'|^2|\ell_2-\ell_2'|^2}\overline{\xi_{\ell_1}}\xi_{\ell_1'}\overline{\xi_{\ell_2'}}\xi_{\ell_2}
\end{aligned}
$$

using that K_n is supported in an annulus of $\mathbb{Z}^3$ of radius 2^n. Taking the expectation gives a factor

$$\mathbb{E}\big[\xi_{k_1}\overline{\xi_{k_1'}}\xi_{k_2'}\overline{\xi_{k_2}\xi_{\ell_1}}\xi_{\ell_1'}\overline{\xi_{\ell_2'}}\xi_{\ell_2}\big]$$

which is the sum of a product of delta correlation functions over all possible pairings, that is 28 elements. A number of terms vanish due to the spectral localization as in the previous proof on $\mathbb{T}^2$, for example the paring $\delta_0(k_1-k_1')\delta_0(k_2-k_2')\delta_0(\ell_1-\ell_1')\delta_0(\ell_2-\ell_2')$. An example of non-vanishing paring is

$$\delta_0(k_1-k_2)\delta_0(k_1'-k_2')\delta_0(\ell_1-\ell_2)\delta_0(\ell_1'-\ell_2')$$

which gives the sum

$$
\begin{aligned}
&\sum_{\substack{2k-2k'=2^n\\ |k|,|k'|\leq\varepsilon^{-1}}}\ \sum_{\substack{2\ell-2\ell'=2^n\\ |\ell|,|\ell'|\leq\varepsilon^{-1}}}\frac{(k\cdot k')^2}{|k|^4|k'|^4}\frac{1}{|k-k'|^2}\frac{(\ell\cdot\ell')^2}{|\ell|^4|\ell'|^4}\frac{1}{|\ell-\ell'|^2}\\
&\qquad = 2^{-2(n-1)}\sum_{\substack{2k-2k'=2^n\\ |k|,|k'|\leq\varepsilon^{-1}}}\ \sum_{\substack{2\ell-2\ell'=2^n\\ |\ell|,|\ell'|\leq\varepsilon^{-1}}}\frac{(k\cdot k')^2}{|k|^4|k'|^4}\frac{(\ell\cdot\ell')^2}{|\ell|^4|\ell'|^4}\\
&\qquad = 2^{-2(n-1)}\left(\sum_{\substack{2k-2k'=2^n\\ |k|,|k'|\leq\varepsilon^{-1}}}\frac{(k\cdot k')^2}{|k|^4|k'|^4}\right)^2 \leq 2^{-2(n-1)}\left(\sum_{\substack{2k-2k'=2^n\\ |k|,|k'|\leq\varepsilon^{-1}}}\frac{1}{|k|^2|k'|^2}\right)^2\\
&\qquad \leq 2^{-2(n-1)}\left(\sum_{|k|\leq\varepsilon^{-1}}\frac{1}{|k|^2|k-2^{n-1}|^2}\right)^2 \leq C2^{-2(n-1)}2^{-(n-1)(1-\kappa)}\left(\sum_{|k|\leq\varepsilon^{-1}}\frac{1}{|k|^{3+\kappa}}\right)^2
\end{aligned}
$$

where the sum over $k\in\mathbb{Z}^3\setminus\{0\}$ is convergent. Since there are 2^{3n} terms in the sum because of the support of K_n, this will indeed gives a bound of the order

$$\mathbb{E}\left|\Delta_n\Big(|\nabla Y_\varepsilon|^2-\mathbb{E}\big[|\nabla Y_\varepsilon|^2\big]\Big)\right|^2 \leq C2^{\kappa n}.$$

The large number of terms appearing here is one of the problems of singular SPDEs and it is the main motivation for the introduction of general algebraic structures based on Hopf algebra of trees and diagrams in regularity structures, see Hairer's seminal work [16]. A control of each term yields the bound

$$\mathbb{E}\left|\Delta_n\Big(|\nabla Y_\varepsilon|^2-\mathbb{E}\big[|\nabla Y_\varepsilon|^2\big]\Big)\right|^p \leq C_p2^{\kappa np}$$

using Gaussian hypercontractivity again and so the sequence is bounded in

$$\mathscr{B}^{-\kappa}_{p,p}(\mathbb{T}^3)\hookrightarrow\mathscr{C}^{-\kappa-\frac{3}{p}}(\mathbb{T}^3).$$

Similar computations give that the sequence is Cauchy and completes the proof. For $\nabla X\diamond\nabla Y$, the computations are similar but simpler since this is only a trilinear functional of the noise while $|\nabla Y|^{2\diamond}$ is a 4-linear functional of the noise. In this case, the renormalization constant is zero because of algebraic cancellations, see for example [14, Section 9] for similar computations. □

We define the three dimensional enhanced noise

$$\Xi = \big(\xi, |\nabla X|^{2\diamond}, |\nabla Y|^{2\diamond}, \nabla X \diamond \nabla Y\big)$$

which belongs to

$$\mathscr{X}^{\kappa}(\mathbb{T}^3) = \mathscr{C}^{-\frac{3}{2}-\kappa}(\mathbb{T}^3) \times \mathscr{C}^{-1-2\kappa}(\mathbb{T}^3) \times \mathscr{C}^{-4\kappa}(\mathbb{T}^3) \times \mathscr{C}^{-\frac{1}{2}-3\kappa}$$

for any $\kappa > 0$. We also have that

$$\Xi_\varepsilon = \big(\xi_\varepsilon, |\nabla X_\varepsilon|^2 - (2\pi)^{-2}\varepsilon^{-1}, |\nabla Y_\varepsilon|^2 - (2\pi)^{-2}\log(\varepsilon), \nabla X_\varepsilon \cdot \nabla Y_\varepsilon\big)$$

converges to Ξ in $\mathscr{X}^{\kappa}(\mathbb{T}^3)$ for any $\kappa > 0$.

2. Construction in two dimensions

It is tempting to define the form of the Anderson operator by

$$\mathfrak{a}(u_1, u_2) = \int_{\mathbb{T}^2} \nabla u_1(x) \cdot \nabla u_2(x)\mathrm{d}x + \int_{\mathbb{T}^2} u_1(x) u_2(x) \xi(\mathrm{d}x)$$

for any $u_1, u_2 \in C^\infty(\mathbb{T}^2)$. However, this is not a natural object since this form is not closable as shown by the recent progress on singular stochastic operators, which can be guessed from the fact that for $u \in \mathscr{H}^1$ the form domain of Δ, the product $u\xi$ is ill-defined. For $\xi_\varepsilon = \xi * \rho_\varepsilon$ a regularization of the noise, consider the regularized form

$$\mathfrak{a}_\varepsilon(u_1, u_2) = \int_{\mathbb{T}^2} \nabla u_1(x) \cdot \nabla u_2(x)\mathrm{d}x + \int_{\mathbb{T}^2} u_1(x) u_2(x)\big(\xi_\varepsilon(x) - c_\varepsilon\big)\mathrm{d}x$$

with c_ε the logarithmic diverging constant defined in the previous section. For any fixed $\varepsilon > 0$, $\mathfrak{a}_\varepsilon$ is a closed symmetric form with domain $\mathscr{H}^1$ and we construct a form $\mathfrak{a}$ such that $\mathfrak{a}_\varepsilon$ converges to $\mathfrak{a}$ as ε goes to 0. With X the random field constructed in the previous section, we consider the new variable $u = e^X v$ and define

$$Hu := -e^X \Delta v - 2e^X \nabla X \cdot \nabla v - e^X |\nabla X|^{2\diamond} v + \xi_0 e^X v$$

for $v \in \mathscr{C}^\infty$. Since $X \in \mathscr{C}^{1-\kappa}$ and $|\nabla X|^{2\diamond} \in \mathscr{C}^{-2\kappa}$, Hu is well-defined as a distribution. The associated form is given by

$$\begin{aligned}
\mathfrak{a}(u_1, u_2) &= \langle Hu_1, u_2 \rangle \\
&= \big\langle He^X v_1, e^X v_2 \big\rangle \\
&= -\big\langle \Delta v_1, v_2 e^{2X}\big\rangle - 2\big\langle \nabla X \cdot \nabla v_1, v_2 e^{2X}\big\rangle - \big\langle |\nabla X|^{2\diamond} v_1, v_2 e^{2X}\big\rangle + \xi_0 \big\langle v_1, v_2 e^{2X}\big\rangle \\
&= \int_{\mathbb{T}^2} \nabla v_1(x) \cdot \nabla v_2(x) e^{2X(x)}\mathrm{d}x - \big\langle |\nabla X|^{2\diamond} v_1, v_2 e^{2X}\big\rangle + \xi_0 \int_{\mathbb{T}^2} v_1(x) v_2(x) e^{2X(x)}\mathrm{d}x
\end{aligned}$$

which is well-defined for $v_1, v_2 \in \mathscr{H}^1$ since

$$\begin{aligned}
\Big|\big\langle |\nabla X|^{2\diamond} e^{2X}, v_1 v_2\big\rangle\Big| &\leq \big\| |\nabla X|^{2\diamond} e^{2X}\big\|_{\mathscr{C}^{-\kappa}} \| v_1 v_2 \|_{\mathscr{B}^{\kappa}_{1,1}} \\
&\leq \big\| |\nabla X|^{2\diamond}\big\|_{\mathscr{C}^{-\kappa}} \| e^{2X} \|_{\mathscr{C}^{2\kappa}} \| v_1 \|_{\mathscr{H}^{2\kappa}} \| v_2 \|_{\mathscr{H}^{2\kappa}} \\
&\leq \big\| |\nabla X|^{2\diamond}\big\|_{\mathscr{C}^{-\kappa}} \| e^{2X} \|_{\mathscr{C}^{1-\kappa}} \| v_1 \|_{\mathscr{H}^{1}} \| v_2 \|_{\mathscr{H}^{1}}
\end{aligned}$$

for $\kappa > 0$ small enough using Propositions 2 and 1.

Definition 7. *The Anderson form is defined by*

$$\mathfrak{a}(u_1, u_2) := \langle \nabla v_1, \nabla v_2 \rangle_{L^2(\mathbb{T}^2, e^{2X}\mathrm{d}x)} - \big\langle |\nabla X|^{2\diamond}, v_1 v_2 e^{2X}\big\rangle + \xi_0 \langle v_1, v_2 \rangle_{L^2(\mathbb{T}^2, e^{2X}\mathrm{d}x)}$$

where $v_i = e^{-X} u_i$ with domain $\mathscr{D}(\mathfrak{a}) := e^X \mathscr{H}^1$ equipped with the norm

$$\| u \|_{\mathfrak{a}}^2 := \| u \|_{L^2}^2 + \| e^{-X} u \|_{\mathscr{H}^1}^2.$$

Since $e^X \in \mathscr{C}^{1-\kappa}$ for any $\kappa > 0$, the domain $\mathscr{D}(\mathfrak{a})$ is dense in $\mathscr{H}^{1-\kappa}$ thus in L^2. The following proposition states that this densely defined form is continuous and bounded from below.

Proposition 8. *There exists a random constant $C > 0$ such that*

$$\left|\mathfrak{a}(u_1, u_2)\right| \le C\|u_1\|_{\mathfrak{a}}\|u_2\|_{\mathfrak{a}}$$

for $u_1, u_2 \in \mathscr{D}(\mathfrak{a})$. The form $\mathfrak{a}$ is quasi-coercive, i.e., there exist random constants $\delta, C' > 0$ such that

$$\mathfrak{a}(u,u) + C'\|u\|_{L^2}^2 \ge \delta\|u\|_{\mathfrak{a}}^2$$

for all $u = e^X v \in \mathscr{D}(\mathfrak{a})$.

Proof. The continuity follows directly from

$$\left|\left\langle |\nabla X|^{2\diamond} e^{2X}, v_1 v_2\right\rangle\right| \le \left\||\nabla X|^{2\diamond}\right\|_{\mathscr{C}^{-\kappa}}\|e^{2X}\|_{\mathscr{C}^{1-\kappa}}\|v_1\|_{\mathscr{H}^1}\|v_2\|_{\mathscr{H}^1}.$$

Now we prove the second statement. Set $u = e^X v$ with $v \in \mathscr{H}^1$. We have for any $\kappa > 0$

$$\begin{aligned}\mathfrak{a}(u,u) - \xi_0\int_{\mathbb{T}^2}\left|v(x)\right|^2 e^{2X(x)}\mathrm{d}x &= \int_{\mathbb{T}^2}\left|\nabla v(x)\right|^2(x)e^{2X(x)}\mathrm{d}x - \left\langle|\nabla X|^{2\diamond} v, ve^{2X}\right\rangle\\ &\ge e^{-\|X\|_{L^\infty}}\int_{\mathbb{T}^2}\left|\nabla v(x)\right|^2(x)\mathrm{d}x - \left\||\nabla X|^{2\diamond}\right\|_{\mathscr{C}^{-\kappa}}\|e^{2X}\|_{\mathscr{C}^{2\kappa}}\|v\|_{\mathscr{H}^{2\kappa}}^2.\end{aligned}$$

For small $\kappa > 0$ we use the standard interpolation inequality, which is valid for every $\varepsilon > 0$,

$$\|v\|_{\mathscr{H}^{2\kappa}} \le \varepsilon\|v\|_{\mathscr{H}^1} + c_\varepsilon\|v\|_{L^2}$$

for some $c_\varepsilon > 0$. We choose ε small enough and insert this inequality in the previous estimates to obtain the statement. □

A consequence of the previous proposition is that the norms $\|\cdot\|_{\mathscr{D}(\mathfrak{a})}$ and $\|e^{-X}\cdot\|_{\mathscr{H}^1}$ are equivalent.

We now prove that the form is closed.

Proposition 9. *The form $\mathfrak{a}$ is closed, that is $\left(\mathscr{D}(\mathfrak{a}), \|\cdot\|_{\mathfrak{a}}\right)$ is a complete space.*

Proof. Let $(u_n)_{n\ge0} \subset \mathscr{D}(\mathfrak{a})$ be a Cauchy sequence. Then $(e^{-X}u_n)_{n\ge0}$ is a Cauchy sequence in $\mathscr{H}^1$ thus converges to a limit $v \in \mathscr{H}^1$ while $(u_n)_{n\ge0}$ is a Cauchy sequence in L^2 thus converges to $u \in L^2$. We have

$$\begin{aligned}\|u - e^X v\|_{L^2} &\le \|u - u_n\|_{L^2} + \|u_n - e^X v\|_{L^2}\\ &\le \|u - u_n\|_{L^2} + \|e^X\|_{L^\infty}\|e^{-X}u_n - v\|_{L^2}\end{aligned}$$

Letting $n \to \infty$ yields $u = e^X v \in \mathscr{D}(\mathfrak{a})$ and this completes the proof. □

Finally, we prove that $\mathfrak{a}$ is the limit in some sense of the renormalized forms $\mathfrak{a}_\varepsilon$.

Proposition 10. *For any $\kappa > 0$, there exists a constant $C > 0$ such that*

$$\left|\mathfrak{a}(u_1,u_2) - \mathfrak{a}_\varepsilon(u_1^\varepsilon, u_2^\varepsilon)\right| \le C\|\Xi - \Xi_\varepsilon\|_{\mathscr{X}^\kappa(\mathbb{T}^2)}\|v_1\|_{\mathscr{H}^1}\|v_2\|_{\mathscr{H}^1}$$

with $u_i^\varepsilon = e^{X_\varepsilon}v_i$ for $\varepsilon \ge 0$.

Proof. Let $v_1, v_2 \in \mathscr{H}^1$ and consider $u_i^\varepsilon = e^{X_\varepsilon}v_i$ for $\varepsilon \ge 0$, i.e., $u_i^\varepsilon \in \mathscr{H}^1$ the form domain of a_ε for any $\varepsilon > 0$ while $u_i \in e^X\mathscr{H}^1$ for $\varepsilon = 0$. We have

$$\mathfrak{a}_\varepsilon(u_1^\varepsilon, u_2^\varepsilon) = \langle\nabla v_1, \nabla v_2\rangle_{L^2(\mathbb{T}^2, e^{2X}\mathrm{d}x)} - \left\langle|\nabla X_\varepsilon|^2 - c_\varepsilon, v_1v_2e^{2X_\varepsilon}\right\rangle + \langle\xi_\varepsilon, 1\rangle\langle v_1, v_2\rangle_{L^2(\mathbb{T}^2, e^{2X_\varepsilon}\mathrm{d}x)}$$

and hence

$$\begin{aligned}\left|\mathfrak{a}(u_1,u_2) - \mathfrak{a}_\varepsilon(u_1^\varepsilon,u_2^\varepsilon)\right| &\le \left|\left\langle|\nabla X_\varepsilon|^2 - c_\varepsilon - |\nabla X|^{2\diamond}, v_1v_2\right\rangle\right| + \left|\left\langle\langle\xi_\varepsilon,1\rangle e^{X_\varepsilon} - \xi_0 e^X, v_1v_2\right\rangle\right|\\ &\le C\|\Xi - \Xi_\varepsilon\|_{\mathscr{X}^\kappa}\|v_1\|_{\mathscr{H}^1}\|v_2\|_{\mathscr{H}^1}\end{aligned}$$

for any $\kappa > 0$ and the proof is complete. □

3. Construction in three dimensions

In three dimensions, the expression

$$\langle \nabla v_1, \nabla v_2\rangle_{L^2(\mathbb{T}^3, e^{2X}\mathrm{d}x)} - \big\langle |\nabla X|^{2\diamond}, v_1 v_2 e^{2X}\big\rangle + \xi_0 \big\langle v_1, v_2 e^{2X}\big\rangle$$

does not make sense anymore for $v_1, v_2 \in \mathscr{H}^1$ since $|\nabla X|^{2\diamond}$ belongs to $\mathscr{C}^{-1-\kappa}$ for any $\kappa > 0$. In this case, one makes the change of variable $u = e^{X+Y} v$ with Y the solution to

$$\Delta Y = |\nabla X|^{2\diamond} - \big\langle |\nabla X|^{2\diamond}, 1\big\rangle$$

which belongs to $\mathscr{C}^{1-\kappa}$. We have

$$Hu = -e^{X+Y}\Delta v - 2e^{X+Y}(\nabla X + \nabla Y)\cdot\nabla v - \Big(|\nabla Y|^{2\diamond} + 2\nabla X \diamond \nabla Y - \big\langle |\nabla X|^{2\diamond}, 1\big\rangle - \xi_0\Big) e^{X+Y} v$$

hence

$$\begin{aligned}
\mathfrak{a}(u_1, u_2) &= \langle Hu_1, u_2\rangle \\
&= \big\langle He^{X+Y} v_1, e^{X+Y} v_2\big\rangle \\
&= -\big\langle \Delta v_1, v_2 e^{2X+2Y}\big\rangle - 2\big\langle \nabla(X+Y)\cdot\nabla v_1, v_2 e^{2X+2Y}\big\rangle - \big\langle |\nabla Y|^{2\diamond}, v_1 v_2 e^{2X+2Y}\big\rangle \\
&\qquad - 2\big\langle \nabla X \diamond \nabla Y, v_1 v_2 e^{2X+2Y}\big\rangle + \big\langle |\nabla X|^{2\diamond}, 1\big\rangle + \big\langle \xi_0, v_1 v_2 e^{2X+2Y}\big\rangle \\
&= \int_{\mathbb{T}^3} \nabla v_1(x)\cdot\nabla v_2(x) e^{2X(x)+2Y(x)}\mathrm{d}x - \big\langle |\nabla Y|^{2\diamond} + 2\nabla X \diamond \nabla Y, v_1 v_2 e^{2X+2Y}\big\rangle \\
&\qquad + \Big(\big\langle |\nabla X|^{2\diamond}, 1\big\rangle + \xi_0\Big)\int_{\mathbb{T}^3} v_1(x) v_2(x) e^{2X(x)+2Y(x)}\mathrm{d}x
\end{aligned}$$

which is well-defined for $v_1, v_2 \in \mathscr{H}^1$ since $|\nabla Y|^{2\diamond} \in \mathscr{C}^{-\kappa}$ and $\nabla X \diamond \nabla Y \in \mathscr{C}^{-\frac{1}{2}-\kappa}$ for any $\kappa > 0$.

Definition 11. *The Anderson form is defined by*

$$\begin{aligned}
\mathfrak{a}(u_1, u_2) := \langle \nabla v_1, \nabla v_2\rangle_{L^2(\mathbb{T}^3, e^{2X+2Y}\mathrm{d}x)} &- \big\langle |\nabla Y|^{2\diamond} + 2\nabla X \diamond \nabla Y, v_1 v_2 e^{2X+2Y}\big\rangle \\
&+ \Big(\big\langle |\nabla X|^{2\diamond}, 1\big\rangle + \xi_0\Big)\langle v_1, v_2\rangle_{L^2(\mathbb{T}^3, e^{2X+2Y}\mathrm{d}x)}
\end{aligned}$$

where $v_i = e^{-X} u_i$ with domain $\mathscr{D}(\mathfrak{a}) := e^{X+Y}\mathscr{H}^1$ equipped with the norm

$$\|u\|_{\mathfrak{a}}^2 := \|u\|_{L^2}^2 + \|e^{-(X+Y)} u\|_{\mathscr{H}^1}^2.$$

Since $e^{X+Y} \in \mathscr{C}^{\frac{1}{2}-\kappa}$ for any $\kappa > 0$, the domain $\mathscr{D}(\mathfrak{a})$ is dense $\mathscr{H}^{\frac{1}{2}-\kappa}$ thus in L^2. The following proposition states that this densely defined form is continuous and bounded from below. The proofs are obtained following the same path as in two dimensions.

Proposition 12. *There exists a random constant $C > 0$ such that*

$$\big|\mathfrak{a}(u_1, u_2)\big| \le C\|u_1\|_{\mathfrak{a}}\|u_2\|_{\mathfrak{a}}$$

for $u_1, u_2 \in \mathscr{D}(\mathfrak{a})$. There exists random constants $\delta, C' > 0$ such that

$$\mathfrak{a}(u, u) + C'\|u\|_{L^2}^2 \ge \delta\|u\|_{\mathfrak{a}}^2$$

for all $u = e^{X+Y} v \in \mathscr{D}(\mathfrak{a})$.

Moreover, the form $\mathfrak{a}$ is closed, that is $\big(\mathscr{D}(\mathfrak{a}), \|\cdot\|_{\mathfrak{a}}\big)$ is a complete space, and we have the following convergence. For any $\kappa > 0$, there exists a constant $C > 0$ such that

$$\big|\mathfrak{a}(u_1, u_2) - \mathfrak{a}_\varepsilon(u_1^\varepsilon, u_2^\varepsilon)\big| \le C\|\Xi - \Xi_\varepsilon\|_{\mathscr{X}^\kappa(\mathbb{T}^3)}\|v_1\|_{\mathscr{H}^1}\|v_2\|_{\mathscr{H}^1}$$

with $u_i^\varepsilon = e^{X_\varepsilon + Y_\varepsilon} v_i$ for $\varepsilon \ge 0$.

4. Positivity and spectral gap

The construction of the form $\mathfrak{a}$ is the same in two and three dimensions. It is densely defined, symmetric bounded from below, continuous and closed. Its associated operator H has domain

$$\mathscr{D}(H) = \Big\{u \in L^2(\mathbb{T}^d)\ ;\ \exists\, v \in L^2(\mathbb{T}^d),\ \forall\, \varphi \in \mathscr{D}(\mathfrak{a}),\ \mathfrak{a}(u,\varphi) = \langle v,\varphi\rangle\Big\}.$$

This is well-known in the study of spectral theory, see for example [25, Section 1.2.3]. The idea is that for $u \in \mathscr{D}(H)$, the bilinear form

$$\varphi \in \mathscr{D}(\mathfrak{a}) \longmapsto \mathfrak{a}(u,\varphi)$$

can be represented as a unique element $v \in L^2(\mathbb{T}^d)$ and has to be interpreted as

$$\varphi \longmapsto \langle v,\varphi\rangle = \langle Hu,\varphi\rangle$$

with H the operator associated to the form $\mathfrak{a}$. The density of the form domain is crucial and one can transfer most property of the form to the operator, the domain of the operator being dense in the form domain. In particular, the operator H is self-adjoint, densely defined and bounded from below. Since $\mathscr{D}(\mathfrak{a})$ is embedded into a Sobolev space of positive regularity, it is compactly embedded in $L^2(\mathbb{T}^d)$. Therefore, H has discrete spectrum

$$\lambda_1 \le \lambda_2 \le \dots$$

and there exists an orthonormal basis of $L^2(\mathbb{T}^d)$ which is given by eigenfunctions of H. An important information is the existence of a spectral gap with a positive ground state. This is already known (see for example [4]) and it is a key to prove two-sided Gaussian bounds for the corresponding heat kernel of H. By the classical Krein–Rutman theorem, the general idea to get a spectral gap with a positive ground state is to prove that the semigroup e^{-tH} is positive and irreducible. This means that for any non-negative (and nontrivial) $f \in L^2(\mathbb{T}^d)$, we have at any time $t > 0$, $e^{-tH} f > 0$ a.e. on $\mathbb{T}^d$. The irreducibility is sometimes called *strict positivity* or *positivity improving*. Unlike [4] which relies on quantitative study of the linear Parabolic Anderson Model equation and an approximation argument, we can obtain positivity and irreducibility readily from the form. These two properties are indeed characterized in terms of the form. See Theorems 2.6 and 2.10 in [25]. Thus, we provide a pedestrian approach to the existence of a spectral gap even in three dimensions which usually relies on involved computations with expansion of order 5 using regularity structures or paracontrolled calculus.

Theorem 13. *The semigroup e^{-tH} is irreducible. In particular, the first eigenvalue is simple, that is $\lambda_1 < \lambda_2$ and there exists a positive ground state $\Psi \in \mathscr{D}(H)$.*

Proof. Both positivity and irreducibility are not changed under multiplication by e^X or e^{X+Y} and so we use the form $\mathfrak{a}$ constructed in the previous sections.

Let $u \in D(\mathfrak{a})$ and $v \in \mathscr{H}^1$ such that $u = e^X v$ if $d = 2$ and $u = e^{X+Y} v$ if $d = 3$. Then clearly, $u^+ = e^X v^+$ (or $e^{X+Y} v^+$) and $u^- = e^X v^-$ (or $e^{X+Y} v^-$). Since $v^+, v^- \in \mathscr{H}^1$, we have $u^+, u^- \in D(\mathfrak{a})$. In addition, it is obviously seen from the definition of the Anderson form that $\mathfrak{a}(u^+, u^-) = 0$. By [25, Theorem 2.6] we conclude that $(e^{-tH})_{t\ge 0}$ is a positive semigroup.

Now we prove irreducibility. We apply [25, Theorem 2.10]. Since H is a local operator, it is enough to prove that if $F \subset \mathbb{T}^d$ is such that

$$\forall\, u \in D(\mathfrak{a}), \quad \mathbb{1}_F u \in D(\mathfrak{a}),$$

then either $|F| = 0$ or $|\mathbb{T}^d \backslash F| = 0$. Clearly, $\mathbb{1}_F u = e^X \mathbb{1}_F v$ if $d = 2$ and $\mathbb{1}_F u = e^{X+Y} \mathbb{1}_F v$ if $d = 3$. This implies

$$\forall\, v \in \mathscr{H}^1, \quad \mathbb{1}_F v \in \mathscr{H}^1.$$

Theorem 2.10 from [25] applied to the Laplacian, whose form domain is $\mathscr{H}^1$, gives that $|F| = 0$ or $|\mathbb{T}^d \setminus F| = 0$. This proves irreducibility.

The rest of the theorem is classical and it is a direct consequence of the Krein–Rutman theorem. □

Remark 14. To have a more precise characterization of the domain, one would need to understand the condition $Hu \in L^2(\mathbb{T}^d)$ for $u \in L^2(\mathbb{T}^d)$. For example in the case $d = 2$, smooth functions are not in the domain of the operator since

$$e^{-X} He^X v = -\Delta v - 2\nabla X \cdot \nabla v - |\nabla X|^{2\diamond} v$$

is not an element of $L^2(\mathbb{T}^2)$ for smooth v since $\nabla X, |\nabla X|^{2\diamond} \in \mathscr{C}^{-\kappa}(\mathbb{T}^2)$ for any $\kappa > 0$. The main idea of the construction of such singular stochastic operator is to consider rough functions depending on the noise such that

$$-\Delta u + u\xi \in L^2(\mathbb{T}^2)$$

hence Δu needs to have the same regularity of the noise, that is $u \in \mathscr{C}^{1-\kappa}(\mathbb{T}^2)$. Using regularity structures or paracontrolled calculus gives a local description of such function to ensure not only to define the operator with a renormalization procedure but also to give explicit functions in the domain with a prescribed local behavior depending on the noise. In the framework of [23], the form domain is described with a first order paracontrolled expansion

$$\mathscr{D}(\mathfrak{a}) = \left\{u \in L^2(\mathbb{T}^2)\,;\; u - \mathsf{P}_u X \in \mathscr{H}^1(\mathbb{T}^2)\right\}$$

where P is a paraproduct, which is equivalent to our formulation. However, the operator domain is obtained with a second order paracontrolled expansion

$$\mathscr{D}(H) = \left\{u \in L^2(\mathbb{T}^2)\,;\; u - \mathsf{P}_u X - \mathsf{P}_u X_2 \in \mathscr{H}^2(\mathbb{T}^2)\right\}$$

with X_2 another suitable function depending on ξ and X which does not involve any renormalization procedure.

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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