ACADÉMIE
DES SCIENCES
INSTITUT DE FRANCE

Comptes Rendus
Mathématique

Cyril Houdayer

Operator algebraic characterization of the noncommutative Poisson boundary

Volume 363 (2025), p. 199-204

Online since: 25 March 2025

https://doi.org/10.5802/crmath.715

CENTRE MERSENNE

The Comptes Rendus. Mathématique *are a member of the Mersenne Center for open scientific publishing*
www.centre-mersenne.org — e-ISSN : 1778-3569

Comptes Rendus. Mathématique
2025, Vol. 363, p. 199-204
https://doi.org/10.5802/crmath.715

Research article / *Article de recherche*
Functional analysis / *Analyse fonctionnelle*

Operator algebraic characterization of the noncommutative Poisson boundary

Caractérisation en algèbres d'opérateurs de la frontière de Poisson noncommutative

Cyril Houdayer [a]

[a] École Normale Supérieure, Département de Mathématiques et Applications, Université Paris-Saclay, 45 rue d'Ulm, 75230 Paris Cedex 05, France

E-mail: cyril.houdayer@ens.psl.eu

Abstract. We obtain an operator algebraic characterization of the noncommutative Furstenberg–Poisson boundary $\mathrm{L}(\Gamma) \subset \mathrm{L}(\Gamma \curvearrowright B)$ associated with an admissible probability measure $\mu \in \mathrm{Prob}(\Gamma)$ for which the (Γ, μ)-Furstenberg–Poisson boundary (B, ν_B) is uniquely μ-stationary. This is a noncommutative generalization of Nevo–Sageev's structure theorem [14]. We apply this result in combination with previous works to provide further evidence towards Connes' rigidity conjecture for higher rank lattices.

Résumé. Nous obtenons une caractérisation en algèbres opérateurs de la frontière de Furstenberg–Poisson noncommutative $\mathrm{L}(\Gamma) \subset \mathrm{L}(\Gamma \curvearrowright B)$ associée à une mesure de probabilité admissible $\mu \in \mathrm{Prob}(\Gamma)$ pour laquelle la (Γ, μ)-frontière de Furstenberg–Poisson (B, ν_B) est uniquement μ-stationnaire. Il s'agit d'une généralisation noncommutative du théorème de structure de Nevo–Sageev [14]. Nous appliquons ce résultat en combinaison avec des travaux antérieurs pour fournir des pistes supplémentaires afin de résoudre la conjecture de rigidité de Connes pour les réseaux de rang supérieur.

Keywords. Connes' rigidity conjecture, higher rank lattices, noncommutative Furstenberg–Poisson boundaries, von Neumann algebras.

Mots-clés. Conjecture de rigidité de Connes, réseaux de rang supérieur, frontières de Furstenberg–Poisson noncommutatives, algèbres de von Neumann.

2020 Mathematics Subject Classification. 20G25, 22D25, 46L10, 46L55.

Funding. The author is supported by ERC Advanced Grant NET 101141693.

Manuscript received 4 November 2024, accepted 19 December 2024.

1. Introduction and statement of the main results

Let Γ be a countable discrete group and $\mu \in \mathrm{Prob}(\Gamma)$ an admissible probability measure in the sense that $\bigcup_{n\geq 1} \mathrm{supp}(\mu)^n = \Gamma$. Denote by (B, ν_B) the (Γ, μ)-Furstenberg–Poisson boundary [3,6]. Recall that (B, ν_B) is the unique (Γ, μ)-space, up to isomorphism, for which the Γ-equivariant Poisson transform

$$\mathscr{P}_\mu \colon \mathrm{L}^\infty(B, \nu_B) \longrightarrow \mathrm{Har}^\infty(\Gamma, \mu) : f \longmapsto \left(\gamma \mapsto \int_B f(\gamma b)\, \mathrm{d}\nu_B(b)\right)$$

ISSN (electronic): 1778-3569

is onto and isometric. Here, $\mathrm{Har}^\infty(\Gamma,\mu) \subset \ell^\infty(\Gamma)$ denotes the subspace of bounded (right) μ-harmonic functions. We say that the (Γ,μ)-Furstenberg–Poisson boundary (B,ν_B) is *μ-uniquely stationary* if there exists a compact metrizable model K of (B,ν_B) for which there exists a unique μ-stationary Borel probability measure $\nu_K \in \mathrm{Prob}(K)$ and $(K,\nu_K) \cong (B,\nu_B)$ as (Γ,μ)-spaces. In [14, Theorem 9.2], Nevo–Sageev showed that if the (Γ,μ)-Furstenberg–Poisson boundary (B,ν_B) is μ-uniquely stationary, then for any amenable (Γ,μ)-space (X,ν_X), there exists a relatively measure preserving Γ-equivariant measurable factor map $(X,\nu_X) \to (B,\nu_B)$.

Denote by $M = \mathrm{L}(\Gamma)$ the group von Neumann algebra and by $\mathscr{B} = \mathrm{L}(\Gamma \curvearrowright B)$ the group measure space construction associated with the nonsingular action $\Gamma \curvearrowright (B,\nu_B)$. Then we have the natural inclusion $M \subset \mathscr{B}$. We also consider the conjugation action $\Gamma \curvearrowright \mathscr{B}$ and we denote by $\mathrm{E}_B : \mathscr{B} \to \mathrm{L}^\infty(B,\nu_B)$ the canonical Γ-equivariant faithful normal conditional expectation. Then the faithful normal state $\psi_B = \nu_B \circ \mathrm{E}_B \in \mathscr{B}_*$ is μ-stationary. Regard $\mathrm{L}^2(M,\tau) \subset \mathrm{L}^2(\mathscr{B},\psi_B)$ as a closed subspace and simply denote by $e_M : \mathrm{L}^2(\mathscr{B},\psi_B) \to \mathrm{L}^2(M,\tau)$ the orthogonal projection. Define the normal state $\varphi_\mu \in \mathrm{B}(\ell^2(\Gamma))_*$ by the formula $\varphi_\mu(T) = \sum_{\gamma\in\Gamma} \mu(\gamma)\langle T\delta_\gamma,\delta_\gamma\rangle$ for $T \in \mathrm{B}(\ell^2(\Gamma))$. Define the normal ucp (unital completely positive) map

$$\Phi_\mu : \mathrm{B}(\ell^2(\Gamma)) \longrightarrow \mathrm{B}(\ell^2(\Gamma)) : T \longmapsto \sum_{\gamma\in\Gamma} \mu(\gamma) J\lambda(\gamma)JTJ\lambda(\gamma)^* J.$$

Then we have $\varphi_\mu(T) = \langle \Phi_\mu(T)\delta_e,\delta_e\rangle$ for all $T \in \mathrm{B}(\ell^2(\Gamma))$. Denote by $\mathrm{Har}(\Phi_\mu) = \{T \in \mathrm{B}(\ell^2(\Gamma)) \mid \Phi_\mu(T) = T\}$ the subspace of Φ_μ-harmonic elements. Following [5,11], the noncommutative Poisson transform

$$\widehat{\mathscr{P}}_\mu : \mathscr{B} \longrightarrow \mathrm{Har}(\Phi_\mu) : T \longmapsto e_M T e_M$$

is a normal onto isometric ucp map such that $\widehat{\mathscr{P}}_\mu|_M = \mathrm{id}_M$. Then we refer to the inclusion $M \subset \mathscr{B}$ as the (M,φ_μ)-noncommutative Furstenberg–Poisson boundary.

Let $M \subset \mathscr{M}$ be an inclusion of von Neumann algebras and consider the conjugation action $\Gamma \curvearrowright \mathscr{M}$. By [2, Proposition 4.2], to any (faithful) normal μ-stationary state $\psi \in \mathscr{M}_*$ corresponds a unique Γ-equivariant (faithful) normal ucp map $\Theta : \mathscr{M} \to \mathrm{L}^\infty(B,\nu_B)$ such that $\nu_B \circ \Theta = \psi$. Moreover, by [5, Proposition 2.1], to any (faithful) normal μ-stationary state $\psi \in \mathscr{M}_*$ corresponds a unique (faithful) normal ucp map $\widehat{\Theta} : \mathscr{M} \to \mathscr{B}$ such that $\widehat{\Theta}|_M = \mathrm{id}_M$, $\psi = \nu_B \circ \mathrm{E}_B \circ \widehat{\Theta}$ and $\Theta = \mathrm{E}_B \circ \widehat{\Theta}$.

Whenever $\Phi : A \to B$ is a ucp map between unital C^*-algebras, we denote by $\mathrm{mult}(\Phi) \subset A$ the multiplicative domain of Φ. Our main result is the following noncommutative generalization of Nevo–Sageev's structure theorem [14, Theorem 9.2].

Theorem 1. *Keep the same notation as above with $M = \mathrm{L}(\Gamma)$ and $\mathscr{B} = \mathrm{L}(\Gamma \curvearrowright B)$. Assume that the (Γ,μ)-Furstenberg–Poisson boundary (B,ν_B) is μ-uniquely stationary. Let $M \subset \mathscr{M}$ be an inclusion of von Neumann algebras. Assume that $\mathscr{M}$ is amenable and that there exists a faithful normal ucp map $\widehat{\Theta} : \mathscr{M} \to \mathscr{B}$ such that $\widehat{\Theta}|_M = \mathrm{id}_M$.*

Then $\widehat{\Theta}|_{\mathrm{mult}(\widehat{\Theta})} : \mathrm{mult}(\widehat{\Theta}) \to \mathscr{B}$ is a unital normal onto $*$-isomorphism. Therefore, we may regard $\mathscr{B} \subset \mathscr{M}$ as a von Neumann subalgebra and $\widehat{\Theta} : \mathscr{M} \to \mathscr{B}$ as a faithful normal conditional expectation.

In particular, letting $\psi = \nu_B \circ \mathrm{E}_B \circ \widehat{\Theta} \in \mathscr{M}_*$ and $\psi_B = \nu_B \circ \mathrm{E}_B \in \mathscr{B}_*$, it follows from [15, Theorem A] and [12] that the inclusion $M \subset \mathscr{M}$ has maximal φ_μ-entropy in the sense of [5], that is,

$$h_{\varphi_\mu}(M \subset \mathscr{M},\psi) = h_{\varphi_\mu}(M \subset \mathscr{B},\psi_B) = h(\mu) = \lim_n \frac{1}{n} H(\mu^{*n}).$$

When Γ is moreover an infinite icc group (infinite conjugacy classes), we deduce the following operator algebraic characterization of the (M,φ_μ)-noncommutative Furstenberg–Poisson boundary $M \subset \mathscr{B}$.

Corollary 2. *Keep the same notation as above with $M = \mathrm{L}(\Gamma)$ and $\mathscr{B} = \mathrm{L}(\Gamma \curvearrowright B)$. Assume that Γ is infinite icc and that the (Γ,μ)-Furstenberg–Poisson boundary (B,ν_B) is μ-uniquely stationary. Let $M \subset \mathscr{M}$ be an inclusion of von Neumann algebras satisfying the following conditions:*

(i) $M' \cap \mathscr{M} = \mathbb{C}1$.
(ii) *There exists a normal ucp map $\widehat{\Theta}\colon \mathscr{M} \to \mathscr{B}$ such that $\widehat{\Theta}|_M = \mathrm{id}_M$.*
(iii) *$\mathscr{M}$ is amenable.*
(iv) *Whenever $M \subset \mathscr{N} \subset \mathscr{M}$ is an intermediate von Neumann subalgebra for which there exists a normal conditional expectation $\mathrm{E}\colon \mathscr{M} \to \mathscr{N}$, we have $\mathscr{N} = \mathscr{M}$.*

Then $\widehat{\Theta}\colon \mathscr{M} \to \mathscr{B}$ is a unital normal onto $*$-isomorphism.

Let us point out that the (M,φ_μ)-noncommutative Furstenberg–Poisson boundary $M \subset \mathscr{B}$ satisfies Conditions (i), (ii), (iii), (iv) by [5]. Thus, Corollary 2 gives an abstract operator algebraic characterization of the inclusion $M \subset \mathscr{B}$.

We also apply Theorem 1 to provide further evidence towards Connes' rigidity conjecture for higher rank lattices.

Connes' rigidity conjecture. *For every $i \in \{1,2\}$, let G_i be a semisimple connected real Lie group with trivial center, no compact factor such that $\mathrm{rk}_{\mathbb{R}}(G_i) \geq 2$ and let $\Gamma_i < G_i$ be an irreducible lattice. If $\mathrm{L}(\Gamma_1) \cong \mathrm{L}(\Gamma_2)$, then $G_1 \cong G_2$ and in particular $\mathrm{rk}_{\mathbb{R}}(G_1) = \mathrm{rk}_{\mathbb{R}}(G_2)$.*

For every $i \in \{1,2\}$, choose a Furstenberg measure $\mu_i \in \mathrm{Prob}(\Gamma_i)$ so that $(G_i/P_i, \nu_{P_i})$ is the (Γ_i,μ_i)-Furstenberg–Poisson boundary [7]. Here, $\nu_{P_i} \in \mathrm{Prob}(G_i/P_i)$ denotes the unique K_i-invariant Borel probability measure, where $K_i < G_i$ is a maximal compact subgroup and $G_i = K_i P_i$. It is well-known that $(G_i/P_i, \nu_{P_i})$ is μ_i-uniquely stationary [8]. Set $M_i = \mathrm{L}(\Gamma_i)$ and $\mathscr{B}_i = \mathrm{L}(\Gamma_i \curvearrowright G_i/P_i)$.

Assume that $M_1 \cong M_2$ and set $M = M_1 = M_2$. Since $\mathscr{B}_2$ is an amenable (hence injective) von Neumann algebra and since $M \subset \mathscr{B}_1$, Arveson's extension theorem implies that there exists a ucp map $\Phi\colon \mathscr{B}_1 \to \mathscr{B}_2$ such that $\Phi|_M = \mathrm{id}_M$. We show that if $\Phi\colon \mathscr{B}_1 \to \mathscr{B}_2$ is normal, then $\mathrm{rk}_{\mathbb{R}}(G_1) = \mathrm{rk}_{\mathbb{R}}(G_2)$.

Theorem 3. *Keep the same notation as above. Assume that there exists a normal ucp map $\Phi\colon \mathscr{B}_1 \to \mathscr{B}_2$ such that $\Phi|_M = \mathrm{id}_M$. Then $\Phi\colon \mathscr{B}_1 \to \mathscr{B}_2$ is a unital normal onto $*$-isomorphism. In particular, we have*

$$\mathrm{rk}_{\mathbb{R}}(G_1) = \mathrm{rk}_{\mathbb{R}}(G_2).$$

2. Proofs of the main results

We record the following well-known fact on ucp maps.

Lemma 4. *Let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ be unital C^*-algebras and $\Phi\colon \mathfrak{A} \to \mathfrak{B}$, $\Psi\colon \mathfrak{B} \to \mathfrak{C}$ be ucp maps. Assume that Ψ is faithful and that $\Psi \circ \Phi$ is a unital $*$-homomorphism. Then Φ is a unital $*$-homomorphism and $\Phi(\mathfrak{A}) \subset \mathrm{mult}(\Psi)$.*

Proof. By Kadison's inequality and since $\Psi \circ \Phi$ is a unital $*$-homomorphism, for every $a \in \mathfrak{A}$, we have

$$\Psi\big(\Phi(a^* a)\big) = \Psi\big(\Phi(a^*)\big)\Psi\big(\Phi(a)\big) \leq \Psi\big(\Phi(a)^*\Phi(a)\big) \leq \Psi\big(\Phi(a^* a)\big).$$

Since Ψ is faithful, it follows that $\Phi(a)^*\Phi(a) = \Phi(a^* a)$. Then $\mathrm{mult}(\Phi) = \mathfrak{A}$ and so Φ is a unital $*$-homomorphism. Moreover, for every $a \in \mathfrak{A}$, we have

$$\Psi\big(\Phi(a)^*\big)\Psi\big(\Phi(a)\big) = \Psi\big(\Phi(a)^*\Phi(a)\big).$$

This further implies that $\Phi(\mathfrak{A}) \subset \mathrm{mult}(\Psi)$. $\square$

Let Γ be a countable discrete group and $\mu \in \mathrm{Prob}(\Gamma)$ an admissible probability measure. Denote by (B, ν_B) the (Γ,μ)-Furstenberg–Poisson boundary. Set $M = \mathrm{L}(\Gamma)$ and $\mathscr{B} = \mathrm{L}(\Gamma \curvearrowright B)$. Denote by $\mathrm{E}_B : \mathscr{B} \to \mathrm{L}^\infty(B,\nu_B)$ the canonical Γ-equivariant faithful normal conditional expectation.

Let $M \subset \mathscr{M}$ be an inclusion of von Neumann algebras. Let $\widehat{\Theta} \colon \mathscr{M} \to \mathscr{B}$ be a faithful normal ucp map such that $\widehat{\Theta}|_M = \mathrm{id}_M$. Then $\Theta = \mathrm{E}_B \circ \widehat{\Theta} \colon \mathscr{M} \to \mathrm{L}^\infty(B,\nu_B)$ is a Γ-equivariant faithful normal ucp map.

Lemma 5. *Keep the same notation as above. We have*

$$\Theta\big(\mathrm{mult}(\Theta)\big) \rtimes \Gamma \subset \widehat{\Theta}\big(\mathrm{mult}(\widehat{\Theta})\big) \subset \mathrm{L}^\infty(B,\nu_B) \rtimes \Gamma = \mathscr{B}.$$

In particular, if $\Theta\big(\mathrm{mult}(\Theta)\big) = \mathrm{L}^\infty(B,\nu_B)$*, then* $\widehat{\Theta}\big(\mathrm{mult}(\widehat{\Theta})\big) = \mathscr{B}$*. In that case, we may regard* $\mathscr{B} \subset \mathscr{M}$ *as a von Neumann subalgebra and* $\widehat{\Theta} \colon \mathscr{M} \to \mathscr{B}$ *as a faithful normal conditional expectation.*

Proof. By definition, $\Theta|_{\mathrm{mult}(\Theta)} \colon \mathrm{mult}(\Theta) \to \mathrm{L}^\infty(B,\nu_B)$ is a Γ-equivariant unital normal $*$-isomorphism and $\widehat{\Theta}|_{\mathrm{mult}(\widehat{\Theta})} \colon \mathrm{mult}(\widehat{\Theta}) \to \mathscr{B}$ is a unital normal $*$-isomorphism such that $M \subset \widehat{\Theta}\big(\mathrm{mult}(\widehat{\Theta})\big) \subset \mathscr{B}$. Since $\mathrm{E}_B \circ \widehat{\Theta} = \Theta$ and since E_B is faithful, Lemma 4 implies that $\widehat{\Theta}|_{\mathrm{mult}(\Theta)} \colon \mathrm{mult}(\Theta) \to \mathscr{B}$ is a unital normal $*$-isomorphism and $\widehat{\Theta}\big(\mathrm{mult}(\Theta)\big) \subset \mathrm{mult}(\mathrm{E}_B)$. Then we have $\mathrm{mult}(\Theta) \subset \mathrm{mult}(\widehat{\Theta})$ and so $M \vee \mathrm{mult}(\Theta) \subset \mathrm{mult}(\widehat{\Theta})$. Since $\mathrm{mult}(\mathrm{E}_B) = \mathrm{L}^\infty(B,\nu_B) \subset \mathscr{B}$, it follows that $\widehat{\Theta}\big(\mathrm{mult}(\Theta)\big) \subset \mathrm{L}^\infty(B,\nu_B) \subset \mathscr{B}$ and that $\Theta|_{\mathrm{mult}(\Theta)} = \widehat{\Theta}|_{\mathrm{mult}(\Theta)}$. Then we have

$$\Theta\big(\mathrm{mult}(\Theta)\big) \rtimes \Gamma \subset \widehat{\Theta}\big(\mathrm{mult}(\widehat{\Theta})\big) \subset \mathrm{L}^\infty(B,\nu_B) \rtimes \Gamma = \mathscr{B}.$$

Moreover, assume that $\Theta\big(\mathrm{mult}(\Theta)\big) = \mathrm{L}^\infty(B,\nu_B)$. Then we necessarily have $\widehat{\Theta}\big(\mathrm{mult}(\widehat{\Theta})\big) = \mathscr{B}$. Define the unital normal $*$-isomorphism $\iota = (\widehat{\Theta}|_{\mathrm{mult}(\widehat{\Theta})})^{-1} \colon \mathscr{B} \to \mathrm{mult}(\widehat{\Theta})$. Then $\iota \circ \Theta \colon \mathscr{M} \to \mathrm{mult}(\widehat{\Theta})$ is a faithful normal conditional expectation. □

Proof of Theorem 1. Keep the same notation as above with $M = \mathrm{L}(\Gamma)$ and $\mathscr{B} = \mathrm{L}(\Gamma \curvearrowright B)$. Assume moreover that the (Γ,μ)-Furstenberg–Poisson boundary (B,ν_B) is μ-uniquely stationary. We still denote by B the compact metrizable model and we assume that $B = \mathrm{supp}(\nu_B)$. Then the identity map $\mathrm{id}_{\mathrm{C}(B)} \colon \mathrm{C}(B) \to \mathrm{L}^\infty(B,\nu_B)$ is the unique Γ-equivariant ucp map (see [13, Corollary VI.2.10] and [9, Theorem 3.4]). Denote by $\lambda \colon \Gamma \to \mathscr{U}(M) : \gamma \mapsto \lambda(\gamma)$ the left regular representation. We naturally identify $\mathrm{L}^2(M) = \ell^2(\Gamma)$. Denote by M' the commutant of M in $\mathrm{B}\big(\mathrm{L}^2(M)\big)$. Since $M \subset \mathscr{M}$, we may regard $\mathrm{L}^2(\mathscr{M})$ as a Hilbert left M-module. Then [1, Proposition 8.2.3] implies that there exists a projection $e \in M' \mathbin{\overline{\otimes}} \mathrm{B}(\ell^2)$ and a unitary mapping $V \colon \mathrm{L}^2(\mathscr{M}) \to e\big(\mathrm{L}^2(M) \otimes \ell^2\big)$ such that $V(a\xi) = (a \otimes 1)V(\xi)$ for every $a \in M$ and every $\xi \in \mathrm{L}^2(\mathscr{M})$. Then the mapping

$$\pi \colon \mathrm{B}\big(\mathrm{L}^2(M)\big) \longrightarrow \mathrm{B}\big(\mathrm{L}^2(\mathscr{M})\big) : T \longmapsto V^* e(T \otimes 1)eV$$

is a normal ucp map such that $\pi(aTb) = a\pi(T)b$ for all $T \in \mathrm{B}\big(\mathrm{L}^2(M)\big)$ and all $a, b \in M$.

Since $\mathscr{M} \subset \mathrm{B}\big(\mathrm{L}^2(\mathscr{M})\big)$ is amenable, there exists a (possibly non normal) conditional expectation $\mathrm{E} \colon \mathrm{B}\big(\mathrm{L}^2(\mathscr{M})\big) \to \mathscr{M}$. Choose a point $b \in B$ and define the Γ-equivariant unital $*$-homomorphism $\theta \colon \mathrm{C}(B) \to \ell^\infty(\Gamma) : F \mapsto \big(\gamma \mapsto F(\gamma b)\big)$. Regard $\ell^\infty(\Gamma) \subset \mathrm{B}\big(\ell^2(\Gamma)\big) = \mathrm{B}\big(\mathrm{L}^2(M)\big)$ and define the ucp map $\iota = \mathrm{E} \circ \pi \circ \theta \colon \mathrm{C}(B) \to \mathscr{M}$. Then for every $F \in \mathrm{C}(B)$ and every $\gamma \in \Gamma$, we have

$$\iota(F \circ \gamma^{-1}) = \mathrm{E}\Big(\pi\big(\theta(F) \circ \gamma^{-1}\big)\Big) = \mathrm{E}\Big(\pi\big(\lambda(\gamma)\theta(F)\lambda(\gamma)^*\big)\Big) = \lambda(\gamma)\iota(F)\lambda(\gamma)^*.$$

Therefore, the ucp map $\iota \colon \mathrm{C}(B) \to \mathscr{M}$ is Γ-equivariant with respect to the conjugation action $\Gamma \curvearrowright \mathscr{M}$.

By composition, $\Theta \circ \iota \colon \mathrm{C}(B) \to \mathrm{L}^\infty(B,\nu_B)$ is a Γ-equivariant ucp map. Then we have $\Theta \circ \iota = \mathrm{id}_{\mathrm{C}(B)}$. Since $\Theta \colon \mathscr{M} \to \mathrm{L}^\infty(B,\nu_B)$ is faithful, Lemma 4 implies that $\iota \colon \mathrm{C}(B) \to \mathscr{M}$ is a unital $*$-homomorphism and $\iota\big(\mathrm{C}(B)\big) \subset \mathrm{mult}(\Theta)$. Set $\psi = \nu_B \circ \Theta \in \mathscr{M}_*$ and note that ψ is a μ-stationary faithful normal state. Since $\psi \circ \iota = \nu_B$, we may uniquely extend $\iota \colon \mathrm{L}^\infty(B,\nu_B) \to \mathscr{M}$ to a faithful normal unital $*$-homomorphism such that $\Theta \circ \iota = \mathrm{id}_{\mathrm{L}^\infty(B,\nu_B)}$. Then we have $\iota\big(\mathrm{L}^\infty(B,\nu_B)\big) \subset \mathrm{mult}(\Theta)$, $\mathrm{L}^\infty(B,\nu_B) = \Theta\big(\iota\big(\mathrm{L}^\infty(B,\nu_B)\big)\big)$ and so $\Theta\big(\mathrm{mult}(\Theta)\big) = \mathrm{L}^\infty(B,\nu_B)$. The moreover part follows from Lemma 5. □

Proof of Corollary 2. Set $M = \mathrm{L}(\Gamma)$ and $\mathscr{B} = \mathrm{L}(\Gamma \curvearrowright B)$. Assume that Γ is infinite icc and that the (Γ,μ)-Furstenberg–Poisson boundary (B,ν_B) is μ-uniquely stationary. Denote by $\lambda\colon \Gamma \to \mathscr{U}(M) : \gamma \mapsto \lambda(\gamma)$ the left regular representation. Let $\widehat{\Theta}\colon \mathscr{M} \to \mathscr{B}$ be a normal ucp map such that $\widehat{\Theta}|_M = \mathrm{id}_M$ as given by Condition (iii). Denote by $p \in \mathscr{M}$ the support projection of $\widehat{\Theta}\colon \mathscr{M} \to \mathscr{B}$. Then for $\gamma \in \Gamma$, we have $\widehat{\Theta}\big(\lambda(\gamma)p\lambda(\gamma)^*\big) = \lambda(\gamma)\widehat{\Theta}(p)\lambda(\gamma)^* = 1$ and so $p \leq \lambda(\gamma)p\lambda(\gamma)^*$. Since this holds for every $\gamma \in \Gamma$, we infer that $p \in \mathrm{L}(\Gamma)' \cap \mathscr{M}$. Condition (i) further implies that $p = 1$. Thus, $\widehat{\Theta}\colon \mathscr{M} \to \mathscr{B}$ is a faithful normal ucp map. By Condition (ii) and Theorem 1, we infer that $\mathscr{B} \subset \mathscr{M}$ can be regarded as a von Neumann subalgebra and $\widehat{\Theta}\colon \mathscr{M} \to \mathscr{B}$ as a faithful normal conditional expectation. Then Condition (iv) finally implies that $\mathscr{B} = \mathscr{M}$ and so $\widehat{\Theta}\colon \mathscr{M} \to \mathscr{B}$ is a unital normal onto $*$-isomorphism. □

Proof of Theorem 3. Consider the inclusion $M \subset \mathscr{B}_1$. Since M is a type II_1 factor, [5, Proposition 2.7] implies that $M' \cap \mathscr{B}_1 = \mathbb{C}1$. By assumption, $\mathscr{B}_1$ is amenable and $\Phi\colon \mathscr{B}_1 \to \mathscr{B}_2$ is a normal ucp map such that $\Phi|_M = \mathrm{id}_M$. By [5, Theorem 4.1], for every intermediate von Neumann subalgebra $M \subset \mathscr{N} \subset \mathscr{B}_1$ for which there exists a normal conditional expectation $\mathrm{E}\colon \mathscr{B}_1 \to \mathscr{N}$, we have $\mathscr{N} = \mathscr{B}_1$. We may apply Corollary 2 to the inclusion $M \subset \mathscr{B}_1$ to conclude that $\Phi\colon \mathscr{B}_1 \to \mathscr{B}_2$ is a unital normal onto $*$-isomorphism. By [10, Corollary F] and [4, Theorem B], we infer that $\mathrm{rk}_{\mathbb{R}}(G_1) = \mathrm{rk}_{\mathbb{R}}(G_2)$. □

Acknowledgments

I am grateful to Amos Nevo for thought provoking discussions during the IHP trimester *Group actions and rigidity: around the Zimmer program* and for pointing out the reference [14].

Declaration of interests

The author does not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and has declared no affiliations other than their research organizations.

References

[1] C. Anantharaman and S. Popa, "An introduction to II_1 factors", 2014. Online at https://www.math.ucla.edu/~popa/books/iiunv15.pdf.

[2] U. Bader, R. Boutonnet, C. Houdayer and J. Peterson, "Charmenability of arithmetic groups of product type", *Invent. Math.* **229** (2022), no. 3, pp. 929–985.

[3] U. Bader and Y. Shalom, "Factor and normal subgroup theorems for lattices in products of groups", *Invent. Math.* **163** (2006), no. 2, pp. 415–454.

[4] R. Boutonnet and C. Houdayer, "The noncommutative factor theorem for lattices in product groups", *J. Éc. Polytech., Math.* **10** (2023), pp. 513–524.

[5] S. Das and J. Peterson, "Poisson boundaries of II_1 factors", *Compos. Math.* **158** (2022), no. 8, pp. 1746–1776.

[6] H. Furstenberg, "A Poisson formula for semi-simple Lie groups", *Ann. Math. (2)* **77** (1963), pp. 335–386.

[7] H. Furstenberg, "Poisson boundaries and envelopes of discrete groups", *Bull. Am. Math. Soc.* **73** (1967), pp. 350–356.

[8] I. Y. Goldsheid and G. A. Margulis, "Lyapunov exponents of a product of random matrices", *Russ. Math. Surv.* **44** (1989), no. 5, pp. 11–71.

[9] Y. Hartman and M. Kalantar, “Tight inclusions of C^*-dynamical systems”, *Groups Geom. Dyn.* **18** (2024), no. 1, pp. 67–90.

[10] C. Houdayer, “Noncommutative ergodic theory of higher rank lattices”, in *ICM—International Congress of Mathematicians. Vol. 4. Sections 5–8*, European Mathematical Society, 2023, pp. 3202–3223.

[11] M. Izumi, “Non-commutative Poisson boundaries”, in *Discrete geometric analysis*, Contemporary Mathematics, American Mathematical Society, 2004, pp. 69–81.

[12] V. A. Kaimanovich and A. Moiseevich, “Random walks on discrete groups: boundary and entropy”, *Ann. Probab.* **11** (1983), no. 3, pp. 457–490.

[13] G. A. Margulis, *Discrete subgroups of semisimple Lie groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, Springer, 1991, pp. x+388.

[14] A. Nevo and M. Sageev, “The Poisson boundary of CAT(0) cube complex groups”, *Groups Geom. Dyn.* **7** (2013), no. 3, pp. 653–695.

[15] S. Zhou, “Noncommutative Poisson boundaries, ultraproducts, and entropy”, *Int. Math. Res. Not.* **2024** (2024), no. 10, pp. 8794–8818.