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Hodge–Lyubeznik numbers

Nombres de Hodge–Lyubeznik

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Abstract. We define a Hodge-theoretical refinement of the Lyubeznik numbers for local rings of complex algebraic varieties. We prove that these numbers are independent of the choices made in their definition and that, for the local ring of an isolated singularity, they can be expressed in terms of the Hodge numbers of the cohomology of the link of the singularity. We give examples of isolated singularities with the same Lyubeznik numbers but different Hodge–Lyubeznik numbers.

Résumé. Nous définissons un raffinement en théorie de Hodge des nombres de Lyubeznik pour les anneaux locaux de variétés algébriques complexes. Nous prouvons que ces nombres sont indépendants des choix faits dans leur définition et que, pour l'anneau local d'une singularité isolée, ils peuvent être exprimés en termes de nombres de Hodge de la cohomologie de l'entrelac de la singularité. Nous donnons des exemples de singularités isolées ayant les mêmes nombres de Lyubeznik mais des nombres de Hodge–Lyubeznik différents.

Keywords. Hodge numbers, Lyubeznik numbers, mixed Hodge modules.

Mots-clés. Nombres de Hodge, nombres de Lyubeznik, modules de Hodge mixtes.

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Introduction

Let A be a local ring for which there exists a surjection $\pi\colon R\to A$, where R is a regular local ring of dimension n containing a field (for instance, A is the local ring of an algebraic variety at a point). Put $I=\ker\pi$, let $\mathfrak{m}$ be the maximal ideal of R, set $k=R/\mathfrak{m}$.

In [7], Lyubeznik proved that the numbers

$$\lambda_{r,s}\coloneqq\dim_k \operatorname{Ext}^r_R\bigl(k,\mathrm{H}^{n-s}_I(R)\bigr)$$

are finite and depend only on the ring A, they are independent of the presentation $A\cong R/I$ (finiteness was known if R contains a field of positive characteristic). They are known as the *Lyubeznik numbers of A*, and have been studied by many authors (see the survey article [9] and the references therein).

Let X be a complex algebraic variety, $x \in X$. Taking an affine chart, we can assume that for some $n \geqslant 1$ we have a closed embedding $i\colon X \hookrightarrow \mathbb{C}^n$ with $i(x) = 0 \in \mathbb{C}^n$.

Set $A = \mathscr{O}_{X,x}$ and let D_n denote the ring of differential operators on $\mathbb{C}^n$ with polynomial coefficients. The embedding i gives a presentation $A \cong \mathscr{O}_{\mathbb{C}^n,0}/I$ and the Lyubeznik number $\lambda_{r,s}(A)$ equals the length of the holonomic D_n-module $H^r_{\mathfrak{m}}\big(H^{n-s}_I(\mathscr{O}_{\mathbb{C}^n,0})\big)$, where $\mathfrak{m}$ is the maximal ideal of $\mathscr{O}_{\mathbb{C}^n,0}$. Let $\mathscr{D}_n$ denote the sheaf of algebraic linear differential operators on $\mathbb{C}^n$. Passing to sheaves, the D_n-module $H^r_{\mathfrak{m}}\big(H^{n-s}_I(\mathscr{O}_{\mathbb{C}^n,0})\big)$ corresponds to a $\mathscr{D}_n$-module $\mathcal{H}^r_{\{0\}}\big(\mathcal{H}^{n-s}_{i(X)}(\mathscr{O}_{\mathbb{C}^n})\big)$ with punctual support, which is part of the datum defining a mixed Hodge module [13]. Since the category of mixed Hodge modules with punctual support is equivalent to the category of mixed Hodge structures, we have a Hodge and a weight filtration on $\mathcal{H}^r_{\{0\}}\big(\mathcal{H}^{n-s}_{i(X)}(\mathscr{O}_{\mathbb{C}^n})\big)$, and we can consider numerical invariants attached to them, say[1]

$$\lambda^{p,q}_{r,s}(\mathscr{O}_{X,x}) := \dim_{\mathbb{C}} \operatorname{Gr}^p_F \operatorname{Gr}^W_{p+q} \mathcal{H}^r_0\Big(\mathcal{H}^{n-s}_{i(X)}\big(\mathbb{Q}^{\mathrm{H}}_n[n](n)\big)\Big). \tag{1}$$

Here, $\mathbb{Q}^{\mathrm{H}}_n[n](n)$ is defined below, following Saito (see [13, (4.5.5)]). It is a Hodge module with $\mathscr{O}_{\mathbb{C}^n}$ as underlying $\mathscr{D}_n$-module, and it will follow from its definition that

$$\lambda_{r,s}(\mathscr{O}_{X,x}) = \sum_{p,q} \lambda^{p,q}_{r,s}(\mathscr{O}_{X,x}).$$

So, for local rings of complex algebraic varieties, the numbers $\lambda^{p,q}_{r,s}(\mathscr{O}_{X,x})$ might be regarded as Hodge-theoretical refinements of the Lyubeznik numbers. We prove that these Hodge numbers are independent of the embedding $X \hookrightarrow \mathbb{C}^n$, and we give examples showing that they do give more information than the Lyubeznik numbers. We show that for isolated singularities they can be computed in terms of the Hodge numbers of the cohomology of the link of X at x, extending the main result in [4].

Notations

We refer to [6] for terminology and notations concerning $\mathscr{D}$-modules and to [13,14] for those concerning mixed Hodge modules, see also the introduction to this theory in [15] and also [10, Chapter 14].[2] Still, we briefly recall a few notations we will use:

(i) All varieties considered in this note are defined over the complex numbers. For $n \geqslant 1$, we denote by $\mathscr{D}_n$ the sheaf of algebraic linear differential operators on $\mathbb{C}^n$. Unless otherwise specified, $\mathscr{D}_n$-modules are always left modules. A mixed Hodge module on a smooth variety Y is a 4-tuple

$$\mathcal{M} = (M, F_\bullet, K, W_\bullet),$$

where M is a holonomic left $\mathscr{D}_Y$-module with regular singularities, K is a rational perverse sheaf on Y together with an isomorphism $\mathrm{DR}(M) \cong K \otimes_{\mathbb{Q}} \mathbb{C}$ (omitted in the notation, but part of the data), $F_\bullet$ is a good filtration of M by $\mathscr{O}_X$-coherent subsheaves and $W_\bullet$ is a finite increasing filtration of K, inducing one on M. These data has to verify a number of compatibilities and conditions, see [13, 2.17] or [15].

(ii) We use the abbreviations MHS for "mixed Hodge structure" and MHM for "mixed Hodge module". We denote by $\mathrm{MHM}(X)$ the abelian category of mixed Hodge modules on an algebraic variety X [13, Section 4]. We will have to consider MHMs on singular varieties embedded in affine spaces. Recall [13, (2.17.5)] that if X is an algebraic variety with a closed immersion $X \hookrightarrow Y$, where Y is smooth, then the category $\mathrm{MHM}(X)$ can be identified with the subcategory $\mathrm{MHM}_X(Y)$ of MHMs on Y with support on X. If X is

[1] The possibility of considering such invariants is mentioned in a remark after Corollary 2 in [11].

[2] In [13–15] *right* $\mathscr{D}$-modules are used, while we will use *left* ones.

a point, then there is an equivalence of categories between $\mathrm{MHM}_X(Y)$ and the category of mixed Hodge structures.

If $\mathcal{A}$ is an abelian category, we denote by $D^b(\mathcal{A})$ its bounded derived category. The symbol $\sim$ denotes an isomorphism in $D^b(\mathcal{A})$. The category $D^b(\mathrm{MHM}(Y))$ is endowed with a six functor formalism [13, Section 4]. If $X \subset Y$ is a subvariety, we denote by $D^b_X(\mathrm{MHM}(Y))$ the full subcategory of $D^b(\mathrm{MHM}(Y))$ which has as objects those complexes whose cohomology is supported on X.

(iii) Following Saito [13], we denote by $\mathbb{Q}^{\mathrm{H}}_{\mathrm{pt}}$ the constant Hodge module $(\mathbb{C}, F, \mathbb{Q}, W)$ on $\{\mathrm{pt}\}$ with $\mathrm{Gr}^i_F = Gr^W_i = 0$ if $i \neq 0$. If $\alpha_X \colon X \to \{\mathrm{pt}\}$ is the constant map, put $\mathbb{Q}^{\mathrm{H}}_X = \alpha^*_X \mathbb{Q}^{\mathrm{H}}_{\mathrm{pt}} \in D^b(\mathrm{MHM}(X))$. If Y is smooth of dimension d_Y, then $\mathbb{Q}^{\mathrm{H}}_Y[d_Y]$ is concentrated in degree zero and $\mathbb{Q}^{\mathrm{H}}_Y[d_Y]_0$ is, by abuse of notation, denoted also

$$\mathbb{Q}^{\mathrm{H}}_Y[d_Y] = (\mathscr{O}_Y, F_\bullet, \mathbb{Q}_Y[d_Y], W_\bullet), \quad \text{where} \quad \begin{cases} \mathrm{Gr}^k_F \mathscr{O}_Y = \{0\} & \text{if } k \neq 0, \\ \mathrm{Gr}^W_k \mathscr{O}_Y = \{0\} & \text{if } k \neq d_Y. \end{cases}$$

For $Y = \mathbb{C}^n$, we put $\mathbb{Q}^{\mathrm{H}}_n := \mathbb{Q}^{\mathrm{H}}_Y$.

(iv) If $\mathcal{M} = (M, F_\bullet, K, W_\bullet)$ is a mixed Hodge module and $k \in \mathbb{Z}$, then its Tate twist by k is the MHM

$$\mathcal{M}(k) := (M, F_{\bullet - k}, (2\pi i)^k K, W_{\bullet - 2k}),$$

see [13, 2.17].

(v) We will consider the duality functor $\mathbb{D}$ in the category of mixed Hodge modules as defined by Saito in [13, 2.6], see also [12, 2.4.3]. On the underlying regular holonomic $\mathscr{D}$-modules, the duality $\mathbb{D}$ is the usual holonomic duality. For $\mathbb{Q}^{\mathrm{H}}_n[n](n)$ we have

$$\mathbb{D}(\mathbb{Q}^{\mathrm{H}}_n[n](n)) \sim \mathbb{Q}^{\mathrm{H}}_n[n]. \tag{2}$$

(vi) Let X be an algebraic variety, $Z \xhookrightarrow{k} X$ a closed subvariety. Deligne defines in [3, (8.3.8)] a mixed Hodge structure on the (topological) local cohomology $\mathrm{H}^\ell_Z(X, \mathbb{Q})$, $\ell \geqslant 0$. If $\alpha_X \colon X \to \{\mathrm{pt}\}$, $\alpha_Z \colon Z \to \{\mathrm{pt}\}$ are the constant maps then, by a theorem of Saito (see [14]), we have isomorphisms of MHS

$$\mathrm{H}^\ell_Z(X, \mathbb{Q})^{\mathrm{Del}} \cong \mathrm{H}^\ell(\mathrm{pt}, (\alpha_X)_* k_* k^! \alpha^*_X \mathbb{Q}) \cong \mathrm{H}^\ell(\mathrm{pt}, (\alpha_Z)_* k^! \alpha^*_X \mathbb{Q}^{\mathrm{H}}_{\mathrm{pt}}),$$

where on the left-hand side the MHS is the one of Deligne. For $\mathcal{M}$ a mixed Hodge module on X, we put

$$\mathbb{R}\Gamma_Z(\mathcal{M}) := (\alpha_X)_* k_* k^! \mathcal{M}.$$

(vii) If $X \hookrightarrow Y$ is a subvariety of a smooth variety Y then $\mathcal{H}^\bullet_X(\mathscr{O}_Y)$, the local algebraic cohomology modules of $\mathscr{O}_Y$ supported on X, regarded as left $\mathscr{D}_Y$-modules, are part of the data defining the mixed Hodge modules $\mathcal{H}^\bullet(\mathbb{R}\Gamma_X(\mathbb{Q}^{\mathrm{H}}_n[n](k)))$, for all $k \in \mathbb{Z}$, see [13].

(viii) If $\mathcal{M}$ is a Hodge module with punctual support (equivalently, a MHS), we put

$$h^{p,q}(\mathcal{M}) = \dim_{\mathbb{C}} (\mathrm{Gr}^p_F Gr^W_{p+q} \mathcal{M}).$$

Hodge–Lyubeznik numbers

Let A be a finitely generated $\mathbb{C}$-algebra, that is, a quotient of a polynomial ring with coefficients in $\mathbb{C}$. Then A is the ring of regular functions of an affine algebraic variety X, and any presentation of A as a quotient of $\mathbb{C}[x_1, \ldots, x_n]$ corresponds to a closed embedding $i \colon X \hookrightarrow \mathbb{C}^n$. For a given $x \in X$, we can assume $i(x) = 0$. We denote by $\mathscr{O}_{X,x}$ the local ring of X at $x \in X$.

Theorem. *Let X be an affine algebraic variety, $x \in X$. Choose a closed embedding $i\colon X \hookrightarrow \mathbb{C}^n$ with $n \geqslant 1$, $i(x) = 0 \in \mathbb{C}^n$. Then:*

(a) *The numbers*

$$\lambda^{p,q}_{r,s}(\mathscr{O}_{X,x}) := \dim_{\mathbb{C}} \mathrm{Gr}^p_F \, Gr^W_{p+q} \, \mathcal{H}^r_0\Big(\mathcal{H}^{n-s}_{i(X)}\big(\mathbb{Q}^{\mathrm{H}}_n[n](n)\big)\Big),$$

where $p, q \in \mathbb{Z}$, are independent of the embedding i.

(b) *Assume the singularity of X at x is isolated and X is of pure dimension $d \geqslant 2$ at x. Denote by $H^i_{\{x\}}(X,\mathbb{C})$ the singular cohomology groups of X with complex coefficients and support on $\{x\}$, endowed with their mixed Hodge structure (see* (vi) *above). Then for $p, q \in \mathbb{Z}$, one has:*

(i) $\lambda^{p,q}_{0,s}(\mathscr{O}_{X,x}) = h^{-p,-q}\big(H^s_{\{x\}}(X,\mathbb{C})\big)$ *for* $1 \le s \le d-1$,

(ii) $\lambda^{p,q}_{r,d}(\mathscr{O}_{X,x}) = h^{p+d,q+d}\big(H^{d+r}_{\{x\}}(X,\mathbb{C})\big)$ *for* $2 \le r \le d$,

(iii) *all other* $\lambda^{p,q}_{r,s}(\mathscr{O}_{X,x})$ *vanish.*

Proof. (a). Following [7], we first prove a special case: assume we have an algebraic embedding $\kappa\colon \mathbb{C}^n \hookrightarrow \mathbb{C}^m$ with $\kappa(0_n) = 0_m$, set $j = \kappa \circ i$, we will compare the embeddings given by i and j.

We have $\mathbb{Q}^{\mathrm{H}}_n[n](n) \sim \kappa^! \mathbb{Q}^{\mathrm{H}}_m[m](m)[m-n]$, so

$$\begin{aligned}\kappa_* \mathbb{R}\Gamma_{i(X)}\big(\mathbb{Q}^{\mathrm{H}}_n[n](n)\big) &\sim \kappa_* \, i_* \, i^! \kappa^! \mathbb{Q}^{\mathrm{H}}_m[m](m)[m-n]\\ &\sim \mathbb{R}\Gamma_{j(X)} \mathbb{Q}^{\mathrm{H}}_m[m](m)[m-n].\end{aligned}$$

Since κ is affine, κ_* is the right derived functor of the cohomological functor $\mathscr{H}^0\kappa_*$, which is exact because κ is a closed embedding. It follows that

$$\begin{aligned}\mathcal{H}^{m-r}\Big(\mathbb{R}\Gamma_{j(X)}\big(\mathbb{Q}^{\mathrm{H}}_m[m](m)\big)\Big) &\cong \mathcal{H}^{m-r}\Big(\kappa_* \mathbb{R}\Gamma_{i(X)}\big(\mathbb{Q}^{\mathrm{H}}_n[n](n)\big)[n-m]\Big)\\ &\cong \kappa_* \, \mathcal{H}^{n-r}\Big(\mathbb{R}\Gamma_{i(X)}\big(\mathbb{Q}^{\mathrm{H}}_n[n](n)\big)\Big),\end{aligned}$$

see also [8, Remark 2.5].

Let $a\colon \{0\} \hookrightarrow \mathbb{C}^n$ and $b = \kappa \circ a$. Then

$$\begin{aligned}\mathbb{R}\Gamma_{\{0\}}\Big(\mathcal{H}^{m-r}\Big(\mathbb{R}\Gamma_{j(X)}\big(\mathbb{Q}^{\mathrm{H}}_m[m](m)\big)\Big)\Big) &\sim b_* \, b^!\Big(\mathcal{H}^{m-r}\Big(\mathbb{R}\Gamma_{j(X)}\big(\mathbb{Q}^{\mathrm{H}}_m[m](m)\big)\Big)\Big)\\ &\sim \kappa_* \, a_* \, a^! \kappa^! \kappa_* \, \mathcal{H}^{n-r}\Big(\mathbb{R}\Gamma_{i(X)}\big(\mathbb{Q}^{\mathrm{H}}_n[n](n)\big)\Big)\\ &\sim \kappa_* \, \mathbb{R}\Gamma_{\{0\}}\Big(\mathcal{H}^{n-r}\Big(\mathbb{R}\Gamma_{i(X)}\big(\mathbb{Q}^{\mathrm{H}}_n[n](n)\big)\Big)\Big),\end{aligned}$$

since $\kappa^!\kappa_* \sim \mathrm{id}$. Thus, for any $e \geqslant 0$,

$$\begin{aligned}\mathcal{H}^e\Big(\mathbb{R}\Gamma_{\{0\}}\Big(\mathcal{H}^{m-r}\Big(\mathbb{R}\Gamma_{j(X)}\big(\mathbb{Q}^{\mathrm{H}}_m[m](m)\big)\Big)\Big)\Big) &\cong \mathcal{H}^e\Big(\kappa_* \mathbb{R}\Gamma_{\{0\}}\Big(\mathcal{H}^{n-r}\Big(\mathbb{R}\Gamma_{i(X)}\big(\mathbb{Q}^{\mathrm{H}}_n[n](n)\big)\Big)\Big)\Big)\\ &\cong \kappa_* \, \mathcal{H}^e\Big(\mathbb{R}\Gamma_{\{0\}}\Big(\mathcal{H}^{n-r}\Big(\mathbb{R}\Gamma_{i(X)}\big(\mathbb{Q}^{\mathrm{H}}_n[n](n)\big)\Big)\Big)\Big).\end{aligned}$$

If $\mathcal{M}$ is a Hodge module on $\mathbb{C}^n$ with punctual support in $\{0\}$ then, since $\mathrm{MHM}_{\{0\}}(\mathbb{C}^n) \simeq \mathrm{MHM}(\{0\}) \simeq \mathrm{MHM}_{\{0\}}(\mathbb{C}^m)$,

$$h^{p,q}(\mathcal{M}) = h^{p,q}(\kappa_*\mathcal{M}),$$

and so the desired equality is proved in this special case. In general, for any $\ell \leqslant N \in \mathbb{N}$ denote by $a_{\ell,N}\colon \mathbb{C}^\ell \hookrightarrow \mathbb{C}^N$ the linear embedding defined by $(x_1 \dots, x_\ell) \to (x_1, \dots, x_\ell, 0, \dots, 0)$. Given embeddings $i\colon X \hookrightarrow \mathbb{C}^n$ and $j\colon X \hookrightarrow \mathbb{C}^m$, by [16, Theorem 2] there is an $N > \max\{n, m\}$ and a polynomial automorphism $\varphi\colon \mathbb{C}^N \to \mathbb{C}^N$ such that $a_{n,N} \circ i = \varphi \circ a_{m,N} \circ j\colon X \hookrightarrow \mathbb{C}^N$. Then, applying the previous argument twice, the result follows.[3]

[3] It might be that to invoke [16] is overkill, it is likely that one could also work with polydisks instead of affine spaces, etc. We prefer to stay within an algebraic framework.

(b). The proof is a translation into the MHM language of the proof given in [4]. Some arguments that produce numerical equalities in loc. cit. have to be promoted to isomorphisms of MHM.

First, it is proved in [4, p. 321] that if we consider an embedding $i\colon X\hookrightarrow\mathbb{C}^n$, with $i(x)=0$, one has an isomorphism of $\mathscr{D}_n$-modules

$$\mathcal{H}^p_{\{0\}}\Big(\big(\mathcal{H}^{-q}_X(\mathscr{O}_{\mathbb{C}^n})\big)^*\Big)\cong\mathcal{H}^{p+q}\Big(\mathbb{R}\Gamma_{\{0\}}\big(\mathbb{R}\Gamma_X(\mathscr{O}_{\mathbb{C}^n})\big)^*\Big),$$

where $*$ denotes duality for holonomic $\mathscr{D}$-modules. The same argument as in loc. cit. gives an isomorphism in the category of mixed Hodge modules, i.e., we have

$$\mathcal{H}^r_{\{0\}}\Big(\mathbb{D}\Big(\mathcal{H}^{-s}_X\big(\mathbb{Q}^{\mathrm{H}}_n[n]\big)\Big)\Big)\cong\mathcal{H}^{r+s}\Big(\mathrm{R}\Gamma_{\{0\}}\Big(\mathbb{D}\Big(\mathrm{R}\Gamma_X\big(\mathbb{Q}^{\mathrm{H}}_n[n]\big)\Big)\Big)\Big).\tag{3}$$

If we denote by $a\colon\{0\}\hookrightarrow\mathbb{C}^n$, $i\colon X\hookrightarrow\mathbb{C}^n$, $k\colon\{x\}\hookrightarrow X$ the inclusion maps, then

$$\begin{aligned}
\mathbb{R}\Gamma_{\{0\}}\mathbb{D}\Big(\mathbb{R}\Gamma_X\big(\mathbb{Q}^{\mathrm{H}}_n[n]\big)\Big)&\sim\mathbb{R}\Gamma_{\{0\}}\mathbb{D}i_*i^!\mathbb{Q}^{\mathrm{H}}_n[n] && \text{by definition of } \mathbb{R}\Gamma_X\\
&\sim\mathbb{R}\Gamma_{\{0\}}i_*\mathbb{D}i^!\mathbb{Q}^{\mathrm{H}}_n[n] && \text{by properness of } i\\
&\sim\mathbb{R}\Gamma_{\{0\}}i_*i^*\mathbb{Q}^{\mathrm{H}}_n[n](n) && \text{by } \mathbb{D}i^!=i^*\mathbb{D} \text{ and (2)}\\
&\sim a_*a^!i_*i^*\mathbb{Q}^{\mathrm{H}}_n[n](n) && \text{by definition of } \mathbb{R}\Gamma_{\{0\}}\\
&\sim a_*a^!i_*\mathbb{Q}^{\mathrm{H}}_X[n](n) && \text{by definition of } \mathbb{Q}^{\mathrm{H}}_X\\
&\sim a_*k^!\mathbb{Q}^{\mathrm{H}}_X[n](n) && \text{by [13, (4.4.3)]}.
\end{aligned}$$

It follows from these quasi-isomorphisms and (3) that

$$\begin{aligned}
h^{p,q}\Big(\mathcal{H}^r_{\{0\}}\Big(\mathbb{D}\Big(\mathcal{H}^{-s}_X\big(\mathbb{Q}^{\mathrm{H}}_n[n]\big)\Big)\Big)\Big)&=h^{p,q}\Big(\mathcal{H}^{r+s}\Big(\mathbb{R}\Gamma_{\{0\}}\mathbb{D}\Big(\mathbb{R}\Gamma_X\big(\mathbb{Q}^{\mathrm{H}}_n[n]\big)\Big)\Big)\Big)\\
&=h^{p,q}\Big(\mathcal{H}^{r+s}\big(a_*k^!\mathbb{Q}^{\mathrm{H}}_X[n](n)\big)\Big)\\
&=h^{p,q}\Big(\mathcal{H}^{r+s}\big(k^!\mathbb{Q}^{\mathrm{H}}_X[n](n)\big)\Big)\\
&=h^{p+n,q+n}\big(\mathcal{H}^{r+s+n}(k^!\mathbb{Q}^{\mathrm{H}}_X)\big).
\end{aligned}\tag{4}$$

By (vi) above, we have $\mathcal{H}^{r+s+n}(k^!\mathbb{Q}^{\mathrm{H}}_X)=H^{r+s+n}_{\{x\}}(X,\mathbb{C})$, and so

$$h^{p,q}\Big(\mathcal{H}^r_{\{0\}}\Big(\mathbb{D}\Big(\mathcal{H}^{-s}_X\big(\mathbb{Q}^{\mathrm{H}}_n[n]\big)\Big)\Big)\Big)=h^{p+n,q+n}\big(H^{r+s+n}_{\{x\}}(X,\mathbb{C})\big).\tag{5}$$

Since the singularity of X at x is isolated, if $r=0$ and $s=n-i$ with $i<d$, we have

$$\begin{aligned}
h^{p,q}\Big(\mathcal{H}^0_{\{0\}}\Big(\mathbb{D}\Big(\mathcal{H}^{-s}_X\big(\mathbb{Q}^{\mathrm{H}}_n[n]\big)\Big)\Big)\Big)&=h^{p,q}\Big(\mathbb{D}\Big(\mathcal{H}^{-s}_X\big(\mathbb{Q}^{\mathrm{H}}_n[n]\big)\Big)\Big)\\
&=h^{-p,-q}\Big(\mathcal{H}^{-s}_X\big(\mathbb{Q}^{\mathrm{H}}_n[n]\big)\Big)\\
&=h^{-p,-q}\Big(\mathcal{H}^0_{\{0\}}\Big(\mathcal{H}^{-s}_X\big(\mathbb{Q}^{\mathrm{H}}_n[n]\big)\Big)\Big)\\
&=h^{-n-p,-n-q}\Big(\mathcal{H}^0_{\{0\}}\Big(\mathcal{H}^{-s}_X\big(\mathbb{Q}^{\mathrm{H}}_n[n](n)\big)\Big)\Big),
\end{aligned}$$

and so,

$$\begin{aligned}
\lambda^{p,q}_{0,i}(\mathscr{O}_{X,x})&=h^{p,q}\Big(\mathcal{H}^0_{\{0\}}\Big(\mathcal{H}^{n-i}_X\big(\mathbb{Q}^{\mathrm{H}}_n[n](n)\big)\Big)\Big)\\
&=h^{-n-p,-n-q}\Big(\mathcal{H}^0_{\{0\}}\Big(\mathbb{D}\Big(\mathcal{H}^{n-i}_X\big(\mathbb{Q}^{\mathrm{H}}_n[n]\big)\Big)\Big)\Big)\\
&=h^{-p,-q}\big(H^i_{\{x\}}(X,\mathbb{C})\big)
\end{aligned}$$

for $i<d$.

For the proof of (b)-(ii), we follow [4] again: let $\mathrm{Sing}(X)$ denote the singular locus of X; since the singularity at x is isolated, there exists a hypersurface H of $\mathbb{C}^n$ containing $\mathrm{Sing}(X)\smallsetminus\{x\}$ and such that $x\notin H$. Replacing X by $X\smallsetminus H\cap X$, we can assume that $U=X\smallsetminus\{x\}$ is smooth.

In [13, 4.5], Saito introduces a Hodge module $\mathrm{IC}_X^{\mathrm{Sai}}$,[4] which is the only object of $\mathrm{MHM}(\mathbb{C}^n)$ such that its restriction to U is $\mathbb{Q}_U^{\mathrm{H}}[d]$ and which has no subobject and no quotient object in $\mathrm{MHM}(\mathbb{C}^n)$ supported at $0 \in \mathbb{C}^n$. Its underlying holonomic $\mathscr{D}_n$-module is the one introduced in [2, Proposition 8.5 and its proof], denoted there by $\mathfrak{L}(X,\mathbb{C}^n)$. Put

$$\begin{aligned}\mathcal{K} &= \mathcal{H}^0_{\{0\}}\Big(\mathbb{D}\Big(\mathcal{H}^{n-d}_X\big(\mathbb{Q}^{\mathrm{H}}_n[n]\big)\Big)\Big),\\ \mathrm{IC}_X &= \mathrm{coker}\Big[\mathcal{K}\to\mathbb{D}\Big(\mathcal{H}^{n-d}_X\big(\mathbb{Q}^{\mathrm{H}}_n[n]\big)\Big)\Big].\end{aligned}\tag{6}$$

The Hodge module IC_X has also no subobjet and (by duality, since $\lambda_{0,d}=0$, see [4]) no quotient object supported at 0. Its restriction to U is $\mathbb{Q}_U^{\mathrm{H}}[d](-n)$. By uniqueness of Saito's Hodge module, we have $\mathrm{IC}_X = \mathrm{IC}_X^{\mathrm{Sai}}(n)$.

Since $\mathcal{K}$ is supported at $\{0\}$, the exact sequence

$$0\longrightarrow\mathcal{K}\longrightarrow\mathbb{D}\Big(\mathcal{H}^{n-d}_X\big(\mathbb{Q}^{\mathrm{H}}_n[n]\big)\Big)\longrightarrow\mathrm{IC}_X\longrightarrow 0$$

yields, for $r\geqslant 1$, isomorphisms of MHMs

$$\mathcal{H}^r_{\{0\}}\Big(\mathbb{D}\Big(\mathcal{H}^{n-d}_X\big(\mathbb{Q}^{\mathrm{H}}_n[n]\big)\Big)\Big)\cong\mathcal{H}^r_{\{0\}}(\mathrm{IC}_X).$$

By [13, 4.5.13], we have $\mathbb{D}\,\mathrm{IC}_X\cong\mathrm{IC}_X(d-2n)$, so we also have an exact sequence of MHMs

$$0\longrightarrow\mathrm{IC}_X(d-2n)\longrightarrow\mathcal{H}^{n-d}_X\big(\mathbb{Q}^{\mathrm{H}}_n[n]\big)\longrightarrow\mathbb{D}\,\mathcal{K}\longrightarrow 0,\tag{7}$$

and so, for $r\geqslant 2$,

$$\mathcal{H}^r_{\{0\}}\,\mathrm{IC}_X(d-2n)\cong\mathcal{H}^r_{\{0\}}\Big(\mathcal{H}^{n-d}_X\big(\mathbb{Q}^{\mathrm{H}}_n[n]\big)\Big).$$

It follows that, for $r\geqslant 2$, setting $s=d-n$ in (4),

$$\begin{aligned}\lambda^{p,q}_{r,d}(\mathscr{O}_{X,x}) &= h^{p,q}\Big(\mathcal{H}^r_{\{0\}}\Big(\mathcal{H}^{n-d}_X\big(\mathbb{Q}^{\mathrm{H}}_n[n](n)\big)\Big)\Big)\\ &= h^{p,q}\big(\mathcal{H}^r_{\{0\}}(\mathrm{IC}_X)(d-n)\big)\\ &= h^{p,q}\Big(\mathcal{H}^r_{\{0\}}\Big(\mathbb{D}\Big(\mathcal{H}^{n-d}_X\big(\mathbb{Q}^{\mathrm{H}}_n[n]\big)\Big)\Big)(d-n)\Big)\\ &= h^{p+d-n,q+d-n}\Big(\mathcal{H}^r_{\{0\}}\Big(\mathbb{D}\Big(\mathcal{H}^{n-d}_X\big(\mathbb{Q}^{\mathrm{H}}_n[n]\big)\Big)\Big)\Big)\\ &= h^{p+d,q+d}\big(H^{d+r}_{\{x\}}(X,\mathbb{C})\big).\end{aligned}$$

□

Remarks.

(1) The spectral sequence

$$E_2^{i,j}=\mathcal{H}^i_{\{0\}}\Big(\mathcal{H}^j_X\big(\mathscr{O}_{\mathbb{C}^n}\big)\Big)\Longrightarrow\mathcal{H}^{i+j}_{\{0\}}\big(\mathscr{O}_{\mathbb{C}^n}\big)$$

is a spectral sequence of MHMs with punctual support. Considering the Hodge–Euler characteristics

$$\chi^{p,q}(E_r)=\sum_{i,j}(-1)^{i+j}h^{p,q}(E_r^{i,j}),$$

and since $\chi^{p,q}(E_r)=\chi^{p,q}(E_{r+1})$, we get

$$\begin{aligned}&\sum_{i,j}(-1)^{i+j}\lambda^{p,q}_{i,n-j}(\mathscr{O}_{X,x})=0\quad\text{if }(p,q)\neq(0,0),\\ &\sum_{i,j}(-1)^{i+j}\lambda^{0,0}_{i,n-j}(\mathscr{O}_{X,x})=1\,.\end{aligned}$$

[4] In [13], Saito considers right $\mathscr{D}$-modules, we consider its left $\mathscr{D}$-module counterpart.

(2) When the singularity of X at x is isolated, the link is a compact manifold of real dimension $2d-1$, and its cohomology is endowed with a natural MHS. From Poincaré duality for the cohomology of the link it follows that, in this case,

$$\lambda_{0,i}^{p,q}(\mathscr{O}_{X,x}) = \lambda_{d-i+1,d}^{p,q}(\mathscr{O}_{X,x})$$

for $2 \leqslant i \leqslant d-1$, $p,q \in \mathbb{Z}$.

(3) Let $Y \hookrightarrow \mathbb{P}^N$ be a projective scheme such that ${}^{\mathrm{p}}\mathcal{H}^j(\mathbb{Q}_Y) = 0$ for $j \neq d$ where $d \in \mathbb{N} \smallsetminus \{0\}$ and ${}^{\mathrm{p}}\mathcal{H}$ denotes perverse cohomology (for example, this holds if Y^{an} is a $\mathbb{Q}$-homology manifold of pure dimension). If C is the affine cone over Y, it is not difficult to transpose the proof of [11, 1.6–1.8] to our setting,[5] and so it follows that the Hodge–Lyubeznik numbers $\lambda_{i,j}^{p,q}(\mathscr{O}_{C,0})$ corresponding to the local ring of C at its vertex are independent of the embedding of Y in projective space.

(4) If k is a field of characteristic $p > 0$ and X is a k-variety with an isolated singularity at $x \in X$, Blickle and Bondu proved in [1, Theorem 1.1] that the Lyubeznik numbers of the local ring $\mathscr{O}_{X,x}$ can be computed in terms of the dimensions over $\mathbb{Z}/p\mathbb{Z}$ of the étale cohomology of X supported at x with coefficients in $\mathbb{Z}/p\mathbb{Z}$. It would be interesting to find some refinement of the Lyubeznik numbers also in this setting.

Examples.

(1) Let $M \subset \mathbb{P}^{n-1}$ be a connected smooth projective variety of dimension $d-1 \geq 1$ and let $X \subset \mathbb{C}^n$ be the cone over M. The exact sequence of (topological) local cohomology

$$\cdots \longrightarrow H^{s-1}(X) \longrightarrow H^{s-1}(X \smallsetminus \{0\}) \longrightarrow H^s_{\{0\}}(X) \longrightarrow H^s(X) \longrightarrow \cdots$$

lifts as a sequence of MHS. Since X is contractible, it induces an isomorphism of mixed Hodge structures $H^{s-1}(X \smallsetminus \{0\}) \cong H^s_{\{0\}}(X)$ if $s \geq 2$, and $H^s_{\{0\}}(X) = 0$ if $s \leq 1$ since M is connected. Furthermore, denoting by L an ample class, the Thom–Gysin sequence in [11, Proposition 1.3], which is a sequence of mixed Hodge structures, yields for any $s \geq 2$ an exact sequence

$$H^{s-3}(M) \xrightarrow{L} H^{s-1}(M)(1) \longrightarrow H^s_{\{0\}}(X) \longrightarrow H^{s-2}(M) \xrightarrow{L} H^s(M)(1).$$

By the hard Lefschetz theorem, we identify $H^s_{\{0\}}(X)$ with

(a) the Tate twisted L-primitive part $H^{s-1}(M)_{\mathrm{prim}}(1)$, which is pure of weight $s-3$, if $s \leq d$ (and $s \geq 2$),

(b) and, if $s \geq d+1$, i.e., $s = d+r$ with $r \geq 1$, the L-coprimitive part $H^{s-2}(M)_{\mathrm{coprim}} = \ker L \colon H^{s-2}(M) \to H^s(M)(1)$, which is pure of weight $s-2$.

The main theorem yields then:

(i) $\lambda_{0,s}^{p,q}(\mathscr{O}_{X,0}) = h^{-p+1,-q+1}\big(H^{s-1}(M)_{\mathrm{prim}}\big)$ for $2 \leq s \leq d-1$,

(ii) $\lambda_{r,d}^{p,q}(\mathscr{O}_{X,0}) = h^{p+d,q+d}\big(H^{d+r-2}(M)_{\mathrm{coprim}}\big)$ for $2 \leq r \leq d$,

(iii) all other $\lambda_{r,s}^{p,q}(\mathscr{O}_{X,0})$ vanish.

(2) It is known that there exist singularities whose links have equal Betti numbers but different Hodge numbers (see e.g. [17, Section 3]). In the case of an isolated singularity, Lyubeznik numbers are not enough to recover all Betti numbers of the link, therefore not all Hodge numbers of the link can be obtained as Hodge–Lyubeznik numbers. Nevertheless, it is not difficult to give examples of singularities with equal Lyubeznik numbers and distinct Hodge–Lyubeznik numbers.

In [19], the authors find a pair of three-dimensional smooth projective complete intersections N_1 and N_2 in $\mathbb{P}^9$ which are diffeomorphic but $h^{0,3}(N_1) \neq h^{0,3}(N_2)$ and $h^{1,2}(N_1) \neq h^{1,2}(N_2)$.[6]

[5] For the Thom–Gysin sequence, this is remarked at the end of Section 1.3 in [11].

[6] In fact, they find several pairs with this property.

For $i = 1,2$, let $M_i \subset \mathbb{P}^{19}$ be the image of $N_i \times \mathbb{P}^1$ by the Segre embedding, let $X_i \subset \mathbb{A}^{20}$ be the affine cone over M_i, and let A_i denote the local ring at the origin of X_i.

Since, for $i = 1,2$, the singularity of X_i at zero is isolated, the Lyubeznik numbers of A_i depend only on the Betti numbers of $X_i \smallsetminus \{0\}$ (see [4]). Topologically, $X_i \smallsetminus \{0\}$ is a $\mathbb{S}^1$-bundle over $M_i \cong N_i \times \mathbb{P}^1$. It follows that $\lambda_{p,\ell}(A_1) = \lambda_{p,\ell}(A_2)$ for all $p, \ell \in \mathbb{N}$.

On the other hand, we apply the results of Example (1) with $d = 5$ and $s = d - 1 = 4$. Thus

$$\lambda_{0,4}^{-p,-q}(A_i) = h^{p+1,q+1}\big(H^3(M_i)_{\mathrm{prim}}\big) = h^{p+1,q+1}\big(H^3(M_i)\big) - h^{p,q}\big(H^1(M_i)\big).$$

But we have $h^{p,q}\big(H^\ell(M_i)\big) = h^{p,q}\big(H^\ell(N_i \times \mathbb{P}^1)\big) = h^{p,q}\big(H^\ell(N_i)\big) + h^{p-1,q-1}\big(H^{\ell-2}(N_i)\big)$, and so, $\lambda_{0,4}^{-p,-q}(A_i) = h^{p+1,q+1}\big(H^3(N_i)\big)$. In particular, if $N_1 = X_3(70,16,16,14,7,6)$ we obtain, according to [19],

$$\lambda_{0,4}^{0,-1}(A_1) = h^{1,2}\big(H^3(N_1)\big) = 3365330286081,$$

while if $N_2 = X_3(56,49,8,6,5,4,4)$,

$$\lambda_{0,4}^{0,-1}(A_2) = h^{1,2}\big(H^3(N_2)\big) = 3343868254721.$$

Remarks.

(1) In [5, Theorem 6.4], which takes place in the setting of Example (1) above, the authors show that the $\mathscr{D}_{\mathbb{C}^n}$-module $\mathcal{H}_X^{n-d}(\mathscr{O}_{\mathbb{C}^n})$ has a maximal simple submodule with support X (this is the minimal extension of its restriction to $\mathbb{C}^n \smallsetminus \{0\}$), and that the quotient is supported at the origin with multiplicity equal to $\dim H^{d-1}(M)_{\mathrm{prim}}$. We can recover this result, and upgrade it to an equality between Hodge numbers as follows. The first point comes from the exact sequence (7). For the second point, we write the mixed Hodge module $\mathcal{Q}$ corresponding to this quotient as $\mathcal{Q} \simeq \mathbb{D}\mathcal{K}$, with $\mathcal{K}$ defined by (6). Then (5) with $s = d - n$ and $r = 0$, together with the previous Example (1)-(a) with $s = d$, yields

$$h^{p,q}\mathcal{K}(-n) = h^{p,q}\big(H^d_{\{0\}}(X)\big) = h^{p+1,q+1}\big(H^{d-1}(M)_{\mathrm{prim}}\big),$$

so that $\mathcal{K}(-n)$ is pure of weight $d-3$. It follows that $\mathcal{Q}(n)$ is pure of weight $3-d$ and $h^{p,q}\mathcal{Q}(n) = h^{-p+1,-q+1}\big(H^{d-1}(M)_{\mathrm{prim}}\big)$.

(2) The calculation above does not make use of any special property of the varieties in [19], the proof can be transposed to other examples of homeomorphic complex projective manifolds with different Hodge numbers (according to [19] there are examples obtained by Xiao and Campana in the eighties, but we didn't get access to their papers).

(3) It is proved in [20] that if X is a projective variety over a field of positive characteristic, the Lyubeznik numbers of the local ring at the vertex of the affine cone over X are independent of the embedding $X \hookrightarrow \mathbb{P}^n$. In contrast, the examples in [11] (see also irreducible examples in [18]) show that this does not hold for varieties over the complex numbers. It would be interesting to determine, in these examples, which Hodge–Lyubeznik numbers are involved in the dependence on the embedding $X \hookrightarrow \mathbb{P}^n$.

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Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

References

[1] M. Blickle and R. Bondu, "Local cohomology multiplicities in terms of étale cohomology", *Ann. Inst. Fourier* **55** (2005), no. 7, pp. 2239–2256.

[2] J.-L. Brylinski and M. Kashiwara, "Kazhdan–Lusztig conjecture and holonomic systems", *Invent. Math.* **64** (1981), no. 3, pp. 387–410.

[3] P. Deligne, "Théorie de Hodge. III", *Publ. Math., Inst. Hautes Étud. Sci.* **44** (1974), pp. 5–77.

[4] R. García López and C. Sabbah, "Topological computation of local cohomology multiplicities", *Collect. Math.* **49** (1998), no. 2-3, pp. 317–324.

[5] R. Hartshorne and C. Polini, "Simple $\mathcal{D}$-module components of local cohomology modules", *J. Algebra* **571** (2021), pp. 232–257.

[6] R. Hotta, K. Takeuchi and T. Tanisaki, *$\mathscr{D}$-modules, perverse sheaves, and representation theory*, Progress in Mathematics, Birkhäuser, 2008, pp. xii+407.

[7] G. Lyubeznik, "Finiteness properties of local cohomology modules (an application of D-modules to commutative algebra)", *Invent. Math.* **113** (1993), no. 1, pp. 41–55.

[8] M. Mustaţă and M. Popa, "Hodge filtration on local cohomology, Du Bois complex and local cohomological dimension", *Forum Math. Pi* **10** (2022), article no. e22 (58 pages).

[9] L. Núñez-Betancourt, E. E. Witt and W. Zhang, "A survey on the Lyubeznik numbers", in *Mexican mathematicians abroad: recent contributions*, Contemporary Mathematics, American Mathematical Society, 2016, pp. 137–163.

[10] C. A. M. Peters and J. H. M. Steenbrink, *Mixed Hodge structures*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, Springer, 2008, pp. xiv+470.

[11] T. Reichelt, M. Saito and U. Walther, "Dependence of Lyubeznik numbers of cones of projective schemes on projective embeddings", *Sel. Math., New Ser.* **27** (2021), no. 1, article no. 6 (22 pages).

[12] M. Saito, "Modules de Hodge polarisables", *Publ. Res. Inst. Math. Sci.* **24** (1988), no. 6, pp. 849–995.

[13] M. Saito, "Mixed Hodge modules", *Publ. Res. Inst. Math. Sci.* **26** (1990), no. 2, pp. 221–333.

[14] M. Saito, "Mixed Hodge complexes on algebraic varieties", *Math. Ann.* **316** (2000), no. 2, pp. 283–331.

[15] C. Schnell, "An overview of Morihiko Saito's theory of mixed Hodge modules", in *Representation theory, automorphic forms & complex geometry*, International Press, 2019, pp. 27–80.

[16] V. Srinivas, "On the embedding dimension of an affine variety", *Math. Ann.* **289** (1991), no. 1, pp. 125–132.

[17] J. H. M. Steenbrink and J. Stevens, "Topological invariance of the weight filtration", *Indag. Math., New Ser.* **46** (1984), no. 1, pp. 63–76.

[18] B. Wang, "Lyubeznik numbers of irreducible projective varieties depend on the embedding", *Proc. Am. Math. Soc.* **148** (2020), no. 5, pp. 2091–2096.

[19] J. Wang, Z. Yu and Y. Wang, "Examples of diffeomorphic complete intersections with different Hodge numbers", *Topology Appl.* **339** (2023), article no. 108690 (6 pages).

[20] W. Zhang, "Lyubeznik numbers of projective schemes", *Adv. Math.* **228** (2011), no. 1, pp. 575–616.