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A remark on the effective nonvanishing of Calabi–Yau varieties

Une remarque sur la non-annulation effective des variétés de Calabi–Yau

Yiming Zhu [a]

[a] Department of Mathematics, Southern University of Science and Technology, 1088 Xueyuan Rd, Shenzhen 518055, China
E-mail: 12131245@mail.sustech.edu.cn

Abstract. For a Calabi–Yau variety X, we prove that there exists a positive integer m, depending on two natural invariants of the fiber of its Albanese map, such that the pluricanonical system $|mK_X|$ is non-empty.

Résumé. Pour une variété de Calabi–Yau X, nous démontrons qu'il existe un entier strictement positif m, dépendant de deux invariants naturels de la fibre de son application d'Albanese, tel que le système pluricanonique $|mK_X|$ soit non vide.

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We work with varieties defined over $\mathbb{C}$.

1. Introduction

In their work [2], Birkar and Zhang proved that for any smooth projective variety X of dimension d with nonnegative Kodaira dimension, there exists a universal constant m that depends on d and two natural invariants of the general fiber of the Iitaka fibration of X such that the pluricanonical system $|mK_X|$ defines the Iitaka fibration.

In this note, we investigate the case of Kodaira dimension zero. We consider minimal models of Kodaira dimension zero, i.e., projective varieties with terminal singularities and numerically trivial canonical divisor (also called $\mathbb{Q}$-Calabi–Yau varieties).

Let X be a $\mathbb{Q}$-Calabi–Yau variety. By [6, Theorem 8.2], $\mathscr{O}_X(mK_X) \cong \mathscr{O}_X$ for some $m \in \mathbb{N}$, and the global index I_X is defined as the minimum of such m. The canonical cover of X is defined as the quasi-étale cover $\widetilde{X} := \operatorname{Spec}\bigoplus_{i=0}^{I_X-1}\mathscr{O}_X(-iK_X) \to X$. By [6, Theorem 8.3], the Albanese map $f\colon X \to A$ of X is an étale fiber bundle. Let F be the fiber of f. Since F has terminal singularities and $K_F = K_X\big|_F$ is numerically trivial, F is also a $\mathbb{Q}$-Calabi–Yau variety. Let $\widehat{F}$ be a smooth model of the canonical cover $\widetilde{F}$ of F, and let $\beta_{\hat{F}} := \dim H^{\dim F}(\widehat{F}, \mathbb{C})$ be the middle Betti number of $\widehat{F}$. Set $m(I_F, \beta_{\hat{F}}) = I_F \cdot \operatorname{lcm}\{m \in \mathbb{N} \mid \varphi(m) \le \beta_{\hat{F}}\}$, where φ is Euler's totient function. We show the following.

Theorem 1. *The pluricanonical system* $\left|m(I_F,\beta_{\hat{F}})K_X\right|$ *is nonempty.*

Note that $m(I_F,\beta_{\hat{F}})$ is solely dependent on F. We have the following.

Corollary 2. *Let* X *be a smooth projective variety of Kodaira dimension zero and* $q(X) = \dim H^1(X,\mathcal{O}_X)$ *be its irregularity.*

(1) *If* $q(X) = \dim(X)-1$, *then* $|12K_X|$ *is non-empty.*
(2) *If* $q(X) = \dim(X)-2$, *then* $|mK_X|$ *is non-empty, where* $m = 2\cdot \mathrm{lcm}\{m\in\mathbb{N} \mid \varphi(m)\le 22\} = 2^6\cdot 3^3\cdot 5^2\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23$.

The existence of such a universal bound for $\mathbb{Q}$-Calabi–Yau 3-folds was initially shown by Kawamata [7, Theorem 3.2]. Using Bogomolov–Beauville decomposition, Beauville showed that for a Calabi–Yau 3-fold X, $|BK_X|$ is nonempty, where $B = \mathrm{lcm}\{m\in\mathbb{N} \mid \varphi(m)\le 20\} = 2^5\cdot 3^3\cdot 5^2\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19$ [1, Proposition 8]. Morrison showed that the same number works in the terminal singular case [9, Corollary]. The proofs of Kawamata and Morrison both rely on analyzing the terminal 3-fold singularities.

2. Preliminaries

2.1. *Boundedness of pluricanonical representation*

Let X be a projective variety with terminal singularities. The pluricanonical representation of the birational automorphism group of X is denoted as $\rho_m\colon \mathrm{Bir}(X)\to \mathrm{Aut}\, H^0(mK_X)$. We adopt the convention $\rho_m(g) := (g^{-1})^*$ so that $\rho_m(g)\cdot\rho_m(h) = \rho_m(g\circ h)$.

Theorem 3 ([11, Theorem 14.10] and [4, Corollary 3.10]). *For a* $\mathbb{Q}$*-Calabi–Yau variety* X*, let* $d = \dim X$ *and* $m = I_X$ *its canonical index, then*

$$\varphi\big(|\mathrm{Im}\,\rho_m|\big) \le \dim H^d(\widehat{X},\mathbb{C}),$$

where φ *is Euler's totient function and* $\widehat{X}$ *is a smooth model of the canonical cover* $\widetilde{X}$ *of* X.

Proof. By [4, Corollary 3.10], $\mathrm{Im}\,\rho_m$ is a finite subgroup of $\mathrm{Aut}\, H^0\big(X,\mathcal{O}_X(mK_X)\big) = \mathbb{C}^*$, hence is cyclic. Choose $g\in \mathrm{Bir}(X)$ and $0\ne\omega\in H^0\big(X,\mathcal{O}_X(mK_X)\big)$. We have $g^{-1*}\omega = \alpha\omega$ and α is a root of unity. On the canonical cover $\pi\colon \widetilde{X}\to X$, we have $\pi^*\omega = \widetilde{\omega}^m$, where $\widetilde{\omega}$ is a holomorphic d-form on $\widetilde{X}$, and g induces a $\widetilde{g}\in\mathrm{Bir}(\widetilde{X})$ such that $\widetilde{g}^{-1*}\pi^*\omega = \pi^*g^{-1*}\omega$. Hence $\widetilde{g}^{-1*}\widetilde{\omega} = \alpha^{1/m}\widetilde{\omega}$. Since $\widetilde{X}$ has terminal singularities, $\widetilde{\omega}$ can be viewed as a holomorphic d-form $\widehat{\omega}$ on $\widehat{X}$, and $\widetilde{g}$ induces a $\widehat{g}\in\mathrm{Bir}(\widehat{X})$ such that $\widehat{g}^{-1*}\widehat{\omega} = \alpha^{1/m}\widehat{\omega}$. By the same argument of [11, first paragraph of the proof of Proposition 14.4], we have $\big[\mathbb{Q}(\alpha^{1/m}):\mathbb{Q}\big]\le \dim H^d(\widehat{X},\mathbb{Q})$, and thus $\varphi\big(|\mathrm{Im}\,\rho_m|\big)\le \dim H^d(\widehat{X},\mathbb{Q})$. □

2.2. *Fiber bundles*

Let F be a projective variety, and let $f\colon X\to Y$ be an F-fiber bundle defined by a cocycle $\phi_{ij}\colon Y_{ij}\to \mathrm{Aut}(F)$, where $\{Y_i\}$ is an open covering of Y with $Y_{ij} := Y_i\cap Y_j$. Assume that X is terminal and Y is smooth. Take m to be a positive integer such that mK_X is Cartier, then $f_*\mathcal{O}_X(mK_{X/Y})$ is a rank $h^0(mK_F) := \dim H^0\big(F,\mathcal{O}_F(mK_F)\big)$ vector bundle. The following Lemma describes this bundle in terms of ϕ_{ij}.

Lemma 4. *The vector bundle* $f_*\mathcal{O}_X(mK_{X/Y})$ *is defined by the cocycle* $\rho_m\circ\phi_{ij}$*, where* $\rho_m\colon \mathrm{Aut}(F)\to \mathrm{Aut}\, H^0\big(F,\mathcal{O}_F(mK_F)\big)$ *is the pluricanonical representation of the automorphism group of* F.

Proof. See also [11, p. 186] for a proof. We have isomorphisms $\alpha_i\colon X_i := f^{-1}(Y_i) \cong Y_i \times F$ over Y_i, where $\alpha_i \circ \alpha_j^{-1} = (\mathrm{id}, \phi_{ij})$. By taking a set of basis $\{\omega_k\}_{k=1,\dots,h^0(mK_F)}$ of $H^0\big(F, \mathscr{O}_F(mK_F)\big)$, we can write each $\rho_m(\phi_{ij})$ as a matrix: $\rho_m(\phi_{ij})\omega_k = \phi_{ji}^*\omega_k = \sum_l \omega_l(\Phi_{ij})_{lk}$, where $(\Phi_{ij})_{lk}$ is a matrix indexed by lk. Let s be a local section of $f(mK_{X/Y})$. Under the isomorphisms α_i, s can be identified on each Y_i with $s_i = \sum_k t_{ik} \otimes \omega_k$ satisfying $s_j = (\mathrm{id}, \phi_{ij})^* s_i$. Where $t_{ik} \in H^0(Y_i, \mathscr{O}_{Y_i})$. Consequently, we find that $\sum_k t_{jk} \otimes \omega_k = \sum_{l,k} t_{ik} \otimes \omega_l (\Phi_{ji})_{lk}$ and thus $t_{ik} = \sum_l (\Phi_{ij})_{kl} t_{jl}$. □

We say an F-fiber bundle $f\colon X \to Y$ is locally constant if there exists a representation $\gamma\colon \pi_1(Y) \to \mathrm{Aut}(F)$, such that $X \cong \widetilde{Y} \times F/\pi_1(Y)$ where $\widetilde{Y}$ is the universal cover of Y, and $\pi_1(Y)$ acts on $\widetilde{Y} \times F$ diagonally. Viewing $\widetilde{Y} \to Y$ as a principal $\pi_1(Y)$-bundle defined by a cocycle u_{ij}, then the fiber bundle $X \to Y$ is defined by the cocycle $\gamma(u_{ij})$, and the vector bundle $f_*\mathscr{O}_X(mK_{X/Y})$ is defined by the cocycle $\rho_m \circ \gamma(u_{ij})$ by Lemma 4. In other words, $f_*\mathscr{O}_X(mK_{X/Y})$ is a local system defined by the representation $\rho_m \circ \gamma$.

2.3. *Structure of Albanese maps*

A morphism between two normal varieties will be called a fibration if it is surjective and has connected fibers. Let X be a projective variety with terminal singularities. The irregularity of X is defined as $q(X) = \dim H^1(X, \mathscr{O}_X)$. Take Y to be any smooth model of X. Since X is terminal, it has rational singularities. Hence $q(X) = q(Y)$. Let $Y \to A$ be the Albanese map of Y, and the Albanese map of X is defined as the rational map $X \dashrightarrow A$ which is a morphism by [10, Proposition 2.3]. If $\kappa(X) = 0$, then $\kappa(Y) = 0$. Then $Y \to A$ is a fibration by [5, Theorem 1]. Hence $X \to A$ is also a fibration.

Now, let X be a $\mathbb{Q}$-Calabi–Yau variety. Kawamata showed that its Albanese map $f\colon X \to A$ is an étale fiber bundle [6, Theorem 8.3]. Cao and later Wang showed more in the smooth and singular case respectively.

Theorem 5 ([3, Theorem 1.2] and [12, Theorem B]). *The fiber bundle $f\colon X \to A$ is locally constant.*

Cao and Wang in fact proved this theorem under a much weaker hypothesis: $-K_X$ and $-(K_X+\Delta)$ (for klt pairs (X,Δ)) is nef instead of numerically trivial. Since the fiber bundle $f\colon X \to A$ is locally constant, it is defined by a representation $\gamma\colon \pi_1(A) \to \mathrm{Aut}(F)$. By Lemma 2.2, we have the following.

Lemma 6. *The vector bundle $f_*\mathscr{O}_X(mK_X)$ is a local system defined by the representation $\rho_m \circ \gamma$.*

3. Proof of Theorem 1 and Corollary 2

Proof of Theorem 1. Let $f\colon X \to A$ be the Albanese map of X. It is a locally constant fiber bundle by Theorem 5. Let F be the fiber of f. Let i_X and i_F be the Cartier index of X and F respectively. We claim that $i_X = i_F$. Since $i_X K_X$ is Cartier, $i_X K_F = i_X K_X|_F$ is Cartier. Hence $i_F \mid i_X$. On the other hand, let $\{A_i\}$ be an open covering of A such that $X_i := f^{-1}A_i \cong A_i \times F$. Since $i_F K_X|_{X_i}$ is Cartier for each X_i, we have $i_F K_X$ is Cartier. Thus $i_X \mid i_F$, and $i_X = i_F$ is proved.

The Cartier index i_F divides the global index I_F, hence $\mathscr{O}_X(I_F K_X)$ is a line bundle. By Theorem 3 and Lemma 6, the line bundle $f_*\mathscr{O}_X(I_F K_X)$ is a local system defined by the representation $\rho_{I_F} \circ \gamma\colon \pi_1(A) \to \mathrm{Aut}\, H^0\big(F, \mathscr{O}_F(I_F K_F)\big)$, and the kernel K of this map is of finite index such that $\varphi\big([\pi_1(A) : K]\big) \le \beta_{\hat{F}}$. This induces a finite étale cover $q\colon A_1 \to A$ with $\varphi(\deg q) \le \beta_{\hat{F}}$, and the pullback of $f_*\mathscr{O}_X(I_F K_X)$ through this cover is trivial. Let $X_1 = A_1 \times_A X$ be the fiber product, and let $f_1\colon X_1 \to X$, $p\colon X_1 \to X$ be the induced morphisms as in the following commutative diagram.

$$\begin{array}{ccc} X_1 & \xrightarrow{p} & X \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f} \\ A_1 & \xrightarrow{q} & A \end{array}$$

Therefore we have:

$$q^* f_* \mathcal{O}_X(I_F K_X) = \mathcal{O}_{A_1} \otimes H^0\big(F, \mathcal{O}_F(I_F K_F)\big). \tag{1}$$

On X_1, we have

$$\begin{aligned} H^0\big(X_1, \mathcal{O}_X(I_F K_{X_1})\big) &= H^0\big(X_1, p^*\mathcal{O}_X(I_F K_X)\big) \\ &= H^0\big(A_1, f_{1*} p^* \mathcal{O}_X(I_F K_X)\big) \\ &= H^0\big(A_1, q^* f_* \mathcal{O}_X(I_F K_X)\big) \\ &= H^0(A_1, \mathcal{O}_{A_1}) \otimes H^0\big(F, \mathcal{O}_F(I_F K_F)\big) \\ &= \mathbb{C}. \end{aligned}$$

The first equality holds as $f\colon X_1 \to X$ is finite étale and thus $K_{X_1} = p^* K_X$. The third equality follows from flat base change. The fourth equality is a consequence of (1). Thus, $I_F K_{X_1} \sim 0$. Since $X_1 \to X$ is étale with the same degree as q, $p_*\mathcal{O}_{X_1}$ is locally free of rank $m := \deg q$. By the projection formula, we have $\wedge^m p_*\mathcal{O}_{X_1} = \wedge^m p_*\mathcal{O}_{X_1}(I_F K_{X_1}) = \wedge^m p_*\big(p^*\mathcal{O}_X(I_F K_X)\big) = \wedge^m p_*\mathcal{O}_{X_1} \otimes \mathcal{O}_X(m I_F K_X)$. Since $\wedge^m p_*\mathcal{O}_{X_1}$ is a line bundle, we conclude that $m I_F K_X \sim 0$. □

Proof Corollary 2. Let $f\colon X \to A$ be the Albanese map of X. By running a K_X-MMP over A, we have a minimal model X' of X over A which is actually minimal over $\mathbb{C}$ by [8, second paragraph of the proof of Theorem 4.2]. By Theorem 5, $X' \to A$ is a locally constant fiber bundle, hence the singular locus of X' is of dim $\geq \dim A$. But X' has at most terminal singularities whose singular locus is of codim ≥ 3, hence the fiber F' of $X' \to A$ and X' are both smooth. We may assume from the beginning that X is already minimal, i.e., that X is a smooth $\mathbb{Q}$-Calabi–Yau variety. In the following, let F be the fiber of the Albanese map of X and $m := m(I_F, \beta_{\hat{F}})$.

If $\dim F = 1$, then F is an elliptic curve, $m = 12$.

If $\dim F = 2$:

- if F is a K3 surface, $m = \operatorname{lcm}\{m \in \mathbb{N} \mid \varphi(m) \leq 22\}$;
- if F is an Enriques surface, its canonical cover is a K3 surface, $m = 2 \cdot \operatorname{lcm}\{m \in \mathbb{N} \mid \varphi(m) \leq 22\}$;
- if F is an Abelian surface, $m = \operatorname{lcm}\{m \in \mathbb{N} \mid \varphi(m) \leq 6\}$;
- if F is a bielliptic surface, its canonical cover is an Abelian surface, $m = 2, 3, 4$ or $6 \cdot \operatorname{lcm}\{m \in \mathbb{N} \mid \varphi(m) \leq 6\}$.

Then we conclude by Theorem 1. □

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Declaration of interests

The author does not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and has declared no affiliations other than their research organizations.

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