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Mathématique


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Volume 363 (2025), p. 593-598

Online since: 5 June 2025

<https://doi.org/10.5802/crmath.741>

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www.centre-mersenne.org — e-ISSN : 1778-3569



Research article / *Article de recherche*
Algebraic geometry / *Géométrie algébrique*

The cuspidal cubic and line arrangements with only triple points

La cubique cuspidale et les arrangements de droites avec seulement des points triples

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Abstract. We describe a new infinite family of line arrangements in the projective plane with only triple point singularities, and we recover previously known examples.

Résumé. Nous décrivons une nouvelle famille infinie d'arrangements de droites dans le plan projectif ne présentant que des singularités triples, et nous retrouvons des exemples déjà connus.

2020 Mathematics Subject Classification. 05B35, 14N20, 52C30.

Manuscript received 30 December 2024, revised 11 March 2025, accepted 14 March 2025.

1. Introduction

Line arrangements, i.e. finite unions of lines in the projective plane, are studied in various domains: algebra, topology, combinatorics... A *double point* (respectively *triple point*) on a line arrangement is a point where exactly 2 (respectively 3) lines of the arrangement meet. The celebrated Sylvester–Gallai Theorem asserts that any line arrangement over the real field has at least one double point, or is the trivial example: the union of lines passing through the same point.

Hirzebruch [2] then proved that over the complex field, a line arrangement is either the trivial example or possesses double or triple points. To date, there exists a unique non-trivial example of a complex line arrangement with only triple points. Even before that result, the construction of line arrangements with many or exclusively triple points had attracted attention, see e.g. [3] for a historical account.

In this paper, we present a new construction of line arrangements over finite fields that exhibit only triple points. Line arrangements with only triple points are also interesting from a combinatorial perspective, as they are linked to Steiner triple systems. A *Steiner triple system* is a set T of 3-element subsets of a finite set, such that any 2-element subset is contained in a unique 3-element subset $t \in T$.

Before presenting our results, let us introduce some notations and recall some properties of line arrangements. If $\mathcal{L} = \ell_1 + \dots + \ell_n$ is a line arrangement labeled by $\{1, \dots, n\}$, the data $M(\mathcal{L})$

of the triples $\{i, j, k\} \subset \{1, \dots, n\}$ such that the lines ℓ_i, ℓ_j and ℓ_k meet at a common point is called the *matroid* associated with \mathcal{L} ; it encodes the combinatorics of \mathcal{L} . If \mathcal{L} has only triple points, then $M(\mathcal{L})$ is a Steiner triple system.

If γ is a projective transformation of the plane, one may define $\gamma\mathcal{L} = \gamma\ell_1 + \dots + \gamma\ell_n$. The line arrangements $\mathcal{L}, \gamma\mathcal{L}$ have the same combinatorics, meaning that $M(\mathcal{L}) = M(\gamma\mathcal{L})$. Given a matroid M , a labelled line arrangement \mathcal{L} such that $M(\mathcal{L}) = M$ is said to be a *realization* of M . Given a field K , there exists a scheme parametrizing the realizations over K , and the quotient $\mathcal{R}(M)_{/K}$ of these realizations modulo the action of $\text{PGL}_3(K)$ is called the *moduli space* of M . A line arrangement over a field K is said to be *rigid* if $\mathcal{R}(M)_{/K}$ is zero-dimensional.

In the present paper, we describe a new infinite family of line arrangements in the plane with only triple points, and study some of their associated moduli spaces:

Theorem 1. *For $q = 3^n, n \geq 1$, there exists a line arrangement \mathcal{L}_q of q lines over \mathbb{F}_q with $\frac{q(q-1)}{6}$ triple points and no other singularities.*

Let $M_q = M(\mathcal{L}_q)$ be the matroid associated to \mathcal{L}_q . For $n \geq 3$, the matroid M_q has no realizations over any field of characteristic $\neq 3$.

The moduli space $\mathcal{R}(M_{27})_{/\overline{\mathbb{F}}_3}$ is 3-dimensional with a 2-dimensional immersed component of multiplicity 2.

One obtains a model of the reduced scheme of $\mathcal{R}(M_{27})_{/\overline{\mathbb{F}}_3}$ as an open sub-scheme of a quadric in \mathbb{P}^4 .

Our first example of a line arrangement \mathcal{L}_q was obtained as the dual of the q points defined over \mathbb{F}_q on the smooth part of the cuspidal cubic $y^2z = x^3$. However the generic realization of M_{27} is such that the 27 points of its dual are not contained in any plane cubic.

The previously known line arrangements with only triple points are:

- (a) The trivial example: three lines meeting at one point.
- (b) Over a field containing a primitive third root of unity (in particular over any algebraically closed field of characteristic $\neq 3$): the Ceva(3) line arrangement with 9 lines, given by

$$(x^3 - y^3)(x^3 - z^3)(z^3 - y^3) = 0.$$

It has 12 triple points and it is the dual of the Hesse configuration, i.e. of the 9 flex points of a smooth cubic. In characteristic 3, there is also a line arrangement of 9 lines having the same associated matroid as Ceva(3). This is the 9 lines in $\mathbb{P}^2(\mathbb{F}_3)$ not containing a given point p in $\mathbb{P}^2(\mathbb{F}_3)$. These line arrangements are rigid.

- (c) The Fano plane: the 7 lines in $\mathbb{P}^2(\mathbb{F}_2)$. It is also a rigid line arrangement.
- (d) The sequence of line arrangements \mathcal{C}'_{2^n-1} : for $n \geq 4$, the line arrangement \mathcal{C}'_{2^n-1} consists of $2^n - 1$ lines defined over $\overline{\mathbb{F}}_2$. It is constructed in [3] and obtained by taking the dual of a generic projection to \mathbb{P}^2 of the $2^n - 1$ points $\mathbb{P}^{n-1}(\mathbb{F}_2)$ contained in \mathbb{P}^{n-1} .
- (e) A rigid example over \mathbb{F}_{11} : a rigid line arrangement with 19 lines and 57 triple points, constructed in [3].

It is an open question [5] whether or not (a) and (b) are the unique examples of complex line arrangements with only triple points.

For any $n \geq 2$, we also obtain line arrangements \mathcal{C}_{2^n-1} over $\overline{\mathbb{F}}_2$ with $2^n - 1$ lines and only triple points. These are constructed by using the dual of the non-zero points in $\mathbb{A}^1(\mathbb{F}_{2^n})$ in the smooth part of the cuspidal cubic curve (which is isomorphic to \mathbb{A}^1).

For $n \geq 4$, we obtain that our line arrangement \mathcal{C}_{2^n-1} and the line arrangement \mathcal{C}'_{2^n-1} of example (d) define isomorphic matroids. Thus \mathcal{C}_{2^n-1} is another construction of \mathcal{C}'_{2^n-1} .

Our construction of \mathcal{C}_n generalizes a bit the one of \mathcal{C}'_n since it holds for $n = 2$ and 3. The line arrangement \mathcal{C}_3 is the trivial example, and \mathcal{C}_7 is the Fano plane (example (c)). Regarding the line arrangement \mathcal{L}_n : for $n = 3$, it is the trivial example, and \mathcal{L}_9 corresponds to example (b). We

checked that the 19 points in the dual of example (e) are not contained in a cubic curve. This example is therefore the only one known that does not appear to be related to cubic curves.

2. Line arrangements using the cuspidal cubic

2.1. The cuspidal cubic

Let $C \hookrightarrow \mathbb{P}^2$ be the cuspidal cubic over a field K , with a flex point O defined over K . Let C^* be the complement of the singular point. As for a smooth cubic curve, one can define a composition law (denoted by $+$) on C^* by chords and tangents, so that O is the neutral element.

Theorem 2 ([4, Chapter III, Proposition 2.5]). *The composition law $+$ makes $C^*(K)$ an abelian group isomorphic to $(K, +)$. Three points $a, b, c \in C^*(K)$ (identified with elements of K) are on a line if and only if*

$$a + b + c = 0 \quad \text{in } K.$$

In particular the tangent to a point a in C^* cuts C^* at a residual point b such that $2a + b = 0$. As a concrete example, one may consider the curve $C^* : \{x^3 - zy^2 = 0\} \hookrightarrow \mathbb{P}^2$ with flex point $(0 : 1 : 0)$ and the parametrization $t \rightarrow (t : 1 : t^3)$.

Now, suppose that K is a finite field of order $\#K = q = p^n$, where p is the characteristic of K . From Theorem 2, the lines containing three distinct points of K correspond bijectively to the sets of the form

$$S_{a,b} = \{a, b, -(a+b)\}$$

that have cardinality 3. One has $|S_{a,b}| = 3$ if and only if

$$a \neq b \quad \text{and} \quad a \neq -(a+b) \quad \text{and} \quad b \neq -(a+b)$$

which is equivalent to condition

$$a \neq b \quad \text{and} \quad b \neq -2a \quad \text{and} \quad a \neq -2b. \tag{1}$$

2.2. Characteristic 3

2.2.1. General construction

Suppose that $\text{Char}(K) = 3$. Let $a, b \in K$ and $S_{a,b} = \{a, b, -(a+b)\}$. Since $-2 = 1$ in K , the conditions in (1) for $\#S_{a,b} = 3$ reduce to $a \neq b$. Therefore the number of lines containing 3 points is $\frac{1}{3} \binom{q}{2} = \frac{q(q-1)}{6}$, and through each point in K , there are $\frac{1}{2}(q-1)$ such lines. This leads to a configuration

$$\left(q \frac{1}{2}(q-1), \left(\frac{q(q-1)}{6} \right)_3 \right)$$

of points and lines. Let \mathcal{L}_q denote the dual of the q points of K .

Theorem 3. *The line arrangement \mathcal{L}_q of q lines has $\frac{q(q-1)}{6}$ triple points and no other singularities.*

Proof. The dual of the lines containing three points of K are triple points of the line arrangement \mathcal{L}_q . Thus \mathcal{L}_q has $\frac{q(q-1)}{6}$ triple points. A line arrangement with q lines can have at most $\frac{1}{3} \frac{q(q-1)}{2}$ triple points, and if that bound is reached there are no other singularities. \square

We note that every point a on C^* is a flex point: for any a one has $3a = 0$. Geometrically, the intersection of the tangent line at a with the cubic curve is three times the point a .

Remark 4. As a group, or \mathbb{F}_p -vector space, \mathbb{F}_q is isomorphic to $(\mathbb{F}_p)^n$, where $q = p^n$. Since the relation $a + b + c = 0$ is preserved under such an isomorphism, the group $\text{GL}_n(\mathbb{F}_p)$ induces symmetries on the matroid M_q associated to \mathcal{L}_q . For $p = 3$, since for any element $t \in \mathbb{F}_q$, we have $(a + t) + (b + t) + (c + t) = a + b + c$, the relation $a + b + c = 0$ is preserved by translation. Thus, the automorphism group of M_{3^n} contains the general affine group of $\mathbb{A}_{/\mathbb{F}_3}^n$, which is the semi-direct product of $(\mathbb{Z}/3\mathbb{Z})^n$ with $\text{GL}_n(\mathbb{F}_3)$.

2.2.2. *Examples in characteristic 3*

- The line arrangement \mathcal{L}_3 consists of 3 lines passing through the same point.
- The line arrangement \mathcal{L}_9 consists of 9 lines with 12 double points. The matroid associated to \mathcal{L}_9 is the same as the matroid associated to the Ceva(3) line arrangement, after appropriately labeling the lines.
- Let M_{27} be the matroid associated with the line arrangement \mathcal{L}_{27} . Using [1] and OSCAR software, one computes that the moduli space $\mathcal{R}(M_{27})$ over $\overline{\mathbb{F}}_3$ of M_{27} is 3-dimensional with a non-reduced immersed component Z of dimension 2 and multiplicity 2. A model for this moduli space is an open sub-scheme of a rank 3 quadric in \mathbb{A}^4 . The equations for this model, as well as the associated 27 normal vectors of the line arrangements, are provided in the ancillary file of the arXiv version of this paper.

Using the generic point of $\mathcal{R}(M_{27})$, one can check using MAGMA software that the 27 points of the dual of the generic realization of M_{27} are not contained on a cuspidal cubic. However, for the generic element of Z , (which is such that Z_{red} is a sub-scheme of $\mathcal{R}(M_{27})$), the 27 points dual to a realization of M_{27} lie on a cuspidal cubic curve.

We now state the following result:

Proposition 5. *For any $q = 3^n$ with $n \geq 3$, the matroid M_q has no realizations over any field of characteristic $\neq 3$.*

Proof. Using [1] and OSCAR software, one computes that the matroid M_{27} has no realizations over any field of characteristic $\neq 3$. The matroid M_q , $q = 3^n$, can also be described as triples $\{a, b, c\}$ in $(\mathbb{F}_3)^n$ such that $a + b + c = 0$. For $n \geq 3$, the group $(\mathbb{F}_3)^3$ is a sub-group of $(\mathbb{F}_3)^n$, and a realization of M_q gives a realization of M_{27} by taking the labels indexed by that sub-group. This forces the realization to be over a field of characteristic 3. □

2.3. *Characteristic 2*

2.3.1. *General construction*

Suppose that $\text{Char}(K) = 2$ and $K = \mathbb{F}_q$, where $q = 2^n$ for $n > 1$. Conditions in (1) for $S_{a,b} = \{a, b, -(a + b)\}$ to be of order 3 are then $a \neq b$ and $a \neq 0, b \neq 0$. Geometrically, since for any $a \in K$, one has $2a = 0$, the tangent line trough a contains 0, and the point a with multiplicity 2, so it contains no other points of the cuspidal cubic. This implies that for any pair $a \neq b$ in K^* , the line passing through a and b intersects the cubic at a third point $c = -(a + b) \notin \{0, a, b\}$. Through each point $a \neq 0$ of K there are $\frac{1}{2}(q - 2)$ lines containing a and another point of K^* . There are $\frac{(q-1)(q-2)}{6}$ such lines, and each line contains 3 points of K^* . This results in a

$$\left((q - 1) \frac{1}{2}(q - 2), \left(\frac{(q-1)(q-2)}{6} \right)_3 \right)$$

configuration of points and lines. Let \mathcal{C}_{q-1} be the dual of the $q - 1$ points in K^* . We obtain:

Theorem 6. *The line arrangement \mathcal{C}_{q-1} of $q - 1$ lines has $\frac{(q-1)(q-2)}{6}$ triple points and no other singularities.*

2.3.2. *Examples in characteristic 2*

- The line arrangement $\mathcal{C}_{2^{2n-1}}$ is 3 lines passing through the same point.
- The line arrangement $\mathcal{C}_{2^{3n-1}}$ is the Fano plane.
- Let $N_{2^{n-1}}$ be the matroid associated to $\mathcal{C}_{2^{n-1}}$; this is the set of triples $\{a, b, a + b\} \subset \mathbb{F}_{2^n}$ with $a \neq b, a \neq 0, b \neq 0$. Using [1], one computes that the moduli space $\mathcal{R}(N_{15})_{/\mathbb{F}_2}$ is an open subscheme in $\mathbb{P}^3_{/\mathbb{F}_2}$, in particular it is 3-dimensional. We checked that the generic element \mathcal{C} in $\mathcal{R}(N_{15})$ has the property that there is a cuspidal cubic containing the 15 points that are dual to the 15 lines of \mathcal{C} .

For $n > 3$, in [3], are constructed some line arrangements $\mathcal{C}'_{2^{n-1}}$ (over the field $\mathbb{F}_{2^{3n-1}}$) with only triple points and with the same number $q - 1 = 2^n - 1$ of lines as $\mathcal{C}_{2^{n-1}}$. The $2^n - 1$ points dual to $\mathcal{C}'_{2^{n-1}}$ are the image of the $2^n - 1$ points of $\mathbb{P}^{n-1}(\mathbb{F}_2)$ in \mathbb{P}^{n-1} by a generic projection to \mathbb{P}^2 .

In fact, one can check that the line arrangements \mathcal{C}_{q-1} and \mathcal{C}'_{q-1} define isomorphic matroids as follows. As a group, the field $K = \mathbb{F}_q$ (for $q = 2^n$) is the group $(\mathbb{F}_2)^n$ and the set $(\mathbb{F}_2)^n \setminus \{0\}$ is contained in $\mathbb{P}^{n-1}(\mathbb{F}_2)$. Two distinct points a, b of $(\mathbb{F}_2)^n \setminus \{0\}$ generate a line in \mathbb{P}^{n-1} which contains the points $a, b, c = a + b$ (so that $a + b + c = 0$), and no other points of $\mathbb{P}^{n-1}(\mathbb{F}_2)$. As shown in [3], a generic projection $f: \mathbb{P}^n \dashrightarrow \mathbb{P}^2$ induces a bijection between the set $\mathbb{P}^{n-1}(\mathbb{F}_2)$ and its image, and f preserves the alignments. Specifically, for $a, b, c \in \mathbb{P}^{n-1}(\mathbb{F}_2)$, one has $a + b + c = 0$ in $(\mathbb{F}_2)^n$ if and only if $f(a), f(b)$ and $f(c)$ are on a line in \mathbb{P}^2 . Thus a bijective morphism η of groups between K and $(\mathbb{F}_2)^n$ defines an isomorphism between the matroids $M(\mathcal{C}_q)$ and $M(\mathcal{C}'_q)$, i.e. sends the triples $\{a, b, c\}$ of $N = M(\mathcal{C}_q)$ bijectively to the triples of $N' = M(\mathcal{C}'_q)$. If $\mathcal{C}'_q = (\ell_a)_{a \in \mathbb{F}_2^n}$ is a realization of N' , then $(\ell_{\eta(a)})_{a \in K}$ is a realization of N .

The construction of \mathcal{C}'_{15} in [3] explains why the moduli space $\mathcal{R}(N_{15})$ is an open sub-scheme of $\mathbb{P}^3_{/\mathbb{F}_2}$, since the line arrangement \mathcal{C}'_{15} is obtained by the projection to \mathbb{P}^2 from a generic point in \mathbb{P}^3 of the 15 points of $\mathbb{P}^3(\mathbb{F}_2)$ in \mathbb{P}^3 .

Remark 7. For $n \geq 3$, the projections $\mathbb{P}^n \dashrightarrow \mathbb{P}^2$ are parametrized by the Grassmannian $G(n - 2, n + 1)$. For $n > 4$, it would be of interest to determine whether, as in the case $n = 3$, an open sub-scheme of the Grassmannian also corresponds to the moduli spaces of the matroid $N_{2^{n-1}}$. For $n = 4$, we have verified that the moduli space $\mathcal{R}(N_{31})$ is 6-dimensional and rational, as $G(2, 5)$.

Remark 8. The line arrangements $\mathcal{C}_{2^{n-1}}$ are defined over \mathbb{F}_{2^n} . This addresses the question posed before Theorem 3.4 in [3] regarding the identification of a smaller field over which \mathcal{C}'_q can be defined.

2.4. *Characteristic $\neq 2, 3$*

Suppose that $\text{Char}(K) = p > 3$. Then for $b = -2a$ and $b = -\frac{1}{2}a$, $S_{a,b}$ has order 2. Thus, the number of lines containing 3 points and passing through $a \in C^*$ is $\frac{1}{2}(q - 3)$. Furthermore, the number of lines containing 3 distinct points of K is

$$\frac{1}{6}q(q - 3)$$

and the number of lines containing exactly two points is q . This leads to a

$$\left(q_{\frac{1}{2}(q-3)}, \binom{q(q-3)}{6} \right)_3$$

configuration of points and lines. The dual is a line arrangement of q lines and $\frac{q(q-3)}{6}$ triple points. It has

$$\frac{q(q-1)}{2} - 3 \frac{q(q-3)}{6} = q$$

double points (corresponding to the q 2-rich lines in the dual space).

Declaration of interests

The author does not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and has declared no affiliations other than their research organizations.

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