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# Comptes Rendus Mathématique

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Volume 363 (2025), p. 599-602

Online since: 5 June 2025

https://doi.org/10.5802/crmath.745

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Research article / *Article de recherche* Harmonic analysis / *Analyse harmonique* 

## A direct proof of the weighted Pólya–Knopp inequality following Carleson's method

### *Une preuve directe de l'inégalité de Pólya–Knopp à poids par la méthode de Carleson*

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**Abstract.** The aim of the paper is to provide a direct proof of the weighted Pólya–Knopp inequality. This inequality (which is a limiting case of the Ariño–Muckenhoupt inequalities), involving non-increasing functions, was initially established by Sbordone–Wik, who proved its validity under the necessary and sufficient condition that the weight satisfies an appropriate doubling condition. Our main contribution is to use Carleson's approach to Carleman's inequality in conjunction with Hardy's lemma and Sbordone–Wik's doubling condition, in order to obtain the weighted Pólya–Knopp inequality.

**Résumé.** L'objectif de cet article est de fournir une preuve directe de l'inégalité de Pólya–Knopp à poids. Cette inégalité (qui est un cas limite des inégalités de Ariño–Muckenhoupt), portant sur des fonctions décroissantes, a été montrée par Sbordone–Wik, qui ont montré qu'elle est valide si elle seulement si le poids satisfait une condition adaptée de doublement. Notre contribution principale est d'utiliser la méthode de Carleson pour la preuve de l'inégalité de Carleman, en conjonction avec le lemme de Hardy et la condition de doublement de Sbordone–Wik, afin de prouver directement l'inégalité de Pólya–Knopp à poids.

Keywords. Hardy's inequality, Pólya–Knopp's inequality, weight, doubling condition, nonincreasing.

Mots-clés. Inégalité de Hardy, inégalité de Pólya–Knopp, poids, doublement, convexité.

2020 Mathematics Subject Classification. 26D15, 26A51.

Manuscript received 31 December 2024, revised 20 February 2025 and 29 March 2025, accepted 27 March 2025.

#### 1. Introduction

Let the Ariño–Muckenhoupt class  $B_p$   $(1 \le p < \infty)$  be defined as the set of all nonnegative functions *W* for which a constant *B* exists such that

$$\int_{r}^{\infty} \left(\frac{r}{x}\right)^{p} W(x) \, \mathrm{d}x \le B \int_{0}^{r} W(x) \, \mathrm{d}x \quad \text{for all } r > 0,$$

and let the class  $B_{\infty}$  be the union of all such  $B_p$ .

Let  $1 \le p < \infty$ . Then, the Ariño–Muckenhoupt theorem [1, Theorem 1.7] asserts the following.

Ariño-Muckenhoupt theorem ([1, Theorem 1.7]). The weighted Hardy inequality

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(t)^{1/p} \, \mathrm{d}t\right)^p W(x) \, \mathrm{d}x \le C \int_0^\infty f(x) \, W(x) \, \mathrm{d}x$$

holds for all nonnegative, nonincreasing functions f on  $[0,\infty)$  if and only if W belongs to the class  $B_p$ .

By Riesz [7],  $(\int_0^x f(t)^{1/p} dt/x)^p$  decreases and tends to  $\exp(\int_0^x \log f(t) dt/x)$  as p increases to  $\infty$ . Therefore, as a limiting case of this theorem, it is natural to try to characterize the class  $B_\infty$  such that the following holds, where we use the notation  $Gf(x) = \exp(\int_0^x \log f(t) dt/x)$ : the weighted Pólya–Knopp inequality

$$\int_0^\infty Gf(x)W(x)\,\mathrm{d}x \le C\int_0^\infty f(x)W(x)\,\mathrm{d}x\tag{1}$$

holds for all positive, nonincreasing functions f on  $[0,\infty)$  if and only if  $W \in B_{\infty}$ .

In Sbordone–Wik [8], the class  $B_{\infty}$  is defined using the doubling condition (D<sub>a</sub>) below. Moreover, it is proved in [8, Theorem 5] that the class  $B_{\infty}$  is the union of all  $B_p$  ( $1 \le p < \infty$ ), and the inequality (1) is proved in [8, Theorem 6].

The aim of this paper is to provide a direct proof of the inequality (1) by making full use of Carleson's approach to Carleman's inequality.

Following Carleson [4], we express a nonincreasing function f(x) on  $(0,\infty)$  by using a nonincreasing function h(x) as  $f(x) = e^{h(x)}$ , where  $h: (0,\infty) \to \mathbb{R} \cup \{\pm \infty\}$ . Then  $Gf(x) = e^{H(x)}$  and the inequality (1) can be written as

$$\int_0^\infty e^{H(x)} W(x) \,\mathrm{d}x \le C \int_0^\infty e^{h(x)} W(x) \,\mathrm{d}x,\tag{2}$$

where

$$H(x) \coloneqq \frac{1}{x} \int_0^x h(t) \,\mathrm{d}t, \quad x \in (0,\infty).$$

The inequality where  $W(x) = x^p (-1 and <math>C = e^{p+1}$  is treated in Carleson [4]. The case where p = 0 is known as the Pólya–Knopp inequality [5]. See also Carleman [3].

Our paper is organized as follows. In Section 2, we restate the theorem to be considered and then give a direct proof of the inequality (2) following Carleson's method. A few comments are mentioned in Section 3.

#### 2. Theorem and a direct proof of the weighted Pólya-Knopp inequality

For the convenience of the reader, we restate the result to be proved.

**Theorem 1.** Let the weight function W belong to the class  $B_{\infty}$ . Then the weighted Pólya– Knopp inequality (2) holds for all nonincreasing function  $h: (0,\infty) \to \mathbb{R} \cup \{\pm \infty\}$ , where  $H(x) = (1/x) \int_0^x h(t) dt$ .

**Proof.** For a nonnegative function W(x), suppose that it satisfies the  $B_{\infty}$  condition, or equivalently, the following doubling condition (D<sub>a</sub>) for some a > 1:

$$\int_0^{ar} W(x) \,\mathrm{d}x \le C_a \int_0^r W(x) \,\mathrm{d}x \quad \text{for all } r > 0. \tag{D_a}$$

For this equivalence, see Sbordone–Wik [8, Theorem 5]. The inequality (2) is now to be proved. Because *h* is nonincreasing, we have  $axH(ax) \le xH(x) + (ax - x)h(x)$  (even in the case where  $h(x) = -\infty$  and thus this inequality becomes  $-\infty \le -\infty$ ). We use this inequality and Hölder's inequality to obtain, for any A > 0,

$$\int_{0}^{A} e^{H(x)} W(x) dx = \int_{0}^{A/a} e^{H(ax)} W(ax) a dx$$

$$\leq \int_{0}^{A} e^{H(x)/a} e^{(a-1)h(x)/a} W(ax) a dx$$

$$\leq \left(\int_{0}^{A} e^{H(x)} W(ax) a dx\right)^{1/a} \left(\int_{0}^{A} e^{h(x)} W(ax) a dx\right)^{(a-1)/a}.$$
(3)

We next note that the functions  $e^{H(x)}$  and  $e^{h(x)}$  are nonincreasing, and that the condition (D<sub>a</sub>) can be written as follows:

$$\int_0^r W(ax)a\,\mathrm{d}x \le C_a \int_0^r W(x)\,\mathrm{d}x \quad \text{for all } r > 0.$$

We now recall Hardy's lemma according to [2, Proposition 3.6 in Chapter 2].

**Hardy's lemma ([2, Proposition 3.6 in Chapter 2]).** Let  $f_1$  and  $f_2$  be nonnegative functions on  $[0,\infty)$ , and suppose

$$\int_0^r f_1(x) \, \mathrm{d}x \le \int_0^r f_2(x) \, \mathrm{d}x \quad \text{for all } r > 0$$

Let g be any nonnegative nonincreasing function on  $[0,\infty)$ . Then,

$$\int_0^A f_1(x)g(x)\,\mathrm{d} x \le \int_0^A f_2(x)g(x)\,\mathrm{d} x \quad \text{for all } A > 0.$$

In our case, Hardy's lemma yields

$$\int_0^A e^{H(x)} W(ax) a \,\mathrm{d}x \le C_a \int_0^A e^{H(x)} W(x) \,\mathrm{d}x,\tag{4}$$

and

$$\int_0^A e^{h(x)} W(ax) a \, \mathrm{d}x \le C_a \int_0^A e^{h(x)} W(x) \, \mathrm{d}x.$$
(5)

Therefore, from (3), (4) and (5), we obtain

$$\int_{0}^{A} e^{H(x)} W(x) \, \mathrm{d}x \le C_{a} \left( \int_{0}^{A} e^{H(x)} W(x) \, \mathrm{d}x \right)^{1/a} \left( \int_{0}^{A} e^{h(x)} W(x) \, \mathrm{d}x \right)^{(a-1)/a}.$$
(6)

As it is assumed in Ariño–Muckenhoupt [1, p. 730], we can also assume that h is constant on [0, d] for some d > 0. We can add this restriction on h without loss of generality by use of the monotone convergence theorem. Then H is also constant on [0, d] and we have

$$\int_0^A e^{H(x)} W(x) \,\mathrm{d}x < \infty.$$

By rearranging the inequality (6) using this finite-valued term, we obtain

$$\int_0^A e^{H(x)} W(x) \, \mathrm{d}x \le C_a^{a/(a-1)} \int_0^A e^{h(x)} W(x) \, \mathrm{d}x.$$

Letting *A* tend to  $\infty$ , we obtain the desired inequality (2) with the constant  $C = C_a^{a/(a-1)}$ .

**Remark 2.** In Carleson [4], the power function  $W(x) = x^p$   $(-1 is used as a weight. Then, the doubling constant is <math>C_a = a^{p+1}$ , and we have that  $\inf_{a>1} C_a^{a/(a-1)} = \lim_{a \to 1} C_a^{a/(a-1)} = e^{p+1}$ , which is known to be the best constant for the power-weighted Pólya–Knopp inequality.

#### 3. Comments

**Comment 3.** For the use of Carleson's approach in the related direction of the weighted Hardy inequality, see Kwon [6].

**Comment 4.** Let  $1 \le p < \infty$ . Then, the Hardy–Littlewood maximal operator is bounded on the classical Lorentz space  $\Lambda_p(W)$  if and only if *W* satisfies the  $B_p$  condition. See Ariño–Muckenhoupt [1]. The question that remains is what is the limiting case of this equivalence when  $p \to \infty$ .

#### Acknowledgments

The authors would like to express their sincere gratitude to an anonymous referee for their very valuable comments on the representation of nonincreasing functions and on the treatment of infinities that may appear in the inequalities. They would also like to express their sincere gratitude to Professors Hitoshi Ishii (Waseda University) and Eiichi Nakai (Ibaraki University) for valuable suggestions and encouragement.

#### **Declaration of interests**

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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