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A note on the compact uniform integrability in metric spaces

Note sur l'intégrabilité uniforme compacte dans les espaces métriques

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Abstract. In this note, we characterize the compact uniform p-th order integrability (CUI(p)) condition for measurable functions taking values in a metric space, where $p \in (0,\infty)$. Based on that, we then introduce the notion of $(\nu_\theta)_\theta$-CUI(p) for a family of metric space valued random elements which not only extends several known notions of CUI(p) in the literature but also provides insight into interpreting them. Under a uniform tightness condition, characterizations of $(\nu_\theta)_\theta$-CUI(p) in terms of the uniform absolute continuity and of the de la Vallée Poussin criterion are discussed. Our approach to the proofs is different from the relevant works.

Résumé. Dans cette note, nous caractérisons la condition d'intégrabilité d'ordre p-uniforme compact (CUI(p)) pour les fonctions mesurables prenant des valeurs dans un espace métrique, où $p \in (0,\infty)$. Sur cette base, nous introduisons la notion de $(\nu_\theta)_\theta$-CUI(p) pour une famille d'éléments aléatoires valués dans un espace métrique qui non seulement étend plusieurs notions connues de CUI(p) dans la littérature, mais fournit également un aperçu de leur interprétation. Sous une condition d'étanchéité uniforme, les caractérisations de $(\nu_\theta)_\theta$-CUI(p) en termes de continuité absolue uniforme et du critère de de la Vallée Poussin sont discutées. Notre approche des preuves est différente de celle des travaux existants.

Keywords. Compact uniform integrability, de la Vallée Poussin criterion, Kolmogorov extension theorem, metric space, uniform tightness.

Mots-clés. Intégrabilité uniforme compacte, critère de la Vallée Poussin, théorème d'extension de Kolmogorov, espace métrique, étanchéité uniforme.

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1. Introduction

The uniform integrability (UI) is a fundamental notion in probability theory which describes a certain type of boundedness for a family of random variables. It is in particular a very useful tool for deriving limit theorems, for example, the celebrated Vitali theorem asserts that, under the UI condition, the convergence in probability is equivalent to the L^1-convergence for a sequence of random variables.

A collection of real-valued random variables $(X_i)_{i\in I}$, which are defined on a probability space $(\Omega,\mathscr{F},\mathbb{P})$, is called UI if $\sup_{i\in I}\mathbb{E}\big[|X_i|\mathbb{1}_{\{|X_i|>a\}}\big]\to 0$ when $a\to\infty$. To make this condition more applicable in different contexts, several equivalent formulations of the UI are established. Among others, we state here some characterizations that are frequently used: a family $(X_i)_{i\in I}$ is UI if and only if one (and hence, all) of the following two conditions holds true:

(i) $\sup_{i\in I}\mathbb{E}\big[|X_i|\big]<\infty$, and in addition, for any $\varepsilon>0$, there exists $\kappa>0$ such that whenever $A\in\mathscr{F}$ with $\mathbb{P}(A)\le\kappa$ one has $\sup_{i\in I}\mathbb{E}\big[|X_i|\mathbb{1}_A\big]\le\varepsilon$;

(ii) there is a non-decreasing function $G\colon[0,\infty)\to[0,\infty)$ with $G(0)=0$ and $G(x)/x\to\infty$ as $x\to\infty$ such that $\sup_{i\in I}\mathbb{E}\big[G\big(|X_i|\big)\big]<\infty$.

Condition (i) is the uniform boundedness in L^1 and the uniform absolute continuity, respectively. Condition (ii) is known as the celebrated *de la Vallée Poussin criterion* which gives an elegant description for UI. We refer to [13, Section 22] for a comprehensive summary about UI.

In the literature, the notion of UI and its characterizations have been extensively studied in various directions depending on which applications are of interest. For random variables taking real values, we can mention, among others, the weighted UI in the Cesàro sense which is then extended to $\{a_{n,k}\}$-UI in [8]. Recently, the notions of B-statistical UI and $\mathscr{J}$-UI, together with their characterizations, have been introduced and discussed in [11] and [5], respectively. As the literature on UI in this line of research is vast, we refer the reader to, for example, the references in the aforementioned articles and their follow-up works.

In another direction, several authors consider random elements $(X_i)_i$ taking values in a separable Banach space which satisfy a compact uniform integrability (CUI) condition, see, e.g., [6]. One can show that if $(X_i)_i$ is CUI, then there is a family of random elements $(Y_i)_i$ taking values in a finite set that approximates $(X_i)_i$, and moreover, the family $(Y_i)_i$ is constructed in a way that it inherits some good distributional properties of $(X_i)_i$. These features are crucial to derive limit theorems for $(X_i)_i$. Note that when the Banach space is especially a finite dimensional space so that every closed ball is compact, then CUI becomes the usual UI aforementioned above. We refer to, e.g., [1,4,9,10,12,14,17] for relevant discussions on CUI and its characterizations with applications in proving limit theorems.

However, in many applications it appears that random elements take values in metric spaces rather than in normed linear spaces. We can mention, for instance, the space of right-continuous with finite left limits functions on $[0,1]$ with Skorokhod metric [3], the metric space of probability distributions with Wasserstein metric [16], and the space of compact subsets of a given Banach space with Hausdorff metric [7]. Therefore, it might be necessary to consider the CUI condition in metric spaces. For this direction, the authors in [15] have recently introduced the notion of CUI in the Cesàro sense for a sequence of random elements with values in a convex combination metric space and established some limit theorems under a CUI condition.

In this article, we study the notion of compact uniform p-th order integrability, abbreviated by CUI(p), for measurable functions or random elements in a separable and complete metric space, where $p\in(0,\infty)$. We first introduce in Definition 1 the CUI(p) condition for an arbitrary family of measurable functions, which are not necessarily defined on the same measure space, by means of its characterizations. It is shown in Theorem 3 that, under the uniform finiteness of the measure spaces and a uniform tightness condition, then CUI(p) is equivalent to an absolute continuity condition, and, also, to a de la Vallée Poussin type condition.

Our approach to the proof is different from the relevant literature, even in case of UI random variables. Specifically, instead of directly proving the equivalences for the UI or CUI condition as in, e.g., [8,12], we construct a suitable way to transform the (apparently) more general setting into a simplified setting where one can apply the known results in a straightforward way. To do that, the Kolmogorov extension theorem plays a key role.

We then discuss in Section 3 the $(\nu_\theta)_\theta$-CUI(p) condition for a collection of random elements $\{X_s, s \in \bigcup_\theta S_\theta\}$ with values in a metric space. Here, ν_θ is a finite measure on the (possibly uncountable) index set S_θ. It is shown via examples that, when ν_θ are in particular discrete measures, then $(\nu_\theta)_\theta$-CUI(p) boils down to several known notions of CUI(p) such as (classical) CUI(p), Cesàro CUI(p), and weighted CUI(p). Moreover, for certain choices of $(\nu_\theta)_\theta$, the $(\nu_\theta)_\theta$-CUI(p) can be seen as the usual CUI(p) for random elements with random indices, see Example 11. We also provide in Theorem 13 some standard equivalences for $(\nu_\theta)_\theta$-CUI(p) whose proof follows directly from Theorem 3 by realizing the measure spaces in a product form.

Notation

Let $\mathbb{N} := \{1,2,\dots\}$ and let $\mathbb{R}^d$ be the d-dimensional Euclidean space equipped with the Borel σ-algebra $\mathscr{B}(\mathbb{R}^d)$. For a set S, we denote by $\mathscr{P}(S)$ the collection of all subsets of S.

Let $(M,\mathscr{M},\mu)$ be a measure space. The indicator function of an $A \in \mathscr{M}$ is denoted by $\mathbb{1}_A$. For $p \in (0,\infty)$, let $L^p(\mu) := L^p(M,\mathscr{M},\mu)$ be the (equivalent) class of $\mathscr{M}/\mathscr{B}(\mathbb{R})$-measurable functions $f\colon M \to \mathbb{R}$ with $\int_M |f(x)|^p \mu(\mathrm{d}x) < \infty$. For a measurable space $(U,\mathscr{U})$, the pushforward measure of μ via an $\mathscr{M}/\mathscr{U}$-measurable function $f\colon M \to U$ is denoted by $\mu \circ f^{-1}$, i.e. $\mu\circ f^{-1}(A) = \mu(\{f \in A\})$ for $A \in \mathscr{U}$. The Dirac measure δ_x concentrated at a point $x \in M$ is defined by $\delta_x(A) = 1$ if $x \in A$ and $\delta_x(A) = 0$ if $x \notin A$ for $A \in \mathscr{M}$. The notation $\int_M f \mathrm{d}\mu$ stands for $\int_M f(x)\mu(\mathrm{d}x)$ when the integration variable x is not important to indicate.

Assume throughout this article that (E,d_E) is a separable and complete metric space which is equipped with the Borel σ-algebra $\mathscr{B}(E)$ induced by the metric d_E. Let $\mathscr{K}(E)$ be the collection of all nonempty compact subsets in E.

For a probability space $(\Omega,\mathscr{F},\mathbb{P})$, an $\mathscr{F}/\mathscr{B}(E)$-measurable function $X\colon \Omega \to E$ is called an (E-valued) random element. In particular, if $(E,\mathscr{B}(E)) = (\mathbb{R},\mathscr{B}(\mathbb{R}))$ then X is called a random variable. The expectation of a random variable X, if it exists, is denoted by $\mathbb{E}[X]$.

2. A result on the compact uniform integrability

Let us fix an element $\mathfrak{e} \in E$ and denote

$$\|x\|_{\mathfrak{e}} := d_E(x,\mathfrak{e}).$$

In particular, if E is a separable Banach space then we typically choose $\mathfrak{e} = 0$.

Definition 1. *Let $\Theta \neq \emptyset$ be a (possibly uncountable) index set and $\{(M_\theta,\mathscr{M}_\theta,\mu_\theta), \theta \in \Theta\}$ a family of finite measure spaces. Suppose $p \in (0,\infty)$. We say that the* compact uniform p-th order integrability criterion (the CUI(p) criterion, for short) holds true on $(E,\mathscr{B}(E))$ for $\{(M_\theta,\mathscr{M}_\theta,\mu_\theta), \theta \in \Theta\}$ *if for any family $\{f_\theta, \theta \in \Theta\}$ of $\mathscr{M}_\theta/\mathscr{B}(E)$-measurable functions $f_\theta\colon M_\theta \to E$, the following four statements (S1)–(S4) are equivalent:*

(S1) $\{f_\theta, \theta \in \Theta\}$ *is* CUI(p), *which means:*

$$\forall\, \varepsilon > 0,\ \exists\, K_{p,\varepsilon} \in \mathscr{K}(E) \quad \textit{such that} \quad \sup_{\theta\in\Theta} \int_{M_\theta} \|f_\theta\|_{\mathfrak{e}}^p \mathbb{1}_{\{f_\theta \notin K_{p,\varepsilon}\}} \mathrm{d}\mu_\theta \le \varepsilon;$$

(S2) (a) $\{f_\theta, \theta \in \Theta\}$ *is* uniformly tight (UT), *which means:*

$$\forall\, \delta > 0,\ \exists\, K'_\delta \in \mathscr{K}(E) \quad \textit{such that} \quad \sup_{\theta\in\Theta} \mu_\theta\big(\{f_\theta \notin K'_\delta\}\big) \le \delta;$$

(b) $\displaystyle\lim_{a\to\infty} \sup_{\theta\in\Theta} \int_{M_\theta} \|f_\theta\|_{\mathfrak{e}}^p \mathbb{1}_{\{\|f_\theta\|_{\mathfrak{e}} > a\}} \mathrm{d}\mu_\theta = 0;$

(S3) (a) $\{f_\theta, \theta \in \Theta\}$ *is UT;*

(b) $R := \sup_{\theta\in\Theta} \int_{M_\theta} \|f_\theta\|_{\mathfrak{e}}^p \,\mathrm{d}\mu_\theta < \infty$*, and for any* $\varepsilon > 0$ *there exists* $\kappa = \kappa(R, p, \varepsilon) > 0$ *such that*

$$\forall\, \theta \in \Theta,\ A_\theta \in \mathscr{M}_\theta :\quad \mu_\theta(A_\theta) \le \kappa \implies \int_{M_\theta} \|f_\theta\|_{\mathfrak{e}}^p \mathbb{1}_{A_\theta} \mathrm{d}\mu_\theta \le \varepsilon;$$

(S4) (a) $\{f_\theta, \theta \in \Theta\}$ *is UT;*

(b) *there is a non-decreasing function* $G\colon [0,\infty) \to [0,\infty)$ *with* $G(0) = 0$ *and* $G(x)/x^p \to \infty$ *as* $x \to \infty$ *such that*

$$\sup_{\theta\in\Theta} \int_{M_\theta} G\big(\|f_\theta\|_{\mathfrak{e}}\big)\mathrm{d}\mu_\theta < \infty.$$

We will show in Proposition 5 below that our main result does not depend on the choice of the reference element $\mathfrak{e}$.

Remark 2.

(1) The tightness conditions (S2)-(a), (S3)-(a) and (S4)-(a) coincide and they are a variant of UT in the literature, see, e.g., [3, p. 9] or [14, Definition 5.2.1] (note that the above measure spaces indexed by Θ are not necessarily identical).

(2) When $(M_\theta, \mathscr{M}_\theta, \mu_\theta)$ does not depend on θ, i.e. $(M_\theta, \mathscr{M}_\theta, \mu_\theta) = (M, \mathscr{M}, \mu)$ for all $\theta \in \Theta$, then the equivalences (S2)-(b) $\Leftrightarrow$ (S3)-(b) $\Leftrightarrow$ (S4)-(b) hold true and they are a UI characterization of $\{\|f_\theta\|_{\mathfrak{e}}^p, \theta \in \Theta\}$, see, e.g., [13, Theorem 22.9, (ii), (ix), (x)].

(3) For (S2), it is easy to check that conditions (a) and (b) do not imply each other. Indeed, (a) is not sufficient to obtain (b) as UT does not imply UI, even in finite dimensional spaces. Conversely, let $E = \ell^1$ be the Banach space of summable real sequences and $\{e_n, n \in \mathbb{N}\}$ the canonical basis of ℓ^1. Set $\Theta = \mathbb{N}$ and let $(M_n, \mathscr{M}_n, \mu_n) = (\Omega, \mathscr{F}, \mathbb{P})$ be a fixed probability space. For $f_n(\omega) := e_n$, the family $\{f_n, n \in \mathbb{N}\}$ is uniformly bounded, and thus, (b) holds. However, (a) fails to hold as $\{e_n, n \in \mathbb{N}\}$ is not compact.

We are now in a position to formulate the main result.

Theorem 3. *Let* $\Theta \neq \emptyset$ *and* $p \in (0,\infty)$*. Then, one has the following:*

(1) *the* CUI(p) *criterion holds true on* $\big(E, \mathscr{B}(E)\big)$ *for all families* $\{(M_\theta, \mathscr{M}_\theta, \mu_\theta), \theta \in \Theta\}$ *satisfying* $\sup_{\theta\in\Theta} \mu_\theta(M_\theta) < \infty$*;*

(2) *if* E *is not a singleton and if the* CUI(p) *criterion holds true on* $\big(E, \mathscr{B}(E)\big)$ *for some family* $\{(M_\theta, \mathscr{M}_\theta, \mu_\theta), \theta \in \Theta\}$*, then* $\sup_{\theta\in\Theta} \mu_\theta(M_\theta) < \infty$.

We need the following lemma.

Lemma 4. *Let* $I \neq \emptyset$ *and let* $\{(\Omega_i, \mathscr{F}_i, \mathbb{P}_i), i \in I\}$ *be a family of probability spaces. Suppose that* $X_i\colon \Omega_i \to \mathbb{R}$ *is* $\mathscr{F}_i/\mathscr{B}(\mathbb{R})$*-measurable with* $R := \sup_{i\in I} \mathbb{E}_i\big[|X_i|\big] < \infty$*, where* $\mathbb{E}_i$ *is the expectation with respect to* $\mathbb{P}_i$*. Then, the following assertions are equivalent:*

(1) *for any* $\varepsilon > 0$*, there exists* $\kappa = \kappa(R, \varepsilon) > 0$ *such that*

$$\forall\ i \in I,\ A_i \in \mathscr{F}_i :\quad \mathbb{P}_i(A_i) \le \kappa \implies \mathbb{E}_i\big[|X_i|\mathbb{1}_{A_i}\big] \le \varepsilon;$$

(2) *for any* $\varepsilon > 0$*, there exists* $\kappa = \kappa(R, \varepsilon) > 0$ *such that*

$$\forall\ i \in I,\ A_i \in \sigma\{|X_i|\} :\quad \mathbb{P}_i(A_i) \le \kappa \implies \mathbb{E}_i\big[|X_i|\mathbb{1}_{A_i}\big] \le \varepsilon.$$

Proof. The implication (1) $\Rightarrow$ (2) is obvious. Now, we assume that (2) is satisfied. Then, for any $\varepsilon > 0$, there exists $\kappa > 0$ such that, for any $i \in I$, $A_i \in \sigma\{|X_i|\}$, $\mathbb{P}_i(A_i) \le \kappa$ implies $\mathbb{E}_i\big[|X_i|\mathbb{1}_{A_i}\big] \le \varepsilon/2$. Let $a > 0$ be sufficiently large such that $R/a \le \kappa$. Then,

$$\sup_{i\in I} \mathbb{P}_i\big(\{|X_i| > a\}\big) \le \frac{1}{a} \sup_{i\in I} \mathbb{E}_i\big[|X_i|\big] = \frac{R}{a} \le \kappa.$$

For $\kappa' := \varepsilon/(2a)$ and for any $A_i \in \mathscr{F}_i$ with $\mathbb{P}_i(A_i) \le \kappa'$, one has

$$\begin{aligned}\mathbb{E}_i\big[|X_i|\mathbb{1}_{A_i}\big] &= \mathbb{E}_i\big[|X_i|\mathbb{1}_{A_i\cap\{|X_i|\le a\}}\big] + \mathbb{E}_i\big[|X_i|\mathbb{1}_{A_i\cap\{|X_i|>a\}}\big]\\ &\le a\mathbb{P}_i(A_i) + \mathbb{E}_i\big[|X_i|\mathbb{1}_{\{|X_i|>a\}}\big]\\ &\le a\kappa' + \varepsilon/2 = \varepsilon.\end{aligned}$$

Hence, statement (1) holds true. □

Proof of Theorem 3.

(1). The proof is proceeded in three steps.

Claim 1. *The* CUI(p) *criterion holds true on* $\big(E,\mathscr{B}(E)\big)$ *for all families* $\big\{(M_\theta,\mathscr{M}_\theta,\mu_\theta),\, \theta\in\Theta\big\}$ *with*

$$(M_\theta,\mathscr{M}_\theta,\mu_\theta) = (M_{\theta'},\mathscr{M}_{\theta'},\mu_{\theta'}) \quad \textit{and} \quad \mu_\theta(M_\theta) = 1, \quad \forall\, \theta,\theta'\in\Theta.$$

In this case, we let $(M_\theta,\mathscr{M}_\theta,\mu_\theta) = (M,\mathscr{M},\mu)$ be the common probability space. The statements (S2)-(a), (S3)-(a) and (S4)-(a) coincide, and, by Remark 2(2), the relation (S2)-(b) ⇔ (S3)-(b) ⇔ (S4)-(b) holds true. The remaining equivalence (S1) ⇔ (S2) also holds due to [17, Lemma 2.2] where we note that the argument in the proof of [17, Lemma 2.2] is also valid for any Θ-indexed family of E-valued random elements and that the linearity of the underlying space is not necessary there.

Claim 2. *The* CUI(p) *criterion holds true on* $\big(E,\mathscr{B}(E)\big)$ *for all families* $\big\{(M_\theta,\mathscr{M}_\theta,\mu_\theta),\, \theta\in\Theta\big\}$ *with*

$$\mu_\theta(M_\theta) = 1, \quad \forall\, \theta\in\Theta.$$

In this case, $\big\{(M_\theta,\mathscr{M}_\theta,\mu_\theta),\, \theta\in\Theta\big\}$ is a family of probability spaces. Let $f_\theta\colon M_\theta\to E$ be $\mathscr{M}_\theta/\mathscr{B}(E)$-measurable for any $\theta\in\Theta$. According to the Kolmogorov extension theorem, which is recalled in Proposition 16 below, there exist a probability space $(\mathbf{M},\mathfrak{M},\boldsymbol{\mu})$ and $\mathfrak{M}/\mathscr{M}_\theta$-measurable functions $\pi_\theta\colon \mathbf{M}\to M_\theta$, $\theta\in\Theta$, such that

$$\boldsymbol{\mu}\circ\pi_\theta^{-1} = \mu_\theta \quad \text{on } \mathscr{M}_\theta. \tag{1}$$

Then, for any non-negative and $\mathscr{M}_\theta/\mathscr{B}(\mathbb{R})$-measurable function $h\colon M_\theta\to\mathbb{R}$, applying the change of variables formula (see, e.g., [13, Theorem 15.1]) we get

$$\int_{M_\theta} h(x)\mu_\theta(\mathrm{d}x) = \int_{M_\theta} h(x)(\boldsymbol{\mu}\circ\pi_\theta^{-1})(\mathrm{d}x) = \int_{\pi_\theta^{-1}(M_\theta)} h\big(\pi_\theta(\boldsymbol{x})\big)\boldsymbol{\mu}(\mathrm{d}\boldsymbol{x}) = \int_{\mathbf{M}} h\big(\pi_\theta(\boldsymbol{x})\big)\boldsymbol{\mu}(\mathrm{d}\boldsymbol{x}). \tag{2}$$

Define $F_\theta\colon \mathbf{M}\to E$ by

$$F_\theta(\boldsymbol{x}) := f_\theta\big(\pi_\theta(\boldsymbol{x})\big), \quad \boldsymbol{x}\in\mathbf{M}.$$

We observe that:

- For (S1), using (2) with $h(x) = \|f_\theta(x)\|_{\mathfrak{e}}^p\mathbb{1}_{\{f_\theta(x)\notin K_{p,\varepsilon}\}}$ we obtain

$$\int_{M_\theta}\|f_\theta\|_{\mathfrak{e}}^p\mathbb{1}_{\{f_\theta\notin K_{p,\varepsilon}\}}\mathrm{d}\mu_\theta = \int_{\mathbf{M}}\|F_\theta\|_{\mathfrak{e}}^p\mathbb{1}_{\{F_\theta\notin K_{p,\varepsilon}\}}\mathrm{d}\boldsymbol{\mu}.$$

- For (S2)-(a), using (2) with $h(x) = \mathbb{1}_{\{f_\theta(x)\notin K'_\delta\}}$ we get

$$\mu_\theta\big(\{f_\theta\notin K'_\delta\}\big) = \int_{M_\theta}\mathbb{1}_{\{f_\theta\notin K'_\delta\}}\mathrm{d}\mu_\theta = \int_{\mathbf{M}}\mathbb{1}_{\{F_\theta\notin K'_\delta\}}\mathrm{d}\boldsymbol{\mu} = \boldsymbol{\mu}\big(\{F_\theta\notin K'_\delta\}\big).$$

  For (S2)-(b), applying (2) with $h(x) = \|f_\theta(x)\|_{\mathfrak{e}}^p\mathbb{1}_{\{\|f_\theta(x)\|_{\mathfrak{e}}>a\}}$ yields to

$$\int_{M_\theta}\|f_\theta\|_{\mathfrak{e}}^p\mathbb{1}_{\{\|f_\theta\|_{\mathfrak{e}}>a\}}\mathrm{d}\mu_\theta = \int_{\mathbf{M}}\|F_\theta\|_{\mathfrak{e}}^p\mathbb{1}_{\{\|F_\theta\|_{\mathfrak{e}}>a\}}\mathrm{d}\boldsymbol{\mu}.$$

- (S3)-(a) coincides with (S2)-(a). For (S3)-(b), applying (2) for $h(x) = \|f_\theta(x)\|_{\mathfrak{e}}^p$ we get
$$\int_{M_\theta} \|f_\theta\|_{\mathfrak{e}}^p \mathrm{d}\mu_\theta = \int_{\mathbf{M}} \big\|f_\theta\big(\pi_\theta(\boldsymbol{x})\big)\big\|_{\mathfrak{e}}^p \boldsymbol{\mu}(\mathrm{d}\boldsymbol{x}) = \int_{\mathbf{M}} \|F_\theta\|_{\mathfrak{e}}^p \mathrm{d}\boldsymbol{\mu},$$
which are finite. For the absolute continuity, we let $\phi\colon E \to \mathbb{R}$ be defined by $\phi(x) := \|x\|_{\mathfrak{e}}^p$ and have the following equivalences:
$$\text{“} \forall\, \varepsilon > 0,\ \exists\, \kappa > 0 : \forall\, A_\theta \in \mathscr{M}_\theta,\ \mu_\theta(A_\theta) \le \kappa \implies \int_{M_\theta} \|f_\theta\|_{\mathfrak{e}}^p \mathbb{1}_{A_\theta} \mathrm{d}\mu_\theta \le \varepsilon\text{”}$$
$$\iff \text{“} \forall\, \varepsilon > 0,\ \exists\, \kappa > 0 : \forall\, A_\theta \in \sigma\{\phi\circ f_\theta\},\ \mu_\theta(A_\theta) \le \kappa \implies \int_{M_\theta} \phi\circ f_\theta\, \mathbb{1}_{A_\theta} \mathrm{d}\mu_\theta \le \varepsilon\text{”}$$
$$\iff \text{“} \forall\, \varepsilon > 0,\ \exists\, \kappa > 0 : \forall\, B_\theta \in \mathscr{B}(\mathbb{R}),\ \mu_\theta\circ f_\theta^{-1}\circ\phi^{-1}(B_\theta) \le \kappa \implies \int_{M_\theta} \phi\circ f_\theta\, \mathbb{1}_{\{\phi\circ f_\theta \in B_\theta\}} \mathrm{d}\mu_\theta \le \varepsilon\text{”}$$
$$\iff \text{“} \forall\, \varepsilon > 0,\ \exists\, \kappa > 0 : \forall\, \widetilde{A}_\theta \in \sigma\{\phi\circ F_\theta\},\ \boldsymbol{\mu}(\widetilde{A}_\theta) \le \kappa \implies \int_{\mathbf{M}} \phi\circ F_\theta\, \mathbb{1}_{\widetilde{A}_\theta} \mathrm{d}\boldsymbol{\mu} \le \varepsilon\text{”}$$
$$\iff \text{“} \forall\, \varepsilon > 0,\ \exists\, \kappa > 0 : \forall\, \widetilde{A} \in \mathfrak{M},\ \boldsymbol{\mu}(\widetilde{A}) \le \kappa \implies \int_{\mathbf{M}} \|F_\theta\|_{\mathfrak{e}}^p \mathbb{1}_{\widetilde{A}} \mathrm{d}\boldsymbol{\mu} \le \varepsilon\text{”}.$$
Here, obtaining the first and the last equivalences is due to Lemma 4, the third is due to $\mu_\theta\circ f_\theta^{-1} = \boldsymbol{\mu}\circ F_\theta^{-1}$ by (1).
- (S4)-(a) coincides with (S2)-(a). For (S4)-(b), applying (2) with $h(x) = G\big(\|f_\theta(x)\|_{\mathfrak{e}}\big)$ we obtain
$$\int_{M_\theta} G\big(\|f_\theta\|_{\mathfrak{e}}\big)\mathrm{d}\mu_\theta = \int_{\mathbf{M}} G\big(\|F_\theta\|_{\mathfrak{e}}\big)\mathrm{d}\boldsymbol{\mu}.$$

Hence, equivalences (S1) ⇔ (S2) ⇔ (S3) ⇔ (S4) hold for $\{F_\theta, \theta\in\Theta\}$ on $(\mathbf{M}, \mathfrak{M}, \boldsymbol{\mu})$ if and only if they hold for $\{f_\theta, \theta\in\Theta\}$ in relation to $\{(M_\theta, \mathscr{M}_\theta, \mu_\theta), \theta\in\Theta\}$. Since the former holds true by Claim 1, the latter, i.e. Claim 2, follows.

Claim 3. *The* CUI(p) *criterion holds true on* $\big(E, \mathscr{B}(E)\big)$ *for all families* $\{(M_\theta, \mathscr{M}_\theta, \mu_\theta), \theta\in\Theta\}$ *with*
$$\sup_{\theta\in\Theta} \mu_\theta(M_\theta) < \infty.$$

We let $\{(M_\theta, \mathscr{M}_\theta, \mu_\theta), \theta\in\Theta\}$ be a family of finite measure spaces with $C := \sup_{\theta\in\Theta}\mu_\theta(M_\theta) < \infty$. Without loss of generality, we may assume $C > 0$, otherwise the equivalences (S1) ⇔ (S2) ⇔ (S3) ⇔ (S4) in Definition 1 are trivial. For $\theta\in\Theta$, denote
$$C_\theta := \mu_\theta(M_\theta) \in [0, C].$$
Let o be an element such that $\mathrm{o} \notin \bigcup_{\theta\in\Theta} M_\theta$. For $\theta\in\Theta$ we set
$$\widetilde{M}_\theta := M_\theta \cup \{\mathrm{o}\}, \quad \widetilde{\mathscr{M}}_\theta := \sigma\big\{\mathscr{M}_\theta, \{\mathrm{o}\}\big\},$$
and for $A \in \widetilde{\mathscr{M}}_\theta$, we define
$$\widetilde{\mu}_\theta(A) := \frac{1}{C}\mu_\theta\big(A\backslash\{\mathrm{o}\}\big) + \frac{C - C_\theta}{C}\delta_{\mathrm{o}}\big(A\cap\{\mathrm{o}\}\big).$$
Note that
$$\widetilde{\mu}_\theta(A) = \frac{1}{C}\mu_\theta(A), \quad A\in\mathscr{M}_\theta. \tag{3}$$
Since
$$\widetilde{\mu}_\theta(\widetilde{M}_\theta) = \frac{1}{C}\mu_\theta(M_\theta) + \frac{C - C_\theta}{C}\delta_{\mathrm{o}}\big(\{\mathrm{o}\}\big) = \frac{C_\theta}{C} + \frac{C - C_\theta}{C} = 1,$$
it implies that $\{(\widetilde{M}_\theta, \widetilde{\mathscr{M}}_\theta, \widetilde{\mu}_\theta), \theta\in\Theta\}$ is a family of probability spaces. Define $\widetilde{f}_\theta\colon \widetilde{M}_\theta \to E$ by
$$\widetilde{f}_\theta(x) := \begin{cases} f_\theta(x) & \text{if } x \in M_\theta, \\ \mathfrak{e} & \text{if } x = \mathrm{o}. \end{cases}$$

Then, for $B \in \mathscr{B}(E)$,

$$\{x \in \widetilde{M}_\theta : \widetilde{f}_\theta(x) \in B\} = \{x \in M_\theta : \widetilde{f}_\theta(x) \in B\} \cup \{x \in \{\mathrm{o}\} : \widetilde{f}_\theta(x) \in B\} = \{x \in M_\theta : f_\theta(x) \in B\} \cup B_\mathrm{o},$$

where $B_\mathrm{o} \in \widetilde{\mathscr{M}}_\theta$ is given by

$$B_\mathrm{o} := \begin{cases} \{\mathrm{o}\} & \text{if } \mathfrak{e} \in B, \\ \emptyset & \text{if } \mathfrak{e} \notin B. \end{cases}$$

Since $\{x \in M_\theta : f_\theta(x) \in B\} \in \mathscr{M}_\theta$, it implies that $\{x \in \widetilde{M}_\theta : \widetilde{f}_\theta(x) \in B\} \in \widetilde{\mathscr{M}}_\theta$. Hence, $\widetilde{f}_\theta$ is $\widetilde{\mathscr{M}}_\theta/\mathscr{B}(E)$-measurable.

For any non-negative and $\mathscr{B}(E)/\mathscr{B}(\mathbb{R})$-measurable function $H\colon E \to \mathbb{R}$ with $H(\mathfrak{e}) = 0$ one has

$$\begin{aligned} \int_{\widetilde{M}_\theta} H\big(\widetilde{f}_\theta(x)\big)\widetilde{\mu}_\theta(\mathrm{d}x) &= \int_{M_\theta} H\big(\widetilde{f}_\theta(x)\big)\widetilde{\mu}_\theta(\mathrm{d}x) + \int_{\{\mathrm{o}\}} H\big(\widetilde{f}_\theta(x)\big)\widetilde{\mu}_\theta(\mathrm{d}x) \\ &= \frac{1}{C}\int_{M_\theta} H\big(f_\theta(x)\big)\mu_\theta(\mathrm{d}x) + \frac{C - C_\theta}{C} H\big(\widetilde{f}_\theta(\mathrm{o})\big)\delta_\mathrm{o}\big(\{\mathrm{o}\}\big) \\ &= \frac{1}{C}\int_{M_\theta} H\big(f_\theta(x)\big)\mu_\theta(\mathrm{d}x). \end{aligned} \tag{4}$$

We observe that:

- For (S1), we use (4) with $H(x) = \|x\|_\mathfrak{e}^p \mathbb{1}_{\{x \notin K_{p,\varepsilon}\}}$ to get

$$\int_{\widetilde{M}_\theta} \|\widetilde{f}_\theta\|_\mathfrak{e}^p \mathbb{1}_{\{\widetilde{f}_\theta \notin K_{p,\varepsilon}\}} \mathrm{d}\widetilde{\mu}_\theta = \int_{\widetilde{M}_\theta} H\big(\widetilde{f}_\theta(x)\big)\widetilde{\mu}_\theta(\mathrm{d}x) = \frac{1}{C}\int_{M_\theta} \|f_\theta\|_\mathfrak{e}^p \mathbb{1}_{\{f_\theta \notin K_{p,\varepsilon}\}} \mathrm{d}\mu_\theta.$$

- For (S2)-(a), we may assume without loss of generality that $\mathfrak{e} \in K'_\delta$, otherwise we can let $\widetilde{K}'_\delta := K'_\delta \cup \{\mathfrak{e}\}$ which is also a compact set in E, and moreover, (S2)-(a) holds for $\widetilde{K}'_\delta$ provided that (S2)-(a) holds for K'_δ. Note that

$$\begin{aligned} \{x \in \widetilde{M}_\theta : \widetilde{f}_\theta(x) \in K'_\delta\} &= \{x \in M_\theta : \widetilde{f}_\theta(x) \in K'_\delta\} \cup \{x \in \{\mathrm{o}\} : \widetilde{f}_\theta(x) \in K'_\delta\} \\ &= \{x \in M_\theta : f_\theta(x) \in K'_\delta\} \cup \{\mathrm{o}\}, \end{aligned}$$

  which implies

$$\{x \in \widetilde{M}_\theta : \widetilde{f}_\theta(x) \notin K'_\delta\} = \big(\widetilde{M}_\theta \setminus \{x \in M_\theta : f_\theta(x) \in K'_\delta\}\big) \cap M_\theta = \{x \in M_\theta : f_\theta(x) \notin K'_\delta\}.$$

  Hence, by (3),

$$\widetilde{\mu}_\theta\big(\{\widetilde{f}_\theta \notin K'_\delta\}\big) = \frac{1}{C}\mu_\theta\big(\{f_\theta \notin K'_\delta\}\big).$$

  For (S2)-(b), we apply (4) with $H(x) = \|x\|_\mathfrak{e}^p \mathbb{1}_{\{\|x\|_\mathfrak{e} > a\}}$ to get

$$\int_{\widetilde{M}_\theta} \|\widetilde{f}_\theta\|_\mathfrak{e}^p \mathbb{1}_{\{\|\widetilde{f}_\theta\| > a\}} \mathrm{d}\widetilde{\mu}_\theta = \frac{1}{C}\int_{M_\theta} \|f_\theta\|_\mathfrak{e}^p \mathbb{1}_{\{\|f_\theta\|_\mathfrak{e} > a\}} \mathrm{d}\mu_\theta.$$

- (S3)-(a) coincides with (S2)-(a). For (S3)-(b), we have $\widetilde{\mu}_\theta(A_\theta) = \frac{1}{C}\mu_\theta(A_\theta)$ by (3) and $\widetilde{f}\big|_A = f$ by the definition of $\widetilde{f}$ for any $A \in \mathscr{M}_\theta \subseteq \widetilde{\mathscr{M}}_\theta$, and hence,

$$\int_{\widetilde{M}_\theta} \|\widetilde{f}_\theta\|_\mathfrak{e}^p \mathbb{1}_{A_\theta} \mathrm{d}\widetilde{\mu}_\theta = \frac{1}{C}\int_{M_\theta} \|f_\theta\|_\mathfrak{e}^p \mathbb{1}_{A_\theta} \mathrm{d}\mu_\theta.$$

- (S4)-(a) coincides with (S2)-(a). For (S4)-(b), it follows from (4) with $H(x) := G\big(\|x\|_\mathfrak{e}\big)$ that

$$\int_{\widetilde{M}_\theta} G\big(\|\widetilde{f}_\theta\|_\mathfrak{e}\big)\mathrm{d}\widetilde{\mu}_\theta = \int_{\widetilde{M}_\theta} H(\widetilde{f}_\theta)\mathrm{d}\widetilde{\mu}_\theta = \frac{1}{C}\int_{M_\theta} H(f_\theta)\mathrm{d}\mu_\theta = \frac{1}{C}\int_{M_\theta} G\big(\|f_\theta\|_\mathfrak{e}\big)\mathrm{d}\mu_\theta.$$

According to those observations, as $C \in (0,\infty)$, the equivalences (S1) $\Leftrightarrow$ (S2) $\Leftrightarrow$ (S3) $\Leftrightarrow$ (S4) in Definition 1 for $\{f_\theta,\, \theta \in \Theta\}$ in relation to $\{(M_\theta, \mathscr{M}_\theta, \mu_\theta),\, \theta \in \Theta\}$ can be derived from the correspondent equivalences for $\{\widetilde{f}_\theta,\, \theta \in \Theta\}$ in relation to $\{(\widetilde{M}_\theta, \widetilde{\mathscr{M}}_\theta, \widetilde{\mu}_\theta),\, \theta \in \Theta\}$. Since the latter equivalences are satisfied by Claim 2, we obtain Claim 3. The proof of item (1) is completed.

(2). Since E is not a singleton, there exists $\widetilde{\mathfrak{e}} \in E$ and $\widetilde{\mathfrak{e}} \neq \mathfrak{e}$. We define $f_\theta(x) := \widetilde{\mathfrak{e}}$ for all $x \in M_\theta$ and $\theta \in \Theta$. Then, f_θ is clearly $\mathscr{M}_\theta/\mathscr{B}(E)$-measurable. For any $\varepsilon > 0$, we choose $K_{p,\varepsilon} := \{\widetilde{\mathfrak{e}}\} \in \mathscr{K}(E)$ to get $\mathbb{1}_{\{f_\theta \notin K_{p,\varepsilon}\}} = 0$, and hence, (S1) is satisfied for $\{f_\theta, \theta \in \Theta\}$. Since, by assumption, the CUI(p) criterion holds true on $(E, \mathscr{B}(E))$ for $\{(M_\theta, \mathscr{M}_\theta, \mu_\theta), \theta \in \Theta\}$, the condition $\sup_{\theta\in\Theta} \int_{M_\theta} \|f_\theta\|_{\mathfrak{e}}^p \mathrm{d}\mu_\theta < \infty$ in (S3)-(b) has to be satisfied, which then yields to $\sup_{\theta\in\Theta} \mu_\theta(M_\theta) < \infty$ as $\|f_\theta\|_{\mathfrak{e}} = \|\widetilde{\mathfrak{e}}\|_{\mathfrak{e}} > 0$. □

It is straightforward to check that the conclusion of Theorem 3(2) does not depend on the reference element $\mathfrak{e}$. We now show that this observation is also true for Theorem 3(1).

Proposition 5. *Let $\mathfrak{e}' \in E$. Then, the conclusion in Theorem 3(1) holds true with respect to $\mathfrak{e}$ if and only if it holds true with respect to $\mathfrak{e}'$.*

Proof. The proof of Theorem 3(1) reveals that Claim 1, Claim 2 and Claim 3 (stated in relation to $\mathfrak{e}$) are equivalent. Hence, it is sufficient to verify that Claim 1 stated in relation to $\mathfrak{e}$ is equivalent to Claim 1 stated in relation to $\mathfrak{e}'$. Indeed, let us consider $(M_\theta, \mathscr{M}_\theta, \mu_\theta) =: (M, \mathscr{M}, \mu)$ a probability space and any $\mathscr{M}/\mathscr{B}(E)$-measurable functions $f_\theta \colon M \to E$, $\theta \in \Theta$. By writing $(\mathrm{S2})_{\mathfrak{e}}$, we mean assertion (S2) stated in relation to $\mathfrak{e}$, and analogously for $(\mathrm{S2})_{\mathfrak{e}'}$. Note that (S2)-(a) does not depend on the choice of $\mathfrak{e}$. In addition, $(\mathrm{S2})_{\mathfrak{e}}$-(b) $\Leftrightarrow$ $(\mathrm{S2})_{\mathfrak{e}'}$-(b) due to $\left|d_E(f_\theta, \mathfrak{e}) - d_E(f_\theta, \mathfrak{e}')\right| \le d_E(\mathfrak{e}, \mathfrak{e}')$. Hence, we get $(\mathrm{S2})_{\mathfrak{e}} \Leftrightarrow (\mathrm{S2})_{\mathfrak{e}'}$, which completes the proof. □

3. $(\nu_\theta)_{\theta\in\Theta}$-compact uniform integrability and characterizations

In this section, we assume that $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space, (E, d_E) is a separable and complete metric space equipped with the Borel σ-algebra $\mathscr{B}(E)$.

Assumption A. Assume that $\Theta \neq \emptyset$ is an arbitrary index set and $\{(S_\theta, \mathscr{S}_\theta, \nu_\theta), \theta \in \Theta\}$ is a family of finite measure spaces satisfying

$$\sup_{\theta\in\Theta} \nu_\theta(S_\theta) < \infty.$$

Let $\{X_s, s \in \bigcup_{\theta\in\Theta} S_\theta\}$ be a family of E-valued random elements. We assume that for all $\theta \in \Theta$ the map $\Omega \times S_\theta \ni (\omega, s) \mapsto X_s(\omega)$ is $\mathscr{F} \otimes \mathscr{S}_\theta/\mathscr{B}(E)$-measurable, i.e.

$$\{(\omega, s) \in \Omega \times S_\theta : X_s(\omega) \in B\} \in \mathscr{F} \otimes \mathscr{S}_\theta, \quad \forall\, B \in \mathscr{B}(E).$$

Remark 6. We give some sufficient conditions for the joint measurability in Assumption A.

(1) If S_θ is countable and $\mathscr{S}_\theta = \mathscr{P}(S_\theta)$ for all $\theta \in \Theta$, then Assumption A is satisfied. Indeed, we show that $\Omega \times S_\theta \ni (\omega, s) \mapsto X_s(\omega)$ is $\mathscr{F} \otimes \mathscr{S}_\theta/\mathscr{B}(E)$-measurable for all $\theta \in \Theta$. For any $B \in \mathscr{B}(E)$, one has

$$\{(\omega, s) \in \Omega \times S_\theta : X_s(\omega) \in B\} = \bigcup_{s\in S_\theta} \left(\{\omega \in \Omega : X_s(\omega) \in B\} \times \{s\}\right).$$

Since $\{s\} \in \mathscr{S}_\theta$ and $\{X_s \in B\} \in \mathscr{F}$, we conclude that the countable union $\bigcup_{s\in S_\theta}$ on the right-hand side belongs to $\mathscr{F} \otimes \mathscr{S}_\theta$. Hence, $\{(\omega, s) \in \Omega \times S_\theta : X_s(\omega) \in B\} \in \mathscr{F} \otimes \mathscr{S}_\theta$.

(2) Let $S_\theta = \mathbb{I}$ and $\mathscr{S}_\theta = \mathscr{B}(\mathbb{I})$ for all $\theta \in \Theta$, where $\mathbb{I} = [0, \infty)$ or $\mathbb{I} = [0, T]$ for some fixed $T \in (0, \infty)$. Assume that $X_s \colon \Omega \to E$ is $\mathscr{F}/\mathscr{B}(E)$-measurable for every $s \in \mathbb{I}$ and that the mapping $\mathbb{I} \ni s \mapsto X_s(\omega)$ is right-continuous for all $\omega \in \Omega$, i.e. $\lim_{s+\varepsilon\in\mathbb{I},\, 0\le\varepsilon\downarrow 0} d_E\big(X_{s+\varepsilon}(\omega), X_s(\omega)\big) = 0$ for all $(\omega, s) \in \Omega \times \mathbb{I}$. Then, $(\omega, s) \mapsto X_s(\omega)$ satisfies the measurability condition in Assumption A. This assertion can be proven by using a standard approximation argument. Note that the joint measurability of X also holds when the right-continuity is replaced by the left-continuity.

Definition 7. *Let $p \in (0,\infty)$. A collection $\{X_s,\ s \in \bigcup_{\theta\in\Theta} S_\theta\}$ satisfying Assumption A is said to be* $(\nu_\theta)_{\theta\in\Theta}$-compactly uniformly p-th order integrable, *or* $(\nu_\theta)_{\theta\in\Theta}$-CUI$(p)$ *for short, if for any $\varepsilon > 0$ there exists $K_{p,\varepsilon} \in \mathcal{K}(E)$ such that*

$$\sup_{\theta\in\Theta} \mathbb{E}\left[\int_{S_\theta} \|X_s\|_{\mathfrak{e}}^p \mathbb{1}_{\{X_s\notin K_{p,\varepsilon}\}} \nu_\theta(\mathrm{d}s)\right] \le \varepsilon. \tag{5}$$

The notion of $(\nu_\theta)_{\theta\in\Theta}$-CUI$(p)$ generalizes several well-known notions of CUI(p) as shown in Examples 8–10 below.

Example 8 (Classical CUI(p)**).** Let $\{X_\theta,\ \theta \in \Theta\}$ be a collection of E-valued random elements. We show that the CUI(p) condition for $\{X_\theta,\ \theta \in \Theta\}$ can be regarded as the $(\nu_\theta)_{\theta\in\Theta}$-CUI$(p)$ for a suitable choice of $\{(S_\theta, \mathcal{S}_\theta, \nu_\theta),\ \theta \in \Theta\}$. Indeed, for every $\theta \in \Theta$ we let $S_\theta := \{\theta\}$, $\mathcal{S}_\theta := \mathcal{P}(S_\theta) = \{\emptyset, \{\theta\}\}$ and $\nu_\theta := \delta_\theta$, where δ_θ is the Dirac delta measure at $\theta \in S_\theta$. Then, we apply Remark 6(1) to find that Assumption A is satisfied. Hence,

$$\int_{S_\theta} \|X_s\|_{\mathfrak{e}}^p \mathbb{1}_{\{X_s\notin K_{p,\varepsilon}\}} \nu_\theta(\mathrm{d}s) = \int_{\{\theta\}} \|X_s\|_{\mathfrak{e}}^p \mathbb{1}_{\{X_s\notin K_{p,\varepsilon}\}} \delta_\theta(\mathrm{d}s) = \|X_\theta\|_{\mathfrak{e}}^p \mathbb{1}_{\{X_\theta\notin K_{p,\varepsilon}\}},$$

and then condition (5) becomes

$$\sup_{\theta\in\Theta} \mathbb{E}\big[\|X_\theta\|_{\mathfrak{e}}^p \mathbb{1}_{\{X_\theta\notin K_{p,\varepsilon}\}}\big] \le \varepsilon,$$

which is the classical CUI(p) of the family $\{X_\theta,\ \theta \in \Theta\}$, see, e.g., [4,6,14] when E is a separable Banach space (with $\mathfrak{e} = 0$).

Example 9 (Cesàro CUI(p) **for multi-indexed random elements).** Let $d \in \mathbb{N}$ and let $(X_{\mathbf{n}})_{\mathbf{n}\in\mathbb{N}^d}$ be a family of E-valued random elements. We set $\Theta := \mathbb{N}^d$, $S_{\mathbf{n}} := \{\mathbf{k} \in \mathbb{N}^d : \mathbf{k} \preccurlyeq \mathbf{n}\}$ and $\mathcal{S}_{\mathbf{n}} := \mathcal{P}(S_{\mathbf{n}})$ for all $\mathbf{n} \in \mathbb{N}^d$. Here, $\mathbf{k} = (k_1,\dots,k_d) \preccurlyeq \mathbf{n} = (n_1,\dots,n_d)$ means that $k_i \le n_i$ for all $i = 1,\dots,d$. For any $\mathbf{n} = (n_1, n_2,\dots,n_d) \in \mathbb{N}^d$ we set $|\mathbf{n}| := n_1 \cdot n_2 \cdots n_d$ and define

$$\nu_{\mathbf{n}} := \frac{1}{|\mathbf{n}|} \sum_{\mathbb{N}^d \ni \mathbf{k} \preccurlyeq \mathbf{n}} \delta_{\mathbf{k}}$$

with $\delta_{\mathbf{k}}$ the Dirac measure at $\mathbf{k}$. Then, $\nu_{\mathbf{n}}(S_{\mathbf{n}}) = 1$. Due to Remark 6(1), one has that Assumption A is satisfied. Then,

$$\int_{\mathbb{N}^d} \|X_s\|_{\mathfrak{e}}^p \mathbb{1}_{\{X_s\notin K_{p,\varepsilon}\}} \nu_{\mathbf{n}}(\mathrm{d}s) = \frac{1}{|\mathbf{n}|} \sum_{\mathbb{N}^d \ni \mathbf{k} \preccurlyeq \mathbf{n}} \int_{\mathbb{N}^d} \|X_s\|_{\mathfrak{e}}^p \mathbb{1}_{\{X_s\notin K_{p,\varepsilon}\}} \delta_{\mathbf{k}}(\mathrm{d}s) = \frac{1}{|\mathbf{n}|} \sum_{\mathbb{N}^d \ni \mathbf{k} \preccurlyeq \mathbf{n}} \|X_{\mathbf{k}}\|_{\mathfrak{e}}^p \mathbb{1}_{\{X_{\mathbf{k}}\notin K_{p,\varepsilon}\}}.$$

Hence, condition (5) becomes

$$\sup_{\mathbf{n}\in\mathbb{N}^d} \frac{1}{|\mathbf{n}|} \sum_{\mathbb{N}^d \ni \mathbf{k} \preccurlyeq \mathbf{n}} \mathbb{E}\big[\|X_{\mathbf{k}}\|_{\mathfrak{e}}^p \mathbb{1}_{\{X_{\mathbf{k}}\notin K_{p,\varepsilon}\}}\big] \le \varepsilon,$$

which is the Cesàro CUI(p) condition for $(X_{\mathbf{n}})_{\mathbf{n}\in\mathbb{N}^d}$, see [15], or see [1,17] when E is a separable Banach space (with $\mathfrak{e} = 0$).

Example 10 (Weighted CUI(p)**).** Let $d, d' \in \mathbb{N}$ and let $(b_{\mathbf{n},\mathbf{n}'})_{\mathbf{n}\in\mathbb{N}^d, \mathbf{n}'\in\mathbb{N}^{d'}}$ be an array of real numbers such that

$$0 < B_\infty := \sup_{\mathbf{n}\in\mathbb{N}^d} B_{\mathbf{n}} < \infty, \quad \text{where } B_{\mathbf{n}} := \sum_{\mathbf{n}'\in\mathbb{N}^{d'}} |b_{\mathbf{n},\mathbf{n}'}|. \tag{6}$$

For $\mathbf{n} \in \Theta := \mathbb{N}^d$, we let $S_{\mathbf{n}} := \mathbb{N}^{d'}$, $\mathcal{S}_{\mathbf{n}} := \mathcal{P}(\mathbb{N}^{d'})$ and define the measure $\nu_{\mathbf{n}}$ on $\mathcal{S}_{\mathbf{n}}$ by

$$\nu_{\mathbf{n}} := \sum_{\mathbf{n}'\in\mathbb{N}^{d'}} |b_{\mathbf{n},\mathbf{n}'}| \delta_{\mathbf{n}'}.$$

Then, $\nu_{\mathbf{n}}(\mathbb{N}^{d'}) = B_{\mathbf{n}}$, and hence, $\sup_{\mathbf{n}\in\mathbb{N}^d} \nu_{\mathbf{n}}(\mathbb{N}^{d'}) = B_\infty < \infty$. In this case, Assumption A is satisfied and condition (5) becomes

$$\sup_{\mathbf{n}\in\mathbb{N}^d} \sum_{\mathbf{n}'\in\mathbb{N}^{d'}} |b_{\mathbf{n},\mathbf{n}'}| \mathbb{E}\big[\|X_{\mathbf{n},\mathbf{n}'}\|_{\mathfrak{e}}^p \mathbb{1}_{\{X_{\mathbf{n},\mathbf{n}'}\notin K_{p,\varepsilon}\}}\big] \le \varepsilon. \tag{7}$$

In particular, if $d = d' = 1$ and $b_{n,j} = 0$ for all $j \notin [u_n, v_n]$ and for some given sequences $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$ of integers with $u_n \le v_n$, then condition (6) simply boils down to $\sup_{n\ge1}\sum_{j=u_n}^{v_n}|b_{n,j}| \in (0,\infty)$ and (7) becomes

$$\sup_{n\ge1}\sum_{j=u_n}^{v_n}|b_{n,j}|\,\mathbb{E}\big[\|X_{n,j}\|_{\mathfrak{e}}^p\,\mathbb{1}_{\{X_{n,j}\notin K_{p,\varepsilon}\}}\big] \le \varepsilon.$$

Condition (7) is called $\{b_{\mathbf{n},\mathbf{n}'}\}$-CUI($p$), see [9] when E is a separable Banach space.

Example 11 (CUI(p) **with random indices).** For $\theta \in \Theta$, let $\tau_\theta\colon \Omega \to S_\theta$ be $\mathscr{F}/\mathscr{S}_\theta$-measurable and assume that $\sigma\{X_s,\ s\in S_\theta\}$ is independent of $\sigma\{\tau_\theta\}$. Then, $\{X_s,\ s\in\bigcup_{\theta\in\Theta}S_\theta\}$ is $(\mathbb{P}\circ\tau_\theta^{-1})_{\theta\in\Theta}$-CUI($p$) if and only if $\{X_{\tau_\theta},\ \theta\in\Theta\}$ is CUI(p), where $X_{\tau_\theta}(\omega) := X_{\tau_\theta(\omega)}(\omega)$.

Indeed, let us fix $\theta\in\Theta$ and define $T_\theta\colon \Omega\to\Omega\times S_\theta$ by setting $T_\theta(\omega) := \big(\omega,\tau_\theta(\omega)\big)$. It is easy to check that T_θ is $\mathscr{F}/\mathscr{F}\otimes\mathscr{S}_\theta$-measurable. Then, $X_{\tau_\theta} = X\circ T_\theta$. Now, for any non-negative and $\mathscr{B}(E)/\mathscr{B}(\mathbb{R})$-measurable function $h\colon E\to\mathbb{R}$, one has

$$\mathbb{E}\big[h(X_{\tau_\theta})\big] = \int_\Omega h(X\circ T_\theta)\mathrm{d}\mathbb{P} = \int_{\Omega\times S_\theta} h\big(X_s(\omega)\big)(\mathbb{P}\circ T_\theta^{-1})(\mathrm{d}\omega,\mathrm{d}s).$$

Since $\sigma\{X_s,\ s\in S_\theta\}$ is independent of $\sigma\{\tau_\theta\}$, the measure $\mathbb{P}\circ T_\theta^{-1}$ restricted on $\sigma\{X_s,\ s\in S_\theta\}\otimes\mathscr{S}_\theta$ is equal to $\mathbb{P}\otimes(\mathbb{P}\circ\tau_\theta^{-1})$. To see this, letting any $A\in\sigma\{X_s,\ s\in S_\theta\}$ and $B\in\mathscr{S}_\theta$ we get

$$\mathbb{P}\circ T_\theta^{-1}(A\times B) = \mathbb{P}\big(T_\theta^{-1}(A\times B)\big) = \mathbb{P}\big(A\cap\{\tau_\theta\in B\}\big) = \mathbb{P}(A)\,\mathbb{P}\circ\tau_\theta^{-1}(B).$$

Hence, using Tonelli–Fubini's theorem (see, e.g., [13, Theorem 14.8]) we obtain

$$\mathbb{E}\big[h(X_{\tau_\theta})\big] = \int_{\Omega\times S_\theta} h\big(X_s(\omega)\big)\,\mathbb{P}\otimes(\mathbb{P}\circ\tau_\theta^{-1})(\mathrm{d}\omega,\mathrm{d}s) = \mathbb{E}\left[\int_{S_\theta} h(X_s)\,\mathbb{P}\circ\tau_\theta^{-1}(\mathrm{d}s)\right],$$

which then implies the desired assertion by choosing $h(x) = \|x\|_{\mathfrak{e}}^p\,\mathbb{1}_{\{x\notin K_{p,\varepsilon}\}}$.

Remark 12. In view of Example 11, the Cesàro CUI(p) condition for $(X_{\mathbf{n}})_{\mathbf{n}\in\mathbb{N}^d}$ in Example 9 can be regarded as the CUI(p) condition for $(X_{\tau_{\mathbf{n}}})_{\mathbf{n}\in\mathbb{N}^d}$, where each $\tau_{\mathbf{n}}$ is a random variable uniformly distributed on $\{\mathbf{k}\in\mathbb{N}^d : \mathbf{k}\preccurlyeq\mathbf{n}\}$ and is independent of $\{X_{\mathbf{k}},\ \mathbf{k}\preccurlyeq\mathbf{n}\}$.

Theorem 13. *Let $p\in(0,\infty)$. Under Assumption A, the following assertions are equivalent:*

(1) $\{X_s,\ s\in\bigcup_{\theta\in\Theta}S_\theta\}$ *is* $(\nu_\theta)_{\theta\in\Theta}$-CUI($p$);

(2) *one has*

$$\forall\,\delta>0,\ \exists\,K'_\delta\in\mathscr{K}(E)\ :\ \sup_{\theta\in\Theta}(\mathbb{P}\otimes\nu_\theta)\Big(\big\{(\omega,s)\in\Omega\times S_\theta : X_s(\omega)\notin K'_\delta\big\}\Big)\le\delta, \tag{8}$$

and

$$\lim_{a\to\infty}\sup_{\theta\in\Theta}\mathbb{E}\left[\int_{S_\theta}\|X_s\|_{\mathfrak{e}}^p\,\mathbb{1}_{\{\|X_s\|_{\mathfrak{e}}>a\}}\nu_\theta(\mathrm{d}s)\right] = 0; \tag{9}$$

(3) *condition* (8) *holds,* $\sup_{\theta\in\Theta}\mathbb{E}\big[\int_{S_\theta}\|X_s\|_{\mathfrak{e}}^p\,\nu_\theta(\mathrm{d}s)\big]<\infty$*, and in addition, for any $\varepsilon>0$ there exists $\kappa>0$ such that*

$$\forall\ A_\theta\in\mathscr{F}\otimes\mathscr{S}_\theta,\quad \sup_{\theta\in\Theta}(\mathbb{P}\otimes\nu_\theta)(A_\theta)\le\kappa \implies \sup_{\theta\in\Theta}\mathbb{E}\left[\int_{S_\theta}\|X_s\|_{\mathfrak{e}}^p\,\mathbb{1}_{A_\theta}\nu_\theta(\mathrm{d}s)\right]\le\varepsilon;$$

(4) *condition* (8) *holds, and in addition, there is a non-decreasing function $G\colon[0,\infty)\to[0,\infty)$ with $G(0)=0$ and $G(x)/x^p\to\infty$ as $x\to\infty$ such that*

$$\sup_{\theta\in\Theta}\mathbb{E}\left[\int_{S_\theta}G\big(\|X_s\|_{\mathfrak{e}}\big)\nu_\theta(\mathrm{d}s)\right]<\infty.$$

Proof. We let $M_\theta := \Omega\times S_\theta$, $\mathscr{M}_\theta := \mathscr{F}\otimes\mathscr{S}_\theta$, $\mu_\theta := \mathbb{P}\otimes\nu_\theta$. Then, under Assumption A one has

$$\sup_{\theta\in\Theta}\mu_\theta(M_\theta) = \sup_{\theta\in\Theta}(\mathbb{P}\otimes\nu_\theta)(\Omega\times S_\theta) = \sup_{\theta\in\Theta}\nu_\theta(S_\theta)<\infty.$$

Using Theorem 3(1) with $f_\theta\colon M_\theta\to E$ defined by $f_\theta(\omega,s) := X_s(\omega)$, together with Tonelli–Fubini's theorem saying that $\int_{\Omega\times S_\theta}F\mathrm{d}\mathbb{P}\otimes\nu_\theta = \mathbb{E}\big[\int_{S_\theta}F\mathrm{d}\nu_\theta\big]$ for any $\mathscr{F}\otimes\mathscr{S}_\theta/\mathscr{B}(\mathbb{R})$-measurable and non-negative F, we obtain the desired conclusion. □

In the particular case when $E=\mathbb{R}^d$ equipped with the usual Euclidean norm (and let $\mathfrak{e}=0$), then the $(\nu_\theta)_{\theta\in\Theta}$-CUI$(p)$ condition of $\{X_s,\, s\in\bigcup_{\theta\in\Theta}S_\theta\}$ is equivalent to the $(\nu_\theta)_{\theta\in\Theta}$-UI$(p)$ condition (9). Moreover, (9) implies (8) in this case, which can be argued by using Markov's inequality and Tonelli–Fubini's theorem

$$\sup_{\theta\in\Theta}(\mathbb{P}\otimes\nu_\theta)\Big(\big\{(\omega,s)\in\Omega\times S_\theta : \big\|X_s(\omega)\big\|>R\big\}\Big)\le\frac{1}{R^p}\sup_{\theta\in\Theta}\mathbb{E}\left[\int_{S_\theta}\|X_s\|^p\mathbb{1}_{\{\|X_s\|>R\}}\nu_\theta(\mathrm{d}s)\right]\xrightarrow{R\to\infty}0.$$

Therefore, we derive from Theorem 13 the following corollary.

Corollary 14. *Let $E=\mathbb{R}^d$ be the d-dimensional Euclidean space with $\mathfrak{e}=0$. Let $p\in(0,\infty)$. Then, under Assumption A, the following assertions are equivalent:*

(1) $\sup_{\theta\in\Theta}\mathbb{E}\big[\int_{S_\theta}\|X_s\|^p\mathbb{1}_{\{\|X_s\|>a\}}\nu_\theta(\mathrm{d}s)\big]\to 0$ *as* $a\to\infty$;

(2) $\sup_{\theta\in\Theta}\mathbb{E}\big[\int_{S_\theta}\|X_s\|^p\nu_\theta(\mathrm{d}s)\big]<\infty$, *and for any* $\varepsilon>0$ *there is* $\kappa>0$ *such that*

$$\forall\, A_\theta\in\mathscr{F}\otimes\mathscr{S}_\theta,\quad \sup_{\theta\in\Theta}(\mathbb{P}\otimes\nu_\theta)(A_\theta)\le\kappa \implies \sup_{\theta\in\Theta}\mathbb{E}\left[\int_{S_\theta}\|X_s\|^p\mathbb{1}_{A_\theta}\nu_\theta(\mathrm{d}s)\right]\le\varepsilon;$$

(3) *there is a non-decreasing function* $G\colon[0,\infty)\to[0,\infty)$ *with* $G(0)=0$ *and* $G(x)/x^p\to\infty$ *as* $x\to\infty$ *such that* $\sup_{\theta\in\Theta}\mathbb{E}\big[\int_{S_\theta}G\big(\|X_s\|\big)\nu_\theta(\mathrm{d}s)\big]<\infty$.

Remark 15.

(1) If E is a separable Banach space and $(\nu_\theta)_{\theta\in\Theta}$ is chosen as in Example 10, then the equivalence (1) $\Leftrightarrow$ (2) in Theorem 13 is similar to [12, Corollary 1].

(2) In Corollary 14, by letting $d=1$ and choosing $(\nu_\theta)_{\theta\in\Theta}$ as in Example 10 we obtain [8, Theorems 2 and 3].

Appendix A. Kolmogorov extension theorem

We recall the Kolmogorov extension theorem presented in [2], where the index set is not necessarily countable.

Proposition 16 ([2, Theorem 9.2]). *Let $\Theta\neq\emptyset$ be an arbitrary index set and $\{(M_\theta,\mathscr{M}_\theta,\mu_\theta),\theta\in\Theta\}$ a family of probability spaces. Then there exist a probability space $(\mathbf{M},\mathfrak{M},\boldsymbol{\mu})$ and $\mathfrak{M}/\mathscr{M}_\theta$-measurable functions $\pi_\theta\colon\mathbf{M}\to M_\theta$, $\theta\in\Theta$, such that:*

(i) $\{\pi_\theta,\,\theta\in\Theta\}$ *is an independent family under* $\boldsymbol{\mu}$;

(ii) $\boldsymbol{\mu}\circ\pi_\theta^{-1}=\mu_\theta$, *i.e.* $\boldsymbol{\mu}(\pi_\theta\in B)=\mu_\theta(B)$ *for all* $B\in\mathscr{M}_\theta$ *and* $\theta\in\Theta$.

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