ACADÉMIE
DES SCIENCES
INSTITUT DE FRANCE

Comptes Rendus
Mathématique

Rodolphe Richard and Andrei Yafaev

Generalised André–Pink–Zannier conjecture for Shimura varieties of abelian type

Volume 363 (2025), p. 873-878

Online since: 21 July 2025

https://doi.org/10.5802/crmath.751

CENTRE MERSENNE

The Comptes Rendus. Mathématique *are a member of the Mersenne Center for open scientific publishing*
www.centre-mersenne.org — e-ISSN : 1778-3569

ACADÉMIE
DES SCIENCES
INSTITUT DE FRANCE

Comptes Rendus. Mathématique
2025, Vol. 363, p. 873-878
https://doi.org/10.5802/crmath.751

Research article / *Article de recherche*
Algebraic geometry, Number theory / *Géométrie algébrique, Théorie des nombres*

Generalised André–Pink–Zannier conjecture for Shimura varieties of abelian type

Conjecture d'André–Pink–Zannier généralisée pour les variétés de Shimura de type abélien

Rodolphe Richard [a] **and Andrei Yafaev** [*, a]

[a] UCL Department of Mathematics, University College London, Gower Street, London, WC1E 6BT, UK
E-mails: r.richard@ucl.ac.uk, yafaev@ucl.ac.uk

Abstract. This note describes the results of [6]. The main result is the proof of the Generalised André–Pink–Zannier conjecture in Shimura varieties of abelian type. The core result is a lower bound, in terms of height functions defined in [7], for the sizes of Galois orbits of points in generalised Hecke orbits, which is *unconditional* for Shimura varieties of abelian type.

Résumé. Dans cette note nous décrivons les résultats de [6]. Le résultat principal est la preuve de la Conjecture d'André–Pink–Zannier genéralisée pour les variétés de Shimura de type abélien. Le résultat central énonce des bornes inférieures, en termes des fontions hauteurs de [7], pour la taille des orbites galoisiennes dans une orbite de Hecke généralisée, qui sont *inconditionnelles* pour les variétés de Shimura de type abelien.

Keywords. Shimura varieties, Hecke orbits, Zilber–Pink, heights, Siegel sets, Mumford–Tate conjecture, adelic linear groups.

Mots-clés. Variétés de Shimura, orbites de Hecke, Zilber–Pink, hauteurs, ensembles de Siegel, conjecture de Mumford–Tate, groupes linéaires adéliques.

2020 Mathematics Subject Classification. 03C64, 11G18, 11G50, 11F80, 14L30, 20G35, 15A16, 14G35.

Funding. This project was funded by Leverhulme Trust grant RPG-2019-180.

Manuscript received 21 December 2022, revised 17 November 2023 and 1 May 2025, accepted 16 April 2025.

1. Statements of results

In this note we describe the results of [6], which is the sequel to [7]. A previous note [8] described the results of [7]. We refer to [2] for the notions related to Shimura varieties, and to [9, Section 3] for the notion of weakly special subvarieties. We study the following conjecture, introduced in the following form in [7]. We refer to the previous note [8] for the conjecture and for the notions involved.

Conjecture 1 (Generalised André–Pink–Zannier Conjecture [7, Conjecture 1.1]). *Let S be a Shimura variety and Σ a subset of a generalised Hecke orbit in S. Then, irreducible components of the Zariski closure of Σ are weakly special subvarieties.*

* Corresponding author

ISSN (electronic): 1778-3569

Our most notable result is the following. We refer to [2] for the notion of Shimura *of abelian type*. This is the most important class of Shimura varieties, and it contains in particular the moduli spaces of abelian varieties $\mathscr{A}_{g,d}$.

Theorem 2. *If S is of abelian type, and Σ is a subset of a generalised Hecke orbit in S, then the Zariski closure of Σ is a finite union of weakly special subvarieties.*

The main strategy underlying the proof of Theorem 2 is the Pila–Zannier strategy in the form laid out in [7]. This relies on a variant [5] of Pila–Wilkie theorem, some functional transcendence results [4] and the so-called "geometric part of André–Oort conjecture" [9]. The article [7] introduces some height functions with good properties which are required for this strategy to work. We refer to [8] for more details.

A main difficulty in this approach is to obtain lower bounds for the sizes of some Galois orbits. This is Theorem 7. The use of Theorem 5, which is deduced from Faltings' theorems, allows us to reduce Theorem 2 to Theorem 6. The proof of Theorem 6 follows from [7] and Theorem 7. Theorem 7 is reduced to Theorem 8, whose proof is discussed in Section 2. This proof relies on the tool presented in Section 3.

1.1. *Main results*

Theorem 2 follows from two statements. The first one, Theorem 5, is a refined version of Faltings theorems' on Tate conjecture for abelian varieties. We encode this refined form into the Definition 3 of "uniform integral Tate property". The second statement, Theorem 6, proves Conjecture 1 assuming that a point $s \in \Sigma$ satisfies this uniform integral Tate property.

In order to state Definition 3 we let (G, X) be the Shimura datum (G, X) and we let K be the open compact subgroup of $G(\mathbb{A}_f)$ such that $S = \mathrm{Sh}_K(G, X)$. We write $s = [x, 1] \in S$ with $x \in X$. We denote by $M \leq G$ the Mumford–Tate group of x, choose a field $E \leq \mathbb{C}$ such that $s \in S(E)$ and such that we have a Galois representation, as in [7, Section 3],

$$\rho_x\colon \mathrm{Gal}(\overline{E}/E) \to M(\mathbb{A}_f) \cap K. \tag{1}$$

The following definition makes sense for two $\mathbb{Q}$-algebraic groups $M \leq G$ and a faithful representation $\rho_G\colon G \to \mathrm{GL}(d)$. It does not depend on the choice of ρ_G.

Definition 3 (Uniform integral Tate property). *Let $U \leq M(\mathbb{A}_f)$ be a compact subgroup. Let $U_p := U \cap M(\mathbb{Q}_p)$ and, for $p \gg 0$, let $U(p) \leq \mathrm{GL}(d, \mathbb{F}_p)$ be the image of U_p.*

We say that U satisfies the uniform integral Tate property with respect to M, G and ρ_G *if the following is satisfied.*

(1) *For every p,*

$$Z_{G_{\mathbb{Q}_p}}(U_p) = Z_{G_{\mathbb{Q}_p}}({U_p}^0) = Z_G(M)_{\mathbb{Q}_p} \tag{2a}$$

and

$$\text{the action of } U_p \text{ on } \mathbb{Q}_p^d \text{ is semisimple.}^1 \tag{2b}$$

(2) *For every D, there exists an integer $M(D)$ such that, for every $p \geq M(D)$ and every $U' \leq U(p)$ of index $[U(p) : U'] \leq D$, we have*

$$Z_{G_{\mathbb{F}_p}}(U') = Z_{G_{\mathbb{F}_p}}(M_{\mathbb{F}_p}) \tag{3a}$$

and

$$\text{the action of } U' \text{ on } \overline{\mathbb{F}_p}^d \text{ is semisimple.} \tag{3b}$$

Definition 4. *We say that x_0* satisfies the uniform integral Tate conjecture *if $U = \rho_{x_0}\big(\mathrm{Gal}(\overline{E}/E)\big)$ satisfies the uniform integral Tate property with respect to M, G in the sense of Definition 3.*

[1] Also known as *completely reducible*.

The following is deduced (see [6, Section 6.2]) from Faltings' theorems, a specialisation argument of Noot, and a result of Serre.

Theorem 5. *In a Shimura variety of abelian type, every point satisfies the uniform integral Tate conjecture in the sense of Definition 4.*

Our main result towards Conjecture 1 is the following. We recall that a generalised Hecke orbit is a finite union of "geometric Hecke orbits" ([7, Definition 2.3 and Theorem 2.4]).

Theorem 6. *Let $s_0 = [x_0, 1]$ be a point in a Shimura variety $\mathrm{Sh}_K(G,X)$, and assume the uniform integral Tate conjecture for x_0 in X in the sense of Definition 4. Let Z be a subvariety whose intersection with the geometric Hecke orbit of s_0 is Zariski dense in Z. Then, Z is a finite union of weakly special subvarieties of S.*

We observe that the "uniform integral Tate property" follows from the "weakly adelic Mumford–Tate property" of [7]. As a result, Theorem 6 supersedes the main result of [7].

2. Polynomial Galois bounds

The proof of Theorem 6 is found in [6, Section 3]. It follows [7], which uses Pila–Zannier strategy, as summarised in [8]. The main new difficulty which is dealt with in [6] is to prove lower bounds on Galois orbits assuming the uniform integral Tate property.

Theorem 7 (Galois bounds in Shimura varieties). *Let $S = \mathrm{Sh}_K(G,X)$ be a Shimura variety, let $M \leq G$ be the Mumford–Tate group of an element $x \in X$, and denote by W the conjugacy class $G\cdot\phi_0 \simeq G/Z_G(M)$ of the inclusion $\phi_0\colon M \to G$ (viewed as a $\mathbb{Q}$-algebraic variety). We choose integral structures $\mathfrak{m}_{\mathbb{Z}}$ of $\mathfrak{m}$ and $\mathfrak{g}_{\mathbb{Z}}$ of $\mathfrak{g}$, and consider the "finite height function" $H_f\colon W(\mathbb{Q}) \to \mathbb{Z}_{\geq 1}$ defined by*

$$H_f(\phi) = \min\left\{k \in \mathbb{Z}_{\geq 1} \,\middle|\, \mathrm{d}\phi(\mathfrak{m}_{\mathbb{Z}}) \leq \tfrac{1}{k}\cdot \mathfrak{g}_{\mathbb{Z}}\right\},$$

where $\mathrm{d}\phi\colon \mathfrak{m} \to \mathfrak{g}$ denotes the differential of $\phi\colon M \to G$.

We define $W^+ := G(\mathbb{R})^+\cdot\phi_0 \subseteq W(\mathbb{R})$ and $W(\mathbb{Q})^+ := W^+ \cap W(\mathbb{Q})$.

Let $E \leq \mathbb{C}$ be a subfield of finite type over $\mathbb{Q}$ and such that x_0 satisfies Definition 4.

Then, as functions of $\phi \in W(\mathbb{Q})^+$,

$$\#\,\mathrm{Gal}(\overline{E}/E)\cdot[\phi\circ x_0, 1] \approx H_f(\phi).$$

We recall that $\left\{[\phi\circ x_0, 1] \,\middle|\, \phi \in W(\mathbb{Q})^+\right\}$ is the geometric Hecke orbit of x_0.

Using [7, Section 3.3], we deduce Theorem 7 from Theorem 8 applied to U as in Definition 4. The assumption (1) in Theorem 8 follows from the properties of canonical models of Shimura varieties. The assumption (2) in Theorem 8 follows from the assumption that x_0 satisfies Definition 4.

We believe Theorem 8 to be of independent interest.

Theorem 8 ([6, Theorem 5.1]). *Let $M \leq G$ be connected reductive $\mathbb{Q}$-groups. Let $U \leq M(\mathbb{A}_f)$ be a subgroup satisfying the following.*

(1) *The image of U in M^{ab} is MT in M^{ab} in the sense of [7, Defintion 6.1].*
(2) *The group U satisfies the uniform integral Tate property as in Definition 3.*

Denote by $\phi_0\colon M \to G$ the identity homomorphism and $W = G\cdot\phi_0$ its conjugacy class. Then as ϕ varies in $W(\mathbb{A}_f)$, for any compact open subgroup $K \leq G(\mathbb{A}_f)$, where $K_M = K \cap M(\mathbb{A}_f)$, we have

$$\left[\phi(U) : \phi(U)\cap K\right] \approx \left[\phi(K_M) : \phi(K_M)\cap K\right] \succcurlyeq H_f(\phi).$$

as functions $W(\mathbb{A}_f) \to \mathbb{Z}_{\geq 1}$.

Theorem 8 can be studied prime by prime. Without loss of generality, we assume $K=\prod_p K_p$ where $K_p:=\mathrm{GL}(d,\mathbb{Z}_p)\cap G(\mathbb{Q}_p)$. We write $M(\mathbb{Z}_p):=\mathrm{GL}(d,\mathbb{Z}_p)\cap M(\mathbb{Q}_p)$. It is enough to prove the following.

- For every prime p, we have

$$\exists\, a(p),\ c(p)\in\mathbb{R}_{>0},\ \forall\,\phi\in W(\mathbb{Q}_p),\quad \left[\phi(U):\phi(U)\cap K\right]\geq a(p)\cdot\left[\phi\big(M(\mathbb{Z}_p)\big):\phi\big(M(\mathbb{Z}_p)\big)\cap K_p\right]^{c(p)}. \tag{4}$$

- There exists $a,c\in\mathbb{R}_{>0}$ such that, for all but finitely many primes,

$$\forall\,\phi\in W(\mathbb{Q}_p),\quad \left[\phi(U_p):\phi(U_p)\cap K_p\right]\geq a\cdot\left[\phi\big(M(\mathbb{Z}_p)\big):\phi\big(M(\mathbb{Z}_p)\big)\cap K_p\right]^{c}. \tag{5}$$

At a given prime, property (2a) ensures that the conjugacy classes of $M(\mathbb{Z}_p)\to G(\mathbb{Q}_p)$ and $U_p\hookrightarrow G(\mathbb{Q}_p)$ are the same variety W. Then (4) follows from the functoriality of height.

The main difficulty is to ensure uniform and sharper estimates (5) for $p\gg 0$, and properties (2) in Definition 3 are crucial. This is the most technical part of [6].

The approach is as follows (see [6, Section 5] for details).

We construct from U_p a vector, say v, in a representation of G. This representation is a product of copies of the adjoint representation $G\to\mathrm{GL}(\mathfrak{g})$ and of a fixed representation $G\to\mathrm{GL}(d)$. In particular the weights are bounded. Moreover, the orbit $(G\cdot v)(\mathbb{Q}_p)$ can be identified with $W(\mathbb{Q}_p)$.

Using the tools of [7], we can bound from below $\left[\phi(U_p):\phi(U_p)\cap K_p\right]$ in terms of the p-adic height $H_p(g\cdot v)=\max\{1;\|g\cdot v\|\}$, where $\phi\in W(\mathbb{Q}_p)$ is identified with $g\cdot v\in(G\cdot v)(\mathbb{Q}_p)$.

Let v' be the vector constructed from the case $U_p=M(\mathbb{Z}_p)$. The local height $H_p(g\cdot v')$ is the local factor of a global height function $W(\mathbb{A}_f)\to\mathbb{Z}_{\geq 1}$. Theorem 10 is used, for $p\gg 0$, to compare the local height $H_p(g\cdot v)=\max\{1;\|g\cdot v\|\}$ with the local height $H_p(g\cdot v')=\max\{1;\|g\cdot v'\|\}$.

Because the constant C in (13) does not depend on p, but only on the weights of the representation, we obtain the uniformity in (5).

3. p-adic Kempf–Ness theorem

Theorem 9 is key in order to make use of part (2) of Definition 3. Theorem 9 can be seen as an analogue of Kempf–Ness theorem [3, Theorem 0.1(b)]. Kempf–Ness theorem deals with closed orbits in linear representations equipped with an archimedean metric. Theorem 9 is a p-adic analogue. While Kempf–Ness theorem is easy to prove, our p-adic analogue requires a fine study of the "good reduction" of the orbit under consideration and the use of group schemes.

We believe this is a result of independent interest.

Theorem 9 (p-adic Kempf–Ness Theorem). *Let $F_{\mathbb{Z}_p}\leq G_{\mathbb{Z}_p}\leq\mathrm{GL}(n)_{\mathbb{Z}_p}$ be smooth reductive group schemes such that $F_{\mathbb{Z}_p}\to G_{\mathbb{Z}_p}\to\mathrm{GL}(n)_{\mathbb{Z}_p}$ are closed immersions and $G_{\mathbb{Z}_p}$ is connected.*

Let $v\in\mathbb{Z}_p^n$, denote by $\overline{v}\in\mathbb{F}_p^n$ its reduction and assume that

$$\mathrm{Stab}_{G_{\mathbb{Q}_p}}(v)=F_{\mathbb{Q}_p}\qquad\textit{and}\qquad\dim\big(\mathrm{Stab}_{G_{\mathbb{F}_p}}(\overline{v})\big)=\dim(F_{\mathbb{F}_p}), \tag{6}$$

(using Krull dimensions) and assume that the orbits

$$G_{\mathbb{Q}_p}\cdot v\subseteq\mathbb{A}^n_{\mathbb{Q}_p}\qquad\textit{and}\qquad G_{\mathbb{F}_p}\cdot\overline{v}\subseteq\mathbb{A}^n_{\mathbb{F}_p} \tag{7}$$

are closed.

Then, denoting by $\mathbb{Z}_p[G/F]:=\mathbb{Z}_p[G]\cap\mathbb{Q}_p[G]^F$ the algebra of F-invariant functions $G\to\mathbb{A}^1$ defined over $\mathbb{Z}_p$, for every $g\in G(\overline{\mathbb{Q}_p})$, we have,

$$g\cdot v\in\overline{\mathbb{Z}_p}^n\quad\textit{if and only if}\quad\forall\,f\in\mathbb{Z}_p[G/F],\ f(g)\in\overline{\mathbb{Z}_p}. \tag{8}$$

Moreover, $\mathrm{Spec}\big(\mathbb{Z}_p[G/F]\big)$ *is smooth over $\mathbb{Z}_p$, and we have*

$$\big(G(\overline{\mathbb{Q}_p})\cdot v\big)\cap\overline{\mathbb{Z}_p}^n=G(\overline{\mathbb{Z}_p})\cdot v. \tag{9}$$

Remarks. Some of the hypotheses can be rephrased as follows.

The $\mathbb{Q}_p$-algebraic groups F and G are reductive, the compact subgroups $F_{\mathbb{Z}_p}(\mathbb{Z}_p) \leq F(\mathbb{Q}_p)$ and $G_{\mathbb{Z}_p}(\mathbb{Z}_p) \leq G(\mathbb{Q}_p)$ are hyperspecial subgroups, and we have $F_{\mathbb{Z}_p}(\mathbb{Z}_p) = F(\mathbb{Q}_p) \cap \mathrm{GL}(n, \mathbb{Z}_p)$ and $G_{\mathbb{Z}_p}(\mathbb{Z}_p) = G(\mathbb{Q}_p) \cap \mathrm{GL}(n, \mathbb{Z}_p)$.

The property (7) is related to semi-stability and residual semi-stability of the vector v in the sense of [1].

In (6), the hypothesis on dimensions means that $\mathrm{Stab}_{G_{\mathbb{F}_p}}(\overline{v})^{0,\mathrm{red}}$ (the reduced subgroup of the neutral component) is equal to $(F_{\mathbb{F}_p})^0$. Equivalently, $\mathrm{Stab}_{G_{\mathbb{F}_p}}(\overline{v})^0(\overline{\mathbb{F}_p}) = F^0(\overline{\mathbb{F}_p})$. This is implied by the stronger condition

$$\mathrm{Stab}_{G(\overline{\mathbb{F}_p})}(\overline{v}) = F(\overline{\mathbb{F}_p}) \tag{10}$$

and the stronger one

$$\mathrm{Stab}_{G_{\mathbb{F}_p}}(\overline{v}) = F_{\mathbb{F}_p} \text{ as group schemes.} \tag{11}$$

3.1. *Comparison of local height functions*

Theorem 9 is a key result in proving the following, which can be seen as a refined, more precise, version of the functoriality of local heights.

Theorem 10 (Local relative stability estimates). *Let $F \leq G \xrightarrow{\rho} \mathrm{GL}(d)$ be as in Theorem 9. Let $v, v' \in \mathbb{Z}_p^d$ be non zero vectors, denote by $\overline{v}, \overline{v'} \in \mathbb{F}_p^d$ their reduction, and assume that*

(1) *the orbits $G_{\mathbb{Q}_p} \cdot v$ and $G_{\mathbb{Q}_p} \cdot v' \subseteq \mathbb{A}^d_{\mathbb{Q}_p}$ are closed subvarieties;*
(2) *the stabiliser groups $F_v := \mathrm{Stab}_G(v), F_{v'} := \mathrm{Stab}_G(v')$ satisfy*

$$F_v = F_{v'} = F;$$

and that

(3) *the orbits $G_{\mathbb{F}_p} \cdot \overline{v}$ and $G_{\mathbb{F}_p} \cdot \overline{v'} \subseteq \mathbb{A}^d_{\mathbb{F}_p}$ are closed subvarieties;*
(4) *the stabiliser groups $F_{\overline{v}} := \mathrm{Stab}_G(v)$, $F_{\overline{v'}} := \mathrm{Stab}_G(v')$ satisfy, as group schemes*[2],

$$F_{\overline{v}} = F_{\overline{v'}} = F_{\mathbb{F}_p}. \tag{12}$$

We denote by $\|\cdot\| : \mathbb{C}_p{}^d \to \mathbb{R}$ the standard ultrametric norm, and we define two functions $G(\mathbb{C}_p) \to \mathbb{R}$ given by

$$H_v : g \mapsto \max\{1; \|g \cdot v\|\} \qquad \textit{and} \qquad H_{v'} : g \mapsto \max\{1; \|g \cdot v'\|\}.$$

Then, the functions $h_v = \log H_v$ and $h_{v'} = \log H_{v'}$ satisfy

$$h_v \leq C \cdot h_{v'} \qquad \textit{and} \qquad h_{v'} \leq C \cdot h_v, \tag{13}$$

in which $C = C\big(\Sigma(\rho)\big)$ depends only on the set of weights of ρ (cf. [6, 6.1.1.2]).

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

[2] It amounts to the property that $F_{\overline{v}}$ and $F_{\overline{v}}$ are smooth.

References

[1] J.-F. Burnol, “Remarques sur la stabilité en arithmétique”, *Int. Math. Res. Not.* (1992), no. 6, pp. 117–127.

[2] P. Deligne, “Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques”, in *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*, Proceedings of Symposia in Pure Mathematics, vol. XXXIII, American Mathematical Society, 1979, pp. 247–289.

[3] G. Kempf and L. Ness, “The length of vectors in representation spaces”, in *Algebraic geometry (Proc. Summer Meeting, Univ. Copenhagen, Copenhagen, 1978)*, Lecture Notes in Mathematics, vol. 732, Springer, 1979, pp. 233–243.

[4] B. Klingler, E. Ullmo and A. Yafaev, “The hyperbolic Ax–Lindemann–Weierstrass conjecture”, *Publ. Math., Inst. Hautes Étud. Sci.* **123** (2016), pp. 333–360.

[5] J. Pila, “On the algebraic points of a definable set”, *Sel. Math., New Ser.* **15** (2009), no. 1, pp. 151–170.

[6] R. Richard and A. Yafaev, “Generalised André–Pink–Zannier Conjecture for Shimura varieties of abelian type”, 2021. Online at https://arxiv.org/abs/2111.11216v2.

[7] R. Richard and A. Yafaev, “Height functions on Hecke orbits and the generalised André–Pink–Zannier conjecture”, 2021. Online at https://arxiv.org/abs/2109.13718v1.

[8] R. Richard and A. Yafaev, “On the Generalised André–Pink–Zannier conjecture”. To appear in *C. R. Math.*

[9] E. Ullmo, “Applications du théorème d’Ax–Lindemann hyperbolique”, *Compos. Math.* **150** (2014), no. 2, pp. 175–190.