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Bedford–McMullen carpets meet nonlinear curves

Les tapis de Bedford–McMullen intersectent des courbes non linéaires

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Abstract. Let f be a regular function (the precise definition will be provided in the paper). Then, for any $\alpha \in (0,2)$ and any $\epsilon > 0$, there exists a Bedford–McMullen carpet K with convex hull $[0,1]^2$ such that

$$\alpha - \epsilon < \dim_H(K) < \alpha \qquad \text{and} \qquad \{(x,y) : y = f(x)\} \cap K = \{(0,1)\},$$

where $\dim_H$ denotes the Hausdorff dimension. In particular, this result holds for $y = \cos(x)$. Unlike classical methods for analyzing slicing sets, our proofs rely exclusively on mathematical induction.

Résumé. Soit f une fonction régulière (la définition précise sera donnée dans l'article). Alors, pour tout $\alpha \in (0,2)$ et tout $\epsilon > 0$, il existe un tapis de Bedford–McMullen K d'enveloppe convexe $[0,1]^2$ tel que

$$\alpha - \epsilon < \dim_H(K) < \alpha \qquad \text{et} \qquad \{(x,y) : y = f(x)\} \cap K = \{(0,1)\},$$

où $\dim_H$ désigne la dimension de Hausdorff. En particulier, ce résultat est valable pour $y = \cos(x)$. Contrairement aux méthodes classiques d'analyse des ensembles de coupe, notre démonstration repose exclusivement sur un raisonnement par récurrence.

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1. Introduction

Let $K \subset \mathbb{R}^2$ be a fractal set and $\Gamma \subset \mathbb{R}^2$ be a curve. In geometric measure theory, determining the Hausdorff dimension of $K \cap \Gamma$ is a challenging problem [2]. Classical results, such as Marstrand's theorem [2], provide upper bounds for the dimension of linear slices. Roughly speaking, for a given fractal set $K \subset \mathbb{R}^2$ and a line $\Gamma \subset \mathbb{R}^2$, generically, the Hausdorff dimension of $K \cap \Gamma$ is $\dim_H(K) - 1$. The reader may refer to Mattila's book [2, Theorem 10.11]. However, the situation becomes significantly more complicated when dealing with nonlinear slicing sets. For instance, when K is a Bedford–McMullen carpet and Γ is a nonlinear curve, the analysis of $K \cap \Gamma$ is generally non-trivial. To date, there has been limited references in this area. In this paper, we focus on the case where K is a Bedford–McMullen carpet and Γ is a nonlinear curve satisfying specific conditions. We prove that although some Bedford–McMullen carpet has large Hausdorff dimension, it simultaneously intersects many smooth curves with only finite points.

We begin by introducing the following definition.

Definition 1. *Let $f \in C^{\infty}(\mathbb{R})$ be a function. Suppose that s is the smallest positive integer such that $f^{(s)}(0) \neq 0$, where $f^{(s)}$ denotes the s-th order of the derivative and $s \in \mathbb{N}_{\geq 2}$. Let $m = n^s, n \geq 10^{100}$. We say that f is a* regular function *if it satisfies the following properties:*

(1) $f(0) = 1, 1 - 1/m > f(1 - 1/n) > f(1) > 1/m$;
(2) f *is strictly decreasing on* $[0,1]$;
(3) $(n-1)^{-s} \leq \frac{-f^{(s)}(x)}{s!} \leq 1 - 1/m, \ \forall\, x \in [0,1/3]$.

Remark 2. We give some remarks on the condition:

$$1 - 1/m > f(1 - 1/n) > f(1) > 1/m.$$

Since f is strictly decreasing, $f(1-1/n) > f(1)$ is clear. If we let $n \to \infty$, then the above inequalities become

$$f(0) = 1 \geq f(1) \geq 0.$$

The first inequality is clear as f is strictly decreasing. If

$$0 \leq a \leq n/3 - 1, \quad 0 \leq b \leq m/3,$$

then the third condition implies that

$$\left| \frac{f^{(s)}(x)}{s!} \frac{(a+1)^s}{n^s} \right| \leq \left| \frac{f^{(s)}(x)}{s!} \frac{1}{3^s} \right| \leq \frac{m-b-1}{m}, \quad x \in [0,1/3].$$

We shall use this inequality in the proof of Theorem 5.

We present some essential definitions that will be used throughout the paper. Let $\mathbb{N}^+$ denote the set of all positive integers. For a given integer $k \in \mathbb{N}^+$, we define $\mathbb{N}_{\geq k} = \{x \in \mathbb{N}^+ : x \geq k\}$. For any positive real number $\alpha \in \mathbb{R}^+$, we denote by $[\alpha]$ the integer part of α. For instance, $[2.025] = 2$.

We recall the definition of Bedford–McMullen carpet. Let $n, m \in \mathbb{N}_{\geq 2}$, $m > n$, and let $\mathscr{A}$ be a subset of $\{0,1,\ldots,n-1\} \times \{0,1,\ldots,m-1\}$. Without loss of generality, we assume that the cardinality of $\mathscr{A}$ is at least 2, and call $\mathscr{A}$ the digit set. We define a class of self-affine functions as follows:

$$g_{\mathbf{a}}(x,y) = \left(\frac{x + x_i}{n}, \frac{y + y_i}{m} \right), \quad \mathbf{a} = (x_i, y_i) \in \mathscr{A}.$$

The collection of $\{g_{\mathbf{a}}\}_{\mathbf{a} \in \mathscr{A}}$ is called an iterated function system (IFS). Hutchinson [1] proved that there exists a unique non-empty compact set K satisfying

$$K = \bigcup_{\mathbf{a} \in \mathscr{A}} g_{\mathbf{a}}(K).$$

Such a set K is referred to as a Bedford–McMullen carpet. For any $k \in \mathbb{N}^+$ and $\mathbf{a}_i \in \mathscr{A}$ with $1 \leq i \leq k$, we define

$$g_{\mathbf{a}_1 \mathbf{a}_2 \cdots \mathbf{a}_k}\left([0,1]^2\right) = g_{\mathbf{a}_1} \circ g_{\mathbf{a}_2} \circ \cdots \circ g_{\mathbf{a}_k}\left([0,1]^2\right)$$

as a basic rectangle at the k-th level. If the curve $y = f(x)$ intersects a basic rectangle I, we denote by $(x,y)^f \in I$.

The following result, due to McMullen [3], provides the Hausdorff dimension of a Bedford–McMullen carpet.

Theorem 3. *Let K be a Bedford–McMullen carpet associated with the IFS $\{g_{\mathbf{a}_i}\}_{\mathbf{a}_i \in \mathscr{A}}$. Then the Hausdorff dimension of K is given by*

$$\dim_H(K) = \frac{\ln\left(\sum_{j=0}^{n-1} t_j^{\log_m n}\right)}{\ln n},$$

where t_j denotes the number of indices i such that $(j,i)\in\mathscr{A}$.

We state the main result of this paper.

Theorem 4. *Let f be a regular function defined in Definition 1. Then for any $\alpha\in(0,2)$ and any $\epsilon>0$, there exists a Bedford–McMullen carpet K with the convex hull $[0,1]^2$ such that*

$$\alpha-\epsilon<\dim_H(K)<\alpha,$$

and that

$$\{(x,y): y=f(x)\}\cap K=\{(0,1)\},$$

where $\dim_H$ stands for the Hausdorff dimension.

Theorem 4 immediately follows from the following result.

Theorem 5. *Let f be a regular function defined in Definition 1. Let K be a Bedford–McMullen carpet defined by the digit set*

$$D=\{(0,0),(0,m-1),(n-1,0),(n-1,m-1),(a_i,b_i)\},\quad 1\le i\le u,\ u\in\mathbb{N}^+.$$

If $m=n^s$ with $n\ge 10^{100}$, and $0\le a_i\le n/3-1$, $0\le b_i\le m/3$, $a_i,b_i\in\mathbb{N}^+\cup\{0\}$, then

$$\{(x,y): y=f(x)\}\cap K=\{(0,1)\}.$$

Applying Theorem 4, we obtain the following example.

Example 6. For any $\alpha\in(0,2)$, $\epsilon>0$ and any closed interval $[\kappa_1,\kappa_2]\subset(0,1)$, there exists a Bedford–McMullen carpet K such that

$$\alpha-\epsilon<\dim_H(K)<\alpha,$$
$$\left(\bigcup_{\beta\in[0.9,1]}\{(x,y): y=\cos(\beta x)\}\right)\cap K=\{(0,1)\},$$

and

$$\left(\bigcup_{\gamma\in[\kappa_1,\kappa_2]}\{(x,y): y=1-\gamma x^2\}\right)\cap K=\{(0,1)\}.$$

Let $u(x)=\cos(\beta x)$, $\beta\in[0.9,1]$ and $v(x)=1-\gamma x^2$, $\gamma\in[\kappa_1,\kappa_2]$. It is easy to check that for these two functions we have $s=2$. Hence, when we construct some Bedford–McMullen carpet, we can assume $m=n^2$. Note that κ_1 and κ_2 are two fixed numbers. Hence, if n, $m=n^2$ are sufficiently large, $v(x)$ and $u(x)$ are regular functions. So, Example 6 follows from Theorem 4 and the proof of Theorem 5.

By Theorem 4, numerous other functions can be readily identified. We do not give more examples. The key insight of this example lies in the existence of a uniform Bedford–McMullen set that intersects a variety of distinct functions at a single unique point.

This paper is organized as follows. In Section 2, we prove Theorem 4 via Theorem 5. Section 3 is entirely devoted to the detailed proof of Theorem 5.

2. Proof of Theorem 4 by using Theorem 5

Proof. For any $\alpha\in(0,1)$, if m is sufficiently large, then $[m^\alpha]\in(0,m/3)$. Thus, we may select $[m^\alpha]$ basic rectangles within the vertical strip with the form $[0,1/n]\times[i/m,(i+1)/m]$, $0\le i\le[m/3]-1$, $m=n^s$. Additionally, we include the four basic rectangles at the corners of the unit square at the first level. We denote the resulting Bedford–McMullen carpet by K_1. In light of Theorem 3, the Hausdorff dimension of K_1 is given by

$$\frac{\log\left(([m^\alpha]+1)^{1/s}+2^{1/s}\right)}{\log n}.$$

Since $m = n^s$, the above dimension converges to α as $n \to \infty$. Furthermore, by means of Theorem 5, the curve $y = f(x)$ meets the Bedford–McMullen carpet K_1 with a unique point, i.e. $(0,1)$.

For $\alpha \in (1,2)$, we have $\alpha - 1 \in (0,1)$. Thus, for sufficiently large m,

$$[m^{\alpha-1}] \in [0, m/3].$$

We can then choose $[n/3]$ distinct vertical strips (each of the form $\big[i/n, (i+1)/n\big] \times [0,1]$, where $0 \le i \le \big[\frac{n}{3}\big] - 2$) in the unit square at the first level. For each strip, we select $[m^{\alpha-1}]$ basic rectangles of the form:

$$\big[i/n, (i+1)/n\big] \times \big[j/m, (j+1)/m\big], \quad 0 \le i \le [n/3] - 2,\ 0 \le j \le [m/3] - 1.$$

We also include the four-cornered basic rectangles at the first level. Let K_2 be the resulting Bedford–McMullen carpet. Applying McMullen's result once again, the Hausdorff dimension of K_2 is

$$\frac{\log\Big(\big([m^{\alpha-1}]+1\big)^{1/s} + \big([m^{\alpha-1}]\big)^{1/s} \times \big(\big[\frac{n}{3}\big] - 2\big) + 2^{1/s}\Big)}{\log n},$$

which converges to α as $n \to \infty$. By virtue of Theorem 5, we conclude that

$$K_2 \cap \big\{(x,y) : y = f(x)\big\} = \big\{(0,1)\big\}.$$

3. Proof of Theorem 5

Proof. The digit set D consists of four digits

$$\big\{(0,0), (0, m-1), (n-1, 0), (n-1, m-1)\big\}.$$

Define the following functions:

$$\begin{aligned} f_1(x,y) &= (x/n, y/m), \\ f_3(x,y) &= \big((x+n-1)/n, y/m\big), \\ f_4(x,y) &= \big((x+n-1)/n, (y+m-1)/m\big), \\ f_5(x,y) &= \big(x/n, (y+m-1)/m\big). \end{aligned}$$

To represent u digits (a_i, b_i), $1 \le i \le u$, we use a uniform function:

$$f_2(x,y) = \big((x+a)/n, (y+b)/m\big), \quad 0 \le a \le n/3 - 1,\ 0 \le b \le m/3,\ a, b \in \mathbb{N}^+ \cup \{0\}.$$

For instance, let $n = 5$, $m = 25$, $a = 1$, $b = 11$, $y = \cos(x)$, the reader may refer to Figure 1 for the first iteration.

In what follows, we let $F(x_1, x_2) = x_2 - f(x_1)$. Suppose $y = f(x)$ and $(x,y) \in K$. Since $f(0) = 1$, it follows that the curve $y = f(x)$ intersects $f_5\big([0,1]^2\big)$. Moreover, it is straightforward to verify that:

$$F\big(f_1(1,1)\big) \le F\big(f_3(1,1)\big) = 1/m - f(1) < 0,$$

as $f(1) > 1/m$. Since f is strictly decreasing and

$$\partial_{x_1} F = -f'(x_1) \ge 0, \quad \partial_{x_2} F = 1 > 0,$$

by the above inequalities, it follows that the four corners of the rectangles, i.e. $f_3\big([0,1]^2\big)$ and $f_1\big([0,1]^2\big)$, are lying below the curve $y = f(x)$. Thus, the curve $y = f(x)$ does not intersect $f_1\big([0,1]^2\big)$ or $f_3\big([0,1]^2\big)$. We shall use similar ideas to prove other cases. Next, observe that:

$$F\big(f_2(1,1)\big) = \frac{b+1}{m} - f\left(\frac{a+1}{n}\right).$$

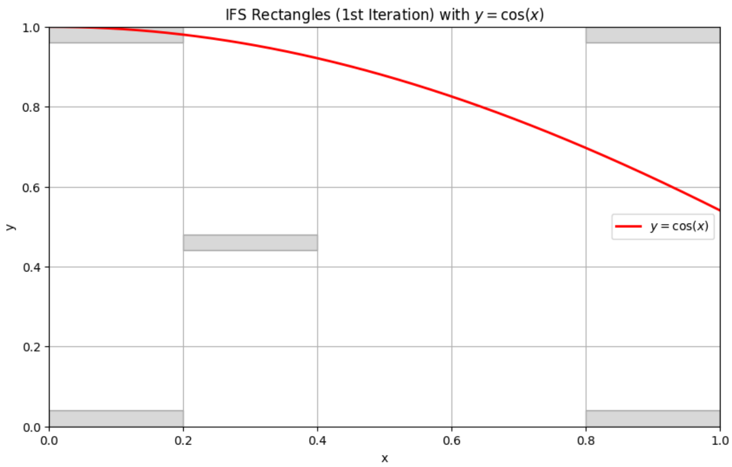

Figure 1. $y = \cos(x)$ intersects basic rectangles at the first level.

By Taylor's formula and the definition of s, there exists $\xi \in \big(0, (a+1)/n\big)$ such that

$$F\big(f_2(1,1)\big) = \frac{b+1}{m} - f(0) - \frac{f^{(s)}(\xi)}{s!}\frac{(a+1)^s}{n^s}.$$

To prove

$$F\big(f_2(1,1)\big) < 0,$$

it suffices to show

$$\left|\frac{f^{(s)}(\xi)}{s!}\frac{(a+1)^s}{n^s}\right| \le \frac{m-b-1}{m}.$$

Since $0 \le a \le n/3 - 1$, $0 \le b \le m/3$, we only need:

$$\left|\frac{f^{(s)}(\xi)}{s!}\frac{(a+1)^s}{n^s}\right| \le \left|\frac{f^{(s)}(\xi)}{s!}\frac{1}{3^s}\right| \le \frac{m-b-1}{m}.$$

The last inequality holds due to Remark 2. Hence, the curve $y = f(x)$ does not intersect $f_2\big([0,1]^2\big)$. Using the conditions of the regular function, we have

$$F\big(f_4(0,0)\big) = 1 - 1/m - f(1-1/n) > 0,$$

and

$$1/m < f(1) < 1 - 1/m.$$

Hence, at the first level, the curve $y = f(x)$ intersects only $f_5\big([0,1]^2\big)$.

If $(x,y)^f \in f_5\big([0,1]^2\big)$, we can show via induction that for any $k \in \mathbb{N}^+$,

$$(x,y)^f \in f_{5^k}\big([0,1]^2\big).$$

Taking $k \to \infty$, we obtain $(x,y) = (0,1)$. To prove this, assume that at the k-th level, $(x,y)^f \in f_{5^k}\big([0,1]^2\big)$, we aim to show $(x,y)^f \in f_{5^{k+1}}\big([0,1]^2\big)$. Since f is strictly decreasing, it suffices to verify:

$$F\big(f_{5^k 3}(1,1)\big) < 0, \quad F\big(f_{5^k 4}(0,0)\big) > 0, \quad F\big(f_{5^k 2}(1,1)\big) < 0,$$

and

$$1 - (m-1)/m^{k+1} < f(1/n^k) < 1 - 1/m^{k+1}.$$

Here $f_{5^k i}(x,y) = \underbrace{f_5 \circ \cdots \circ f_5}_{k} \circ f_i(x,y)$, $i = 2,3,4$. Note that

$$F\big(f_{5^k 3}(1,1)\big) = 1 - 1/m^k + 1/m^{k+1} - f\left(\frac{1}{n^k}\right).$$

By Taylor's theorem, there exists some $\varsigma \in (0, 1/n^k)$ such that

$$F\big(f_{5^k 3}(1,1)\big) = 1 - 1/m^k + 1/m^{k+1} - 1 - \frac{f^{(s)}(\varsigma)}{s!}\frac{1}{n^{ks}}.$$

Hence, to show $F\big(f_{5^k 3}(1,1)\big) < 0$, it suffices to prove

$$-\frac{f^{(s)}(\varsigma)}{s!} < 1 - 1/m, \quad \varsigma \in (0, 1/n^k) \subset (0, 1/3).$$

However, this is one of the conditions for the regular function.

For the second inequality, we have

$$F\big(f_{5^k 4}(0,0)\big) = 1 - 1/m^{k+1} - f\left(\frac{n-1}{n^{k+1}}\right).$$

Employing Taylor's theorem:

$$F\big(f_{5^k 4}(0,0)\big) = 1 - 1/m^{k+1} - \left(1 + \frac{f^{(s)}(\eta)}{s!}\left(\frac{n-1}{n^{k+1}}\right)^s\right),$$

where $\eta \in \left(0, \frac{n-1}{n^{k+1}}\right)$. To show $F\big(f_{5^k 4}(0,0)\big) > 0$, it suffices to have

$$-f^{(s)}(x)/s! > (n-1)^{-s}$$

for any $x \in [0, 1/3]$. This is guaranteed by the conditions of the regular function. For the third inequality:

$$F\big(f_{5^k 2}(1,1)\big) = \frac{b+1}{m^{k+1}} + 1 - \frac{1}{m^k} - f\left(\frac{a+1}{n^{k+1}}\right).$$

Utilizing Taylor's theorem:

$$\frac{b+1}{m^{k+1}} + 1 - \frac{1}{m^k} - f\left(\frac{a+1}{n^{k+1}}\right) = \frac{b+1}{m^{k+1}} + 1 - \frac{1}{m^k} - \left(1 + \frac{f^{(s)}(\kappa)}{s!}\left(\frac{a+1}{n^{k+1}}\right)^s\right),$$

where $\kappa \in \big(0, (a+1)/n^{k+1}\big)$. However, by the following conditions, i.e.

$$m = n^s, \quad 0 \le a \le n/3 - 1 \quad \text{and} \quad 0 \le b \le m/3,$$

the inequality $F\big(f_{5^k 2}(1,1)\big) < 0$ holds from

$$\left|f^{(s)}(x)/s!\right| \le 3^s\big((m-b-1)/m\big), \quad \forall\, x \in [0, 1/3].$$

This is a consequence of the regular function, as noted in Remark 2. Finally, we verify:

$$1 - (m-1)/m^{k+1} < f(1/n^k) < 1 - 1/m^{k+1}.$$

In terms of Taylor's theorem again,

$$f(1/n^k) = f(0) + \frac{f^{(s)}(\xi)}{s!}\frac{1}{n^{ks}} = 1 + \frac{f^{(s)}(\xi)}{s!}\frac{1}{n^{ks}},$$

where $\xi \in (0, 1/n^k)$. Assuming:

$$\frac{1}{n^s} = 1/m < \frac{-f^{(s)}(x)}{s!} < 1 - 1/m$$

for any $x \in [0, 1/3]$, the inequalities hold. This is ensured by the conditions of the regular function.

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Declaration of interests

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