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Skein recursion for holomorphic curves and invariants of the unknot

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Abstract. We determine the skein-valued Gromov–Witten partition functions for single toric Lagrangian branes in C^3 or in the resolved conifold. We first show geometrically they must satisfy certain skein-theoretic recursions, and then solve these equations. The recursion is a skein-valued quantization of the equation of the mirror curve. The solution is the expected hook-content formula.

Keywords. Holomorphic curve, HOMFLYPT skein module, Lagrangian submanifold, multiple-cover formula.

2020 Mathematics Subject Classification. 53D45, 53D42, 57K10.

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1. Introduction

In [14,15] we give foundations for the enumeration of holomorphic curves with Lagrangian boundary in Calabi–Yau 3-folds. The basic idea is that the obstruction to invariance arising from codimension one boundaries in moduli can be exactly identified with the HOMFLYPT framed skein relations (Figure 1) on the boundaries of the holomorphic curves themselves. Thus we retain invariance by counting curves with boundary on L by the isotopy class of their boundary in the framed skein module $\mathrm{Sk}(L)$, i.e., the free module generated by framed links in a three-manifold L, modulo the skein relations. That is, the invariant is an element of the framed skein module.

That there should be some such marriage of holomorphic curve counting and knot theory was long predicted by the string theorists [17,26,29], who moreover made predictions of the resulting curve counts [2,3]. These predictions have in some sense been mathematically confirmed: although a theory of open Gromov–Witten invariants was missing, one can nevertheless formally

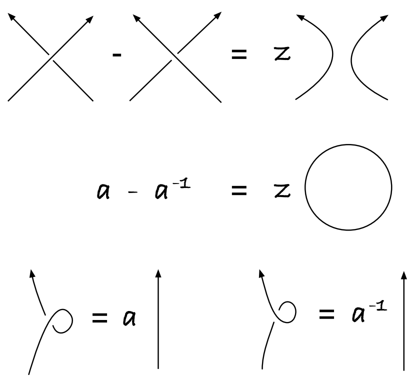

Figure 1. The HOMFLYPT skein relations. Here we will take $z = q^{1/2} - q^{-1/2}$.

compute by equivariant localization [7,18,20,21]. The computations typically reduce to the Hodge integral formula of [16].

In our new setting we have a definition, but no longer have access to equivariant localization: to define skein valued invariants we must perturb the holomorphic curve equation so that the boundaries of curves are embedded; necessarily breaking the $\mathbb{C}^*$ action as the fixed points of the latter are generally multiple covers.

On the other hand we have a new tool: equations in the skein coming from the study of 1-dimensional moduli spaces of holomorphic curves. Indeed, by the same arguments guaranteeing invariance of the skein valued Gromov–Witten invariant, the boundary of such a 1-dimensional moduli space must vanish in the skein. Thus if we can compute this boundary by some other means, we obtain an equation. This idea was proposed and studied (nonrigorously) in [1,10,13]; our new technology [14,15] makes it rigorous and more general (the previous works can in retrospect be understood as valued in a specialization of the skein). In the language of these papers the result of this paper is a skein valued recursion relation for the skein valued Gromov–Witten partition function, see Section 3 for a discussion.

When the Calabi–Yau X and Lagrangian L are noncompact with ideal boundary $(\partial X, \partial L)$ where ∂L is a Legendrian in the contact manifold ∂X, we may study moduli of holomorphic curves with positive punctures asymptotic to Reeb chords of L. The simplest imaginable case is when there are only Reeb chords of Conley–Zehnder index ≥ 1. In this case, the boundary of the moduli of curves with one positive puncture splits as a 'product' of disks in the symplectization of $(\partial X, \partial L)$, and curves without punctures in the interior of (X, L), see Proposition 9. Examples of this kind include the one we study here: $X = \mathbb{C}^3$ or the resolved conifold, and L a toric brane. Toric branes have the topology of a solid torus, and there is one topological type of such for each 'leg' of the toric diagram [3].

Let us clarify what is meant by product. Recall that the framed skein module $\mathrm{Sk}(L)$ is the quotient of framed isotopy classes of embedded links in L by the skein relations [27,28]. Note that $\mathrm{Sk}(\partial L) := \mathrm{Sk}\big(\partial L \times [0,1]\big)$ is an algebra, by concatenation of intervals. For a similar reason this algebra acts on $\mathrm{Sk}(L)$. The disks in the symplectization (after choosing capping paths) determine some element $\mathbf{A}_L \in \mathrm{Sk}(\partial L)$. The curves in the interior determine an element $\Psi_L \in \mathrm{Sk}(L)$. Using the above action we can consider $\mathbf{A}_L \Psi_L \in \mathrm{Sk}(L)$; the fact that it arises as a boundary gives us the equation $\mathbf{A}_L \Psi_L = 0$.

Thus if we can determine $\mathbf{A}_L$, and solve equations in the skein, we can recover Ψ_L. Determining $\mathbf{A}_L$ is essentially the subject of (a generalization of) Legendrian contact homology [11,12]; in the cases at hand, finding these curves is elementary.

Skein modules of 3-manifolds are generally complicated, but here L is a solid torus $\mathbf{T}$ and its skein is very well studied, see [25]. Denote by $\mathbb{T}$ the product of a torus and the interval; then $\mathrm{Sk}(\mathbb{T})$ acts on $\mathrm{Sk}(\mathbf{T})$. Fixing a choice of longitude on $\partial\mathbf{T}$, we denote by $P_{1,0}$ the meridian, $P_{0,1}$ the longitude, and more generally $P_{a,b}$ the curve of slope a/b for a, b relatively prime. The key point for us is that diagonalizing the action of $P_{0,1}$ on $\mathrm{Sk}(\mathbf{T})$ yields the basis corresponding to irreducible quantum group representations and indexed by pairs of partitions, see [25, Section 4]. In fact our invariants can be seen geometrically to live in the 'positive part' $\mathrm{Sk}^+(\mathbf{T})$ in which one of these partitions is empty. We denote the corresponding basis of this positive part by W_λ, see [4,19] for algebraic and geometric properties of these basis elements.

The only nontrivial facts we need from skein module theory are the following identities, which can be found in e.g., [22,24] (we use the notation of [25]:

$$P_{1,0}W_\lambda = \big(\bigcirc + a(q^{1/2}-q^{-1/2})c_\lambda(q)\big)\cdot W_\lambda, \tag{1}$$

$$P_{0,1}W_\lambda = \sum_{\lambda+\square=\mu} W_\mu. \tag{2}$$

Here, $\bigcirc$ is an unknot (circle bounding a disk and framed by the normal to the disk), and $c_\lambda(q)$ is the content polynomial of the partition λ. We recall basic notations for partitions (e.g., the content polynomial, the hook length, $\square\in\lambda$, and $\mu=\lambda+\square$) below in Section 4.

We now state our results.

Theorem 1. *Let L be a toric brane in $\mathbb{C}^3$. Then the skein-valued holomorphic curve count $\Psi\in\mathrm{Sk}^+(L)$ satisfies the relation*

$$(\bigcirc - P_{1,0} + a_L\gamma P_{0,1})\Psi = 0,$$

where a_L is the HOMFLY-PT framing variable of L (the framing variable is denoted a in the skein relation of Figure 1). Here γ is a signed monomial in the framing variable a_L, i.e., $\gamma=\pm a_L^p$, depending on framing choices. This equation has a unique solution of the form $1+\cdots$, namely:

$$\Psi = \sum_\lambda \gamma^{|\lambda|} W_\lambda \prod_{\square\in\lambda} \frac{q^{-c(\square)/2}}{q^{h(\square)/2}-q^{-h(\square)/2}},$$

where c denotes the content and h the hook-length. This is the skein-valued count of curves in $\mathbb{C}^3$ ending on L.

The proof of Theorem 1 consists in three parts: finding the geometric disks in the symplectization in Proposition 5; determining the coefficients with which they appear in Proposition 10, and finally solving the skein equation in Proposition 13. The resulting Ψ should be compared to [20, Theorem 7.1].

We treat similarly the case of the unknot conormal.

Theorem 2. *Let L be the unknot conormal in T^*S^3 and let σ be an orientation of S^3. Let $\Psi_\sigma\in\mathrm{Sk}^+(L)\otimes\mathrm{Sk}(S^3)$ denote the corresponding skein valued holomorphic curve count. Then, for one of the orientations σ:*

$$\big(\bigcirc - P_{1,0} + \gamma(a_L a P_{0,1} - a^{-1}P_{1,1})\big)\Psi_\sigma = 0 \quad\Longrightarrow\quad \Psi_\sigma = \sum_\lambda \gamma^{|\lambda|} W_\lambda \prod_{\square\in\lambda} \frac{aq^{-c(\square)/2}-a^{-1}q^{c(\square)/2}}{q^{h(\square)/2}-q^{-h(\square)/2}},$$

and for the opposite orientation $-\sigma$:

$$\big(\bigcirc - P_{1,0} - \gamma(a_L a^{-1} P_{0,1} - aP_{1,1})\big)\Psi_{-\sigma} = 0 \quad\Longrightarrow\quad \Psi_{-\sigma} = \sum_\lambda \gamma^{|\lambda|} W_\lambda \prod_{\square\in\lambda} \frac{aq^{c(\square)/2}-a^{-1}q^{-c(\square)/2}}{q^{h(\square)/2}-q^{-h(\square)/2}}.$$

Here γ is a signed monomial in the framing variables (a, a_L) (i.e., $\gamma=\pm a^p a_L^q$), depending on framing choices. The solutions $\Psi_{\pm\sigma}$ are interchanged by reversing the orientation of S^3 (and correspondingly taking $a\mapsto a^{-1}$ and $q^{1/2}\mapsto -q^{1/2}$), and give the skein valued curve counts.

We determine the symplectization disks for Theorem 2 in Proposition 7 (known by another method already in [11]), find their coefficients in Proposition 11, and then solve the skein equation in Proposition 14.

Remark 3. The substitution $Q = a^2$ in Theorem 2 gives the corresponding formula for curves in the resolved conifold ending on a toric brane on an external leg.

Remark 4. In [14,15], we count curves of Euler characteristic χ by $z^{-\chi}$. In this article we set $z = q^{1/2} - q^{-1/2}$. We do this because the eigenvalues of $P_{1,0}$ are naturally expressed in the variable q, but would be some complicated power series in the variable z. In particular, the poles in the formula of Theorem 1 imply the existence of bare curves of arbitrarily low Euler characteristic of any given non-minimal area.

2. Holomorphic curves in the symplectization

We determine the holomorphic curves with boundary on the Lagrangians under consideration in the $\mathbb{R}$-invariant regions of $\mathbb{C}^3$ and T^*S^3, respectively. We find them by viewing the contact manifold under consideration as a pre-quantization bundle and studying the Lagrangian projection of the torus. Alternatively, one can find the curves via Morse theory: draw the front of the Legendrian in suitable coordinate systems on the contact manifold and study flow trees, see Remarks 6 and 8.

Consider $\mathbb{C}^3$ with coordinates $(x_1 + iy_1, x_2 + iy_2, x_3 + iy_3)$ and symplectic form $\omega = \sum_{j=1}^3 \mathrm{d}x_j \wedge \mathrm{d}y_j$ as a Weinstein manifold with Liouville vector field $V = \sum_{j=1}^3 (x_j \partial_{x_j} + y_j \partial_{y_j})$. Then the ideal contact boundary of $\mathbb{C}^3$ is the standard contact 5-sphere S^5 with contact form $\sum_j (x_j\,\mathrm{d}y_j - y_j\,\mathrm{d}x_j)$. The Reeb vector field is then $i \cdot V$ and the Reeb flow is periodic with orbits given by intersections of S^5 with complex lines through the origin or in other words fibers of the Hopf fibration $\pi\colon S^5 \to \mathbb{C}P^2$. The toric Lagrangian in $\mathbb{C}^3$ is parameterized by $(\alpha, \beta, r) \in S^1 \times S^1 \times [0, \infty)$ by the formula $(\alpha, \beta, r) \mapsto \left((r^2+\delta)^{\frac{1}{2}} e^{i\alpha}, re^{i\beta}, re^{-i(\alpha+\beta+\frac{\pi}{2})}\right)$, for some fixed $\delta > 0$ [3]. The Lagrangian is asymptotic to a Legendrian torus we denote $\mathbb{T}_{AV} \subset S^5$.

Proposition 5. *After generic perturbation, the Legendrian torus $\mathbb{T}_{AV}$ has a single Reeb chord of index one, and all other Reeb chords of higher index. There are three rigid curves at infinity with one positive puncture asymptotic to this chord; all are disks and after appropriate choice of capping paths, the boundaries of these three curves are: (1) contractible, (2) the longitude, and (3) the meridian of $\mathbb{T}_{AV}$.*

Proof. Consider the Hopf map $\pi\colon S^5 \to \mathbb{C}P^2$. It acts as follows on $\mathbb{T}_{AV}$, parameterized by $(\alpha, \beta) \in (S^1)^2$ as above, in projective coordinates:

$$\pi(\alpha, \beta) = \left[e^{i(2\alpha+\beta)} : e^{i(2\beta+\alpha)} : i\right].$$

Thus the image $\pi(\mathbb{T}_{AV})$ is the Clifford torus in $\mathbb{C}P^2$, and the map has degree $\left|\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right| = 3$ and is a covering map. Thus, for any $\zeta \in \pi(\mathbb{T}_{AV})$, the pre-image $\pi^{-1}(\zeta)$ in $\mathbb{T}_{AV}$ consists of three points on a Hopf fiber, where the second and third points are obtained from the first by rotation $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$, respectively.

Thus, any point $p \in \mathbb{T}_{AV}$ is the starting point of infinitely many Reeb chords of $\mathbb{T}_{AV}$ of actions $k \cdot \frac{2\pi}{3}$, $k > 0$, and there are no other Reeb chords starting at p. It follows that any Reeb chord of $\mathbb{T}_{AV}$ has action $k \cdot \frac{2\pi}{3}$, some $k > 0$ and, furthermore, that the collection of Reeb chords of action $k_0 \cdot \frac{2\pi}{3}$ for fixed k_0 is a Bott family with Bott manifold $\mathbb{T}_{AV}$ (take the starting point of the chord as Bott parameter).

It is straightforward to check that the Maslov class of $\mathbb{T}_{AV}$ equals zero, that the first Bott family of Reeb chords of action $\frac{2\pi}{3}$, corresponding to $k_0 = 1$, has Conley–Zehnder index equal to 1, and that Bott families corresponding to $k_0 > 1$ have indices > 1. This means that moduli spaces of holomorphic curves in the symplectization $\mathbb{R} \times S^5 \approx \mathbb{C}^3 \setminus \{0\}$ with boundary on $\mathbb{R} \times \mathbb{T}_{AV}$ that are not trivial Reeb chord strips have dimension ≥ 1 and dimension $= 1$ only if the curve has exactly one positive puncture at a Reeb chord c of action $\frac{2\pi}{3}$.

Consider a holomorphic curve u in $\mathbb{C}^3 \setminus \{0\}$ with positive puncture at a fixed Reeb chord c of action $\frac{2\pi}{3}$. Then the projection of u, $\pi \circ u$, is a holomorphic curve in $\mathbb{C}P^2$ with boundary on the Clifford torus, where the boundary passes through the point $\pi(c)$. By Stokes' theorem the area of $\pi(c)$ equals one third of the area of $\mathbb{C}P^1$ and by monotonicity of the Clifford torus the Maslov index of the curve then equals 2. Consequently, holomorphic curves with positive puncture at c are lifts of Maslov index 2 curves with boundary on the Clifford torus which passes through a given point $\pi(c)$. For fixed $\zeta = \pi(c)$, the only such curves are the three disks depicted in Figure 2. The boundaries of these disks are well-known and easy to find from the S^1-actions. This then gives the three curves described above. □

Remark 6. It is also possible to find the disks in the proof of Proposition 5 via flow trees of Legendrian fronts. The Legendrian torus $\mathbb{T}_{AV}$ was studied in detail in [9] and the flow trees were determined in [8].

We turn to the unknot conormal, and write $\mathbb{T}_{\bigcirc}$ for the Legendrian conormal torus at infinity.

Proposition 7. *After generic perturbation, the Legendrian torus $\mathbb{T}_{\bigcirc}$ has a single Reeb chord of index one, and all other Reeb chords of higher index. There are four rigid curves at infinity; all disks, each with a positive puncture at the index one Reeb chord. After appropriate choice of capping paths, the boundaries of these four curves are: (1) contractible, (2) the longitude, (3) the meridian, and (4) the slope $(1,1)$ curve of $\mathbb{T}_{\bigcirc}$.*

Proof. Consider $ST^*\mathbb{R}P^3$ as the pre-quantization bundle of local $\mathbb{C}P^1 \times \mathbb{C}P^1$ and ST^*S^3 as its double cover. Here the Legendrian unknot conormal projects to the product (Clifford) torus in $\mathbb{C}P^1 \times \mathbb{C}P^1$ (which itself lifts to the Legendrian conormal of the real projective line) and is a double cover of this projection. As for the toric brane above, holomorphic curves project to holomorphic curves with boundary on the product torus and there are four such disks, see Figure 2 (right). The boundaries of these curves are evident from the moment map (and well known). We deduce the result by taking their lifts to $\mathbb{T}_{\bigcirc}$. □

Remark 8. Proposition 7 can also be proved using flow tree methods: represent ST^*S^3 as J^1S^2, the front of the unknot conormal and corresponding flow trees (and holomorphic disks) were described in [11].

Figure 2. Holomorphic curves at infinity can be seen from toric geometry: the Legendrian torus at infinity of $\mathbb{C}^3$ and T^*S^3 project 3-to-1 and 2-to-1 to the Clifford torus in $\mathbb{C}P^2$ and $\mathbb{C}P^1 \times \mathbb{C}P^1$, respectively. The red dot indicates the Clifford torus and the blue lines holomorphic disks in the moment polytopes of $\mathbb{C}P^2$ (left) and $\mathbb{C}P^1 \times \mathbb{C}P^1$ (right).

3. Relations from infinity

We study one-dimensional moduli spaces of disconnected holomorphic curves with boundary on a Lagrangian L in the symplectic manifold X where (X, L) has ideal contact boundary $(\partial X, \partial L)$. Our curves will have one positive puncture at an index one Reeb chord of ∂L at infinity. By SFT-compactness [5], this moduli space has (assuming transversality) two kinds of boundaries: (1) those arising from SFT-splittings into two level curves, an $\mathbb{R}$-invariant component in the symplectization joined at Reeb chords at its negative end to rigid curves with positive punctures in (X, L), and (2) those arising from boundary degenerations of holomorphic curves in (X, L). Exactly as in [14,15] the second type of boundary can be canceled by counting curves by their boundaries in the skein module of the L. Hence the first type of boundary must itself vanish, when counted appropriately (with signs) in the skein module. When the Legendrian boundary $\partial L \subset \partial X$ has the property that all Reeb chords have non-negative grading (as is the case for knot conormals in T^*S^3) then the curves in the $\mathbb{R}$-invariant region give equations in degree zero Reeb chords with coefficients in the skein. If the Reeb chords can be eliminated from this system of equations we obtain an element in the skein of $\partial L \times [0,1]$ that annihilates the element in $\mathrm{Sk}(L)$ given by all rigid curves.

In the cases at hand, there are no Reeb chords of degree zero, so no elimination is necessary. In addition the single Reeb chord of degree one has the least action among all Reeb chords, so we need not argue for transversality in order to invoke SFT compactness.

Proposition 9. *Let (X, L) be the toric brane in $\mathbb{C}^3$ or the unknot conormal in T^*S^3 as above. Then after arbitrarily small perturbation, all Reeb chords of ∂L have degrees ≥ 1 and there is a unique degree one Reeb chord c. The moduli space $\mathcal{M}(c)$ of holomorphic curves with positive puncture at c is one dimensional, and any SFT-type boundary must correspond to a single curve in the $\mathbb{R}$-invariant region, plus curves without punctures in (X, L).*

Proof. Since c has minimal action, no curve in $\mathcal{M}(c)$ can have any negative puncture. The result then follows by SFT-compactness. □

Let us write $\Psi \in \mathrm{Sk}(\mathbf{T})$ for the count in the skein of the interior curves and $\mathbf{A} \in \mathrm{Sk}(\mathbb{T})$ for the count in the skein of the outside disks. We have the equation $\mathbf{A}\Psi = 0$, where the product means the action of $\mathrm{Sk}(\mathbb{T})$ on $\mathrm{Sk}(\mathbf{T})$.

We distinguish the two cases above by writing $\Psi_{AV}, \mathbf{A}_{AV}$ for the toric brane in $\mathbb{C}^3$, and $\Psi_{\bigcirc}, \mathbf{A}_{\bigcirc}$ for the case of the unknot.

Above we have determined the holomorphic curves which contribute to $\mathbf{A}_{AV}, \mathbf{A}_{\bigcirc}$. We should also determine the coefficients of the corresponding terms. The geometric multiplicities are in each case ± 1, but we have not yet determined the sign. In the skein we must also remember the framing of the boundary and the related 4-chain intersection of the curves. In principle these could be computed directly; instead we will determine them from the equation $\mathbf{A}\Psi = 0$, using the first (easy to compute) term of Ψ. We write a_L for the framing variable in the skein of the toric brane or of the conormal of the unknot. We will use the filtration on the positive skein module $\mathrm{Sk}_+(\mathbf{T})$ given by homological degree. Note that the degree of a basis element W_λ in $\mathrm{Sk}(\mathbf{T})$ equals the number of boxes in the partition λ.

Proposition 10. *We have*

$$\Psi_{AV} = 1 + \frac{\gamma}{q^{1/2} - q^{-1/2}} W_{\square} + \mathcal{O}(2)$$

where $\gamma = \pm a_L^k$ is some signed power of a_L depending on framing choices. For the same choices,

$$\pm (a_L)^m \mathbf{A}_{AV} = \bigcirc - P_{1,0} + a_L \gamma P_{0,1},$$

for some integer m.

Proof. Let us first explain the formula for Ψ_{AV}. The zeroeth term in the count of disconnected curves is 1 by definition. The first term counts the embedded disk sitting above the corresponding 'leg' of the toric diagram. More precisely, the toric brane on the first leg is

$$(\alpha,\beta,r)\mapsto\left((r^2+\delta)^{\frac{1}{2}}e^{i\alpha},re^{i\beta},re^{-i(\alpha+\beta+\frac{\pi}{2})}\right),$$

and the disk is the radius $\sqrt{\delta}$-disk in the first complex coordinate line. The boundary of this disk is the longitude of the Lagrangian solid torus, hence gives $W_\square$. By definition disks are counted by $(q^{1/2}-q^{-1/2})^{-1}$. We absorb framing, 4-chain, and sign conventions in γ, which is a signed monomial in a_L.

For $\mathbf{A}_{AV}$, we know what disks must contribute; hence:

$$\mathbf{A}_{AV}=\pm a_L^{n_{0,0}}\bigcirc\pm a_L^{n_{1,0}}P_{1,0}\pm a_L^{n_{0,1}}P_{0,1}.$$

We rescale so the coefficient of the $\bigcirc$ term is 1:

$$\mathbf{A}_{AV}\sim\bigcirc+\gamma_{1,0}P_{1,0}+\gamma_{0,1}P_{0,1}.$$

Here the coefficients $\gamma_{i,j}$ are signed monomials in a_L. We solve for them using (1) and (2). Consider the zeroeth order term in $\mathbf{A}_{AV}\Psi_{AV}$, which $P_{0,1}$, as an $\mathcal{O}(1)$-term in $\mathrm{Sk}(\mathbb{T})$, cannot affect. This is $(1+\gamma_{1,0})\bigcirc$, so we find $\gamma_{1,0}=-1$. Now we have:

$$(\bigcirc-P_{1,0}+\gamma_{0,1}P_{0,1})\left(1+\gamma(q^{1/2}-q^{-1/2})^{-1}W_\square+\mathcal{O}(2)\right)=0.$$

After expanding the product, the coefficient of $W_\square$ is $\gamma_{0,1}-a_L\gamma$. We learn $\gamma_{0,1}=a_L\gamma$. □

Let us treat similarly the second case. We write a for the framing variable in S^3 and use filtration by homological degree in $\mathrm{Sk}(\mathbf{T})$.

Proposition 11. *We have*

$$\Psi_\bigcirc=1+\gamma\frac{a-a^{-1}}{q^{1/2}-q^{-1/2}}W_\square+\mathcal{O}(2)$$

where $\gamma=\pm a_L^k$ is some signed power of a_L depending on framing choices. For the same choices, one of the following holds:

$$\pm a_L^m\mathbf{A}_\bigcirc=\bigcirc-P_{1,0}+\gamma(a_LaP_{0,1}-a^{-1}P_{1,1})$$

or

$$\pm a_L^m\mathbf{A}_\bigcirc=\bigcirc-P_{1,0}+\gamma(-a_La^{-1}P_{0,1}+aP_{1,1}),$$

for some integer m. Note these possibilities differ by $a\mapsto-a^{-1}$ (which however will also change the value of γ).

Proof. Let us explain the formula for $\Psi_\bigcirc$. Here the first term arises because as we discuss in [14] there is a unique cylinder, whose boundary is the meridian in the conormal and the unknot in S^3. Evaluating the unknot in the skein of S^3 gives the term $\frac{a-a^{-1}}{q^{1/2}-q^{-1/2}}$, and γ contains any extra 4-chain intersections, framing of the knots, etc.

We have seen what disks contribute, so up to scalar multiple

$$\mathbf{A}_\bigcirc\sim\bigcirc+\gamma_{1,0}P_{1,0}+\gamma_{0,1}P_{0,1}+\gamma_{1,1}P_{1,1}.$$

Here the $\gamma_{i,j}$ are signed monomials in the framing variables. We solve for the $\gamma_{i,j}$ in terms of γ. Exactly as for the previous case, we learn from the degree zero term that $\gamma_{1,0}=-1$. Then from the $W_\square$ term we learn

$$\gamma a_L(a-a^{-1})=\gamma_{0,1}+a_L\gamma_{1,1}.$$

Recall the γ are all monomials. Thus either: (1) $\gamma_{0,1}=\gamma a_La$ and $\gamma_{1,1}=-\gamma a^{-1}$, or (2) $\gamma_{0,1}=-\gamma a_La^{-1}$ and $\gamma_{1,1}=\gamma a$. This yields the two possibilities stated in the proposition. □

4. Reminders of partition combinatorics

By a partition λ we mean a finite nonincreasing sequence of integers $\lambda_1 \geq \lambda_2 \geq \cdots$. We write $|\lambda| := \sum \lambda_i$, and $\ell(\lambda)$ for the 'number of parts' i.e. the number of nonzero λ_i. We discuss partitions in terms of their Young diagrams; see Figure 3. We discuss a square in the diagram by writing $\square \in \lambda$. Each square has an *arm*, *leg*, *coarm*, and *coleg* as depicted in Figure 3.

If λ is a partition and $\square \in \lambda$ then its *hook*, $\mathrm{Hook}_\lambda(\square)$, is the union of the square itself, and its arm and leg. The *hooklength* $h(\square)$ is the total number of boxes in the hook, i.e. the length of the arm plus the length of the leg plus one. The *content* $c(\square)$ is the length of the coarm minus that of the coleg. See Figure 4.

Figure 3. Left: The Young diagram of the partition $6+4+2$. Right: The arm, leg, coarm, and coleg of the black square are indicated in red, green, blue, and yellow, respectively.

0	1	2	3	4	5
−1	0	1	2		
−2	−1				

8	7	5	4	2	1
5	4	2	1		
2	1				

Figure 4. Left: In each square we write the value of $c(\square)$. The content polynomial is $q^{-2}+2q^{-1}+2+2q+2q^2+q^3+q^4+q^5$. Right: In each square we write the value of $h(\square)$.

As practice with these notions, and because we need it later, let us prove the following result.

Lemma 12. $\sum_{\square\in\lambda} c(\square)+h(\square)+1 \equiv 0 \pmod 2$.

Proof. The quantity in question is $\sum_{\square\in\lambda} \mathrm{coarm}(\square) - \mathrm{coleg}(\square) + \mathrm{arm}(\square) + \mathrm{leg}(\square)$, which mod 2 agrees with $\sum_{\square\in\lambda} \mathrm{coarm}(\square) + \mathrm{coleg}(\square) + \mathrm{arm}(\square) + \mathrm{leg}(\square)$. Each term is the length of the row containing the box plus the height of the column containing the box, hence we are summing the squares of the row lengths and column heights. This has the same parity as the sum of the row lengths and the column heights, which is $2|\lambda|$, hence zero mod 2. □

We need some q-numbers. We write $[n]_q := 1+q+\cdots+q^{n-1}$. Consider a partition λ. The *content polynomial* of λ is

$$c_\lambda(q) := \sum_{\square\in\lambda} q^{c(\square)}.$$

Note $c_\lambda(1) = |\lambda|$.

If $\square \in \lambda$ then its *q-hooklength* is

$$h_\square(q) := \sum_{\blacksquare\in\mathrm{Hook}_\lambda(\square)} q^{c(\blacksquare)},$$

where the sum runs over all squares $\blacksquare \in \mathrm{Hook}_\lambda(\square) \subset \lambda$. Note that $h_\square(1) = h(\square)$ and that in fact is some power of q times the quantum integer of the hook length, $h_\square(q) = q^{p_\square}[h(\square)]_q$. The *hook polynomial* is

$$h_\lambda(q) := \prod_{\square\in\lambda} h_\square(q) = q^{\sum_{\square\in\lambda} p_\square} \prod_{\square\in\lambda} [h_\square(1)]_q = \prod_{\square\in\lambda} q^{\frac{3}{2}c(\square)}[h_\square(1)]_q. \tag{3}$$

We write $\lambda + \square = \mu$ to indicate that μ is a partition whose Young diagram can be obtained by adding one box to that of λ. We will need the formula:

$$\frac{c_\mu(q)}{h_\mu(q)} = \sum_{\lambda+\square=\mu} \frac{q^{-c(\square)}}{h_\lambda(q)}, \tag{4}$$

where the content $c(\square)$ in the right-hand side is the content of $\square \in \lambda$.

This formula is the specialization of the 'weighted hook length branching rule' of [6] to $x_i = q$ and $y_j = q^{-1}$. (It is asserted there that some q-specialization, presumably this one, can be obtained from the branching rule for Hall–Littlewood polynomials on [23, p. 243].) Note at $q = 1$ and multiplied by $|\lambda|!$, this is just the usual branching rule for dimensions of representations of the symmetric group.

Actually we will use this formula after multiplying both sides by $q^{c(\mu)}$, where $c(\mu) = \sum_{\square\in\mu} c(\square)$. Then the formula reads:

$$c_\mu(q) \cdot \frac{q^{c(\mu)}}{h_\mu(q)} = \sum_{\lambda+\square=\mu} \frac{q^{c(\lambda)}}{h_\lambda(q)}. \tag{5}$$

5. Calculation

Proposition 13. *Any solution of* $(\bigcirc - P_{1,0} + a_L \gamma P_{0,1})\Psi = 0$ *is a scalar multiple of*

$$\Psi = \sum_\lambda \gamma^{|\lambda|} W_\lambda \prod_{\square\in\lambda} \frac{q^{-c(\square)/2}}{q^{h(\square)/2} - q^{-h(\square)/2}}$$

Proof. Note that $(P_{1,0} - \bigcirc)$ is diagonal in the W_λ basis, with eigenvalues proportional to the content polynomials. In particular it is invertible, and we may rewrite the equation as $\Psi = a_L\gamma(P_{1,0} - \bigcirc)^{-1} P_{0,1}\Psi$. Combining (1) and (2), we have:

$$a_L\gamma(P_{1,0} - \bigcirc)^{-1} P_{0,1} W_\lambda = \frac{\gamma}{(q^{1/2} - q^{-1/2})} \sum_{\lambda+\square=\mu} \frac{W_\mu}{c_\mu(q)}. \tag{6}$$

We write $\Psi = \sum_\lambda \psi_\lambda W_\lambda$. From (6), the equation for Ψ becomes the following recursion for its coefficients:

$$\psi_\mu = \frac{\gamma}{c_\mu(q)(q^{-1/2} - q^{1/2})} \sum_{\lambda+\square=\mu} \psi_\lambda. \tag{7}$$

In particular, we see that each ψ_μ is determined by the ψ_λ with fewer boxes, so the solution is unique up to scalar multiple.

It remains to show that our proposed Ψ satisfies the recursion (7). We rewrite the coefficients (using the last equality in (3)):

$$\psi_\nu = \gamma^{|\nu|} \prod_{\square\in\nu} \frac{q^{-c(\square)/2}}{q^{h(\square)/2} - q^{-h(\square)/2}} = \gamma^{|\nu|} \prod_{\square\in\nu} \frac{q^{-c(\square)/2}}{\left[h(\square)\right]_q (q^{1/2} - q^{-1/2})} = \frac{\gamma^{|\nu|}}{(q^{1/2} - q^{-1/2})^{|\nu|}} \frac{q^{c(\nu)}}{h_\nu(q)}.$$

After cancelling the appropriate powers of γ, $(q^{1/2} - q^{-1/2})$, the recursion (7) (and hence the proposition) now reduces to (5). $\square$

With only slightly more work we can also show the following.

Proposition 14. *Any solution to the equation* $\big(\bigcirc - P_{1,0} + \gamma(a_L a P_{0,1} - a^{-1} P_{1,1})\big)\Psi_\sigma = 0$ *is a scalar multiple of*

$$\Psi_\sigma = \sum_\lambda \gamma^{|\lambda|} W_\lambda \prod_{\square\in\lambda} \frac{a q^{-c(\square)/2} - a^{-1} q^{c(\square)/2}}{q^{h(\square)/2} - q^{-h(\square)/2}}.$$

Meanwhile any solution to the equation $\big(\bigcirc - P_{1,0} + \gamma(-a_L a^{-1} P_{0,1} + a P_{1,1})\big)\Psi_{-\sigma} = 0$ *is a scalar multiple of*

$$\Psi_{-\sigma} = \sum_\lambda \gamma^{|\lambda|} W_\lambda \prod_{\square\in\lambda} \frac{aq^{c(\square)/2} - a^{-1}q^{-c(\square)/2}}{q^{h(\square)/2} - q^{-h(\square)/2}}.$$

The products in the two cases are interchanged by taking $a \mapsto a^{-1}$ *and* $q^{1/2} \mapsto -q^{1/2}$.

Proof. We expand $\Psi := \Psi_\sigma$ in coefficients $\Psi = \sum_\lambda \psi_\lambda W_\lambda$. Again using invertibility of $(P_{1,0} - \bigcirc)^{-1}$ we rewrite the equation as

$$\Psi = \gamma(P_{1,0} - \bigcirc)^{-1}(a_L a P_{0,1} - a^{-1}P_{1,1})\Psi.$$

We abbreviate $\Omega := (P_{1,0} - \bigcirc)^{-1}(a_L a P_{0,1} - a^{-1}P_{1,1})$. From the skein relation we have:

$$(q^{1/2} - q^{-1/2})P_{1,1} = [P_{1,0}, P_{0,1}]$$

and noting moreover that $[P_{1,0}, P_{0,1}] = [P_{1,0} - \bigcirc, P_{0,1}]$, we compute

$$\Omega = a_L a(P_{1,0} - \bigcirc)^{-1}P_{0,1} - \frac{a^{-1}}{q^{1/2} - q^{-1/2}}\big(P_{0,1} - (P_{1,0} - \bigcirc)^{-1}P_{0,1}(P_{1,0} - \bigcirc)\big).$$

Again using (1) and (2), we have:

$$\begin{aligned}\Omega W_\lambda &= \frac{1}{q^{1/2} - q^{-1/2}} \sum_{\lambda+\square=\mu} W_\mu \cdot \left(\frac{a}{c_\mu(q)} - a^{-1}\left(1 - \frac{c_\lambda(q)}{c_\mu(q)}\right)\right) \\ &= \frac{1}{q^{1/2} - q^{-1/2}} \sum_{\lambda+\square=\mu} W_\mu \cdot \frac{a - a^{-1}q^{c(\square)}}{c_\mu(q)}.\end{aligned}$$

Thus Ψ is a solution if

$$\psi_\mu = \frac{\gamma}{c_\mu(q)(q^{-1/2} - q^{1/2})} \sum_{\lambda+\square=\mu} (a - a^{-1}q^{c(\square)})\psi_\lambda. \tag{8}$$

Using (3) we rearrange our proposed solution slightly:

$$\begin{aligned}\psi_\nu &= \gamma^{|\nu|} \prod_{\square\in\nu} \frac{aq^{-c(\square)/2} - a^{-1}q^{c(\square)/2}}{q^{h(\square)/2} - q^{-h(\square)/2}} \\ &= \gamma^{|\nu|} \prod_{\square\in\nu} \frac{aq^{-c(\square)/2} - a^{-1}q^{c(\square)/2}}{\big[h(\square)\big]_q (q^{1/2} - q^{-1/2})} \\ &= \frac{\gamma^{|\nu|}}{(q^{1/2} - q^{-1/2})^{|\nu|}} \frac{1}{h_\lambda(q)} \prod_{\square\in\nu} (aq^{c(\square)} - a^{-1}q^{2c(\square)}).\end{aligned}$$

It is now immediate from (5) that our proposed solution satisfies the recursion (8). The proof of the second formula, for $\Psi_{-\sigma}$, is similar.

It is clear that the formulas are interchanged by $a \mapsto a^{-1}$ and $q^{1/2} \mapsto -q^{1/2}$ up to a sign given by the parity of $\sum_{\square\in\lambda} c(\square) + h(\square) + 1$; which was computed in Lemma 12. □

Remark 15. Let us write $\langle W_\lambda(\bigcirc)\rangle \in \mathrm{Sk}(S^3)$ for the element in the skein corresponding to the W_λ cable of a standard unknot. It is well known that under the identification of $\mathrm{Sk}(S^3)$ with an appropriately localized polynomial ring in $a, q^{1/2}$, we have

$$\langle W_\lambda(\bigcirc)\rangle = \prod_{\square\in\lambda} \frac{aq^{c(\square)/2} - a^{-1}q^{-c(\square)/2}}{q^{h(\square)/2} - q^{-h(\square)/2}}.$$

Thus our formula may also be written

$$\Psi' = \sum_\lambda \gamma^{|\lambda|} W_\lambda \cdot \langle W_\lambda(\bigcirc)\rangle.$$

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Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

References

[1] M. Aganagic, T. Ekholm, L. Ng and C. Vafa, "Topological strings, D-model, and knot contact homology", *Adv. Theor. Math. Phys.* **18** (2014), no. 4, pp. 827–956.

[2] M. Aganagic, A. Klemm, M. Mariño and C. Vafa, "The topological vertex", *Commun. Math. Phys.* **254** (2005), no. 2, pp. 425–478.

[3] M. Aganagic and C. Vafa, "Mirror symmetry, D-branes and counting holomorphic discs", 2000. Online at https://arxiv.org/abs/hep-th/0012041.

[4] A. K. Aiston and H. R. Morton, "Idempotents of Hecke algebras of type A", *J. Knot Theory Ramifications* **7** (1998), no. 4, pp. 463–487.

[5] F. Bourgeois, Y. Eliashberg, H. Hofer, K. Wysocki and E. Zehnder, "Compactness results in symplectic field theory", *Geom. Topol.* **7** (2003), pp. 799–888.

[6] I. Ciocan-Fontanine, M. Konvalinka and I. Pak, "The weighted hook length formula", *J. Comb. Theory, Ser. A* **118** (2011), no. 6, pp. 1703–1717.

[7] D.-E. Diaconescu, V. Shende and C. Vafa, "Large N duality, Lagrangian cycles, and algebraic knots", *Commun. Math. Phys.* **319** (2013), no. 3, pp. 813–863.

[8] G. Dimitroglou Rizell, "Knotted Legendrian surfaces with few Reeb chords", *Algebr. Geom. Topol.* **11** (2011), no. 5, pp. 2903–2936.

[9] G. Dimitroglou Rizell and R. Golovko, "Legendrian submanifolds from Bohr–Sommerfeld covers of monotone Lagrangian tori", 2019. Online at https://arxiv.org/abs/1901.08415.

[10] T. Ekholm, "Knot contact homology and open Gromov–Witten theory", in *Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. II. Invited lectures* (B. Sirakov, P. N. de Souza and M. Viana, eds.), World Scientific, 2018, pp. 1063–1086.

[11] T. Ekholm, J. B. Etnyre, L. Ng and M. G. Sullivan, "Knot contact homology", *Geom. Topol.* **17** (2013), no. 2, pp. 975–1112.

[12] T. Ekholm, J. B. Etnyre and M. G. Sullivan, "Legendrian contact homology in $P \times \mathbb{R}$", *Trans. Am. Math. Soc.* **359** (2007), no. 7, pp. 3301–3335.

[13] T. Ekholm and L. Ng, "Higher genus knot contact homology and recursion for colored HOMFLY-PT polynomials", *Adv. Theor. Math. Phys.* **24** (2020), no. 8, pp. 2067–2145.

[14] T. Ekholm and V. Shende, "Skeins on branes", 2019. Online at https://arxiv.org/abs/1901.08027.

[15] T. Ekholm and V. Shende, "Counting bare curves", 2024. Online at https://arxiv.org/abs/2406.00890.

[16] C. Faber and R. Pandharipande, "Hodge integrals and Gromov–Witten theory", *Invent. Math.* **139** (2000), no. 1, pp. 173–199.

[17] R. Gopakumar and C. Vafa, "On the gauge theory/geometry correspondence", 1998. Online at https://arxiv.org/abs/hep-th/9811131.

[18] T. Graber and E. Zaslow, “Open-string Gromov–Witten invariants: calculations and a mirror theorem”, 2001. Online at https://arxiv.org/abs/hep-th/0109075.

[19] R. J. Hadji and H. R. Morton, “A basis for the full Homfly skein of the annulus”, *Math. Proc. Camb. Philos. Soc.* **141** (2006), no. 1, pp. 81–100.

[20] S. Katz and C.-C. M. Liu, “Enumerative geometry of stable maps with Lagrangian boundary conditions and multiple covers of the disc”, in *The interaction of finite-type and Gromov–Witten invariants (BIRS 2003)* (D. Auckly and J. Bryan, eds.), Geometry and Topology Monographs, vol. 8, Geometry and Topology Publications, 2006, pp. 1–47.

[21] C.-C. M. Liu, “Moduli of J-holomorphic curves with Lagrangian boundary condition and open Gromov–Witten invariants for an S^1-equivariant pair”, 2002. Online at https://arxiv.org/abs/math/0210257.

[22] S. G. Lukac, “Idempotents of the Hecke algebra become Schur functions in the skein of the annulus”, *Math. Proc. Camb. Philos. Soc.* **138** (2005), no. 1, pp. 79–96.

[23] I. G. Macdonald, *Symmetric functions and Hall polynomials.*, 2nd edition, Clarendon Press, 1998.

[24] H. R. Morton and S. G. Lukac, “The Homfly polynomial of the decorated Hopf link”, *J. Knot Theory Ramifications* **12** (2003), no. 3, pp. 395–416.

[25] H. R. Morton and P. Samuelson, “The HOMFLYPT skein algebra of the torus and the elliptic Hall algebra”, *Duke Math. J.* **166** (2017), no. 5, pp. 801–854.

[26] H. Ooguri and C. Vafa, “Knot invariants and topological strings”, *Nucl. Phys., B* **577** (2000), no. 3, pp. 419–438.

[27] J. H. Przytycki, “Skein modules of 3-manifolds”, *Bull. Pol. Acad. Sci., Math.* **39** (1991), no. 1–2, pp. 91–100.

[28] V. G. Turaev, “The Conway and Kauffman modules of a solid torus”, *Zap. Nauchn. Semin. Leningr. Otd. Mat. Inst. Steklova* **167** (1988), pp. 79–89, 190.

[29] E. Witten, “Chern–Simons gauge theory as a string theory”, in *The Floer memorial volume* (H. Hofer, C. H. Taubes, A. Weinstein and E. Zehnder, eds.), Progress in Mathematics, vol. 133, Birkhäuser, 1995, pp. 637–678.