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Balancedness of complete cohomology with respect to subcategories

Propriété d'équilibre de la cohomologie complète par rapport aux sous-catégories

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Abstract. We study the balancedness of complete cohomology with respect to a special preenveloping/precovering subcategory. As an application, a balanced result for Tate cohomology is given.

Résumé. Dans cet article, nous étudions la propriété d'équilibre de la cohomologie complète par rapport à une sous-catégorie spéciale pré-enveloppante/pré-couvrante. Comme application, nous démontrons une propriété d'équilibre pour la cohomologie de Tate.

Keywords. Satellite, balancedness, complete cohomology, Tate cohomology.

Mots-clés. Satellite, équilibre, cohomologie complète, cohomologie de Tate.

2020 Mathematics Subject Classification. 18G25, 16E30, 16E05.

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Introduction

Tate cohomology can be traced back to the work of Tate in 1952 (see [16]), which was initially defined for representations of finite groups. Avramov and Martsinkovsky [1] extended the definition so that it can work well for finitely generated modules of finite Gorenstein dimension over a noetherian ring. Sather-Wagstaff, Sharif and White [15] further investigated Tate cohomology of objects admitting a Tate resolution in an abelian category.

As a broad generalization of Tate cohomology to the realm of infinite group algebras or even associative rings, complete cohomology was introduced by Vogel and Goichot [9], Mislin [11] and Benson and Carlson [3] independently, and was further treated by Avramov and Veliche [2] and Nucinkis [12]. Complete cohomology is an important subject in representation theory, commutative algebra and homological algebra. We mention that the 0-th complete cohomology group $\widehat{\mathrm{Ext}}^0_R(M,N)$ for R-modules M and N is actually the group of homomorphisms of objects in a stabilization of the category of R-modules, so Avramov and Veliche referred to complete cohomology as stable cohomology to emphasize this fact; see [2]. Recently, the first two authors of this paper introduced and studied the complete cohomology with respect to a special preenveloping/precovering subcategory via stable functors in [10], where some properties including vanishing were given. This paper is a follow-up to [10]; we aim to investigate the balancedness of the relative complete cohomology.

We notice that the balancedness of the relative complete cohomology is given relying on the concept of balanced pairs introduced by Chen in [5]; the name "balanced pair" was settled to emphasize the capacity to induce balancedness in relative cohomology groups; see Remark 15. Many examples of balanced pairs are from complete hereditary cotorsion triplets, as it was proved by Estrada, Pérez and Zhu [7, Proposition 4.2] that if $(\mathsf{W},\mathsf{Z},\mathsf{V})$ is a complete hereditary cotorsion triplet then (W,V) is a balanced pair. In the following we give more details on the main results in the paper.

Throughout this paper, R denotes an associative unital ring. By an R-module we mean a left R-module. The category of R-modules is denoted R-Mod. We use the term "subcategory" to mean a full and additive subcategory that is closed under isomorphisms.

Let V be a special preenveloping subcategory of R-Mod and W a special precovering subcategory of R-Mod, and let M and N be R-modules. We defined in [10, Definitions 3.4 and 3.7] the complete cohomology groups $\widetilde{\mathrm{Ext}}^n_{\mathsf{V}}(M,N)$ and $\widehat{\mathrm{Ext}}^n_{\mathsf{W}}(M,N)$ as

$$\widetilde{\mathrm{Ext}}^n_{\mathsf{V}}(M,N) = \mathrm{colim}_i\, \overline{\mathrm{Hom}}^{\mathsf{V}}_R(\Theta^i_{\mathsf{V}}M, \Theta^{i+n}_{\mathsf{V}}N)$$

and

$$\widehat{\mathrm{Ext}}^n_{\mathsf{W}}(M,N) = \mathrm{colim}_i\, \underline{\mathrm{Hom}}^{\mathsf{W}}_R(\Omega^{\mathsf{W}}_{i+n}M, \Omega^{\mathsf{W}}_iN).$$

In the paper, we study the relation between the group $\widetilde{\mathrm{Ext}}^n_{\mathsf{V}}(M,N)$ and the group $\widehat{\mathrm{Ext}}^n_{\mathsf{W}}(M,N)$, and prove the next result; see Theorems 19 and 22.

Main Theorem. *Let* (W,V) *be a balanced pair with* W *special precovering and* V *special preenveloping. Assume that* W *and* V *are closed under direct summands and* $\mathrm{Ext}^{\geq 1}_R(\mathsf{W},\mathsf{W}^{\perp}) = 0 = \mathrm{Ext}^{\geq 1}_R({}^{\perp}\mathsf{V},\mathsf{V})$. *Then the following statements are equivalent.*

- (i) *All R-modules in* V *have finite* W*-projective dimension and all R-modules in* W *have finite* V*-injective dimension.*
- (ii) $\mathsf{W}\text{-pd}(\mathsf{V}) = \mathsf{V}\text{-id}(\mathsf{W}) < \infty$.
- (iii) *Each R-module has a Tate* V*-coresolution.*
- (iv) *Each R-module has a Tate* W*-resolution.*
- (v) *For all R-modules M and N and each $n \in \mathbb{Z}$, there is a natural isomorphism*

$$\widetilde{\mathrm{Ext}}^n_{\mathsf{V}}(M,N) \cong \widehat{\mathrm{Ext}}^n_{\mathsf{W}}(M,N).$$

(vi) *For each R-module M in V or W, there is a natural isomorphism*

$$\widetilde{\mathrm{Ext}}^0_{\mathsf{V}}(M,M) \cong \widehat{\mathrm{Ext}}^0_{\mathsf{W}}(M,M).$$

Here, $\mathsf{W}\text{-}\mathrm{pd}(\mathsf{V}) = \sup\{\mathsf{W}\text{-}\mathrm{pd}_R(C) \mid C \in \mathsf{V}\}$ and $\mathsf{V}\text{-}\mathrm{id}(\mathsf{W}) = \sup\{\mathsf{V}\text{-}\mathrm{id}_R(D) \mid D \in \mathsf{W}\}$. The definitions of W-projective dimension $\mathsf{W}\text{-}\mathrm{pd}_R$ and V-injective dimension $\mathsf{V}\text{-}\mathrm{id}_R$ are given in Remark 1.

The above theorem is proved relying on the completion of cohomology sequences studied in Section 2 and some vanishing results given in [10]. As an application, we give a new balanced result for the Tate cohomology; it is proved that under the equivalent conditions in Main Theorem there is a natural isomorphism $\widehat{\mathrm{Ext}}^n_{\mathsf{W}R}(M,N) \cong \widehat{\mathrm{Ext}}^n_{R\mathsf{V}}(M,N)$ (see Theorem 22).

1. Preliminaries

In this section we mainly recall some necessary notions and definitions.

Special preenveloping/precovering subcategories. Given a subcategory X of R-Mod, we write

$${}^{\perp}\mathsf{X} = \{M \mid \mathrm{Ext}^1_R(M,X) = 0 \text{ for all } X \in \mathsf{X}\} \qquad \text{and} \qquad \mathsf{X}^{\perp} = \{N \mid \mathrm{Ext}^1_R(X,N) = 0 \text{ for all } X \in \mathsf{X}\}.$$

Here $\mathrm{Ext}^1_R(-,-)$ is the 1st right derived functor of $\mathrm{Hom}_R(-,-)$. Following Enochs and Jenda [6], a *special* X*-preenvelope* of an R-module N is an exact sequence $0 \to N \to X \to C \to 0$ with $X \in \mathsf{X}$ and $C \in {}^{\perp}\mathsf{X}$. Dually, a *special* X*-precover* of an R-module M is an exact sequence $0 \to K \to X' \to M \to 0$ with $K \in \mathsf{X}^{\perp}$ and $X' \in \mathsf{X}$. Recall that a subcategory X of R-Mod is *special preenveloping* if each R-module has a special X-preenvelope. Dually a subcategory X of R-Mod is called *special precovering* if each R-module has a special X-precover. It is clear that the subcategory Inj of injective R-modules is special preenveloping, and the subcategory Prj of projective R-modules is special precovering.

Proper (co)resolutions. Let Y be a subcategory of R-Mod. A *proper* Y*-coresolution* of an R-module N is a complex I of R-modules in Y such that $I^{-n} = 0 = \mathrm{H}^n(I)$ for all $n > 0$ and $\mathrm{H}^0(I) \cong N$, and the associated exact sequence

$$I^+ \equiv 0 \longrightarrow N \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

is $\mathrm{Hom}_R(-,\mathsf{Y})$-exact (that is, it remains exact after applying the functor $\mathrm{Hom}_R(-,Y)$ to it for each $Y \in \mathsf{Y}$), which is always denoted $N \xrightarrow{\simeq} I$.

Let X be a subcategory of R-Mod. A *proper* X*-resolution* of an R-module M is a complex P of R-modules in X such that $P_{-n} = 0 = \mathrm{H}^n(P)$ for all $n > 0$ and $\mathrm{H}^0(P) \cong M$, and the associated exact sequence

$$P^+ \equiv \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

is $\mathrm{Hom}_R(\mathsf{X},-)$-exact, which is always denoted $P \xrightarrow{\simeq} M$.

Remark 1 (Dimensions and relative cohomology). The Y*-injective dimension* of an R-module N is the quantity

$$\mathsf{Y}\text{-}\mathrm{id}_R(N) = \inf\Big\{\sup\{n \geq 0 \mid I_n \neq 0\} \mid N \xrightarrow{\simeq} I \text{ is a proper } \mathsf{Y}\text{-coresolution of } N\Big\}.$$

When Y is the subcategory of injectives, $\mathsf{Y}\text{-}\mathrm{id}_R(N)$ is the classical injective dimension.

Let N be an R-module with $N \xrightarrow{\simeq} I$ a proper Y-coresolution. Then for each R-module M and every $i \in \mathbb{Z}$, the i-th *relative* Y*-cohomology* of N with coefficients in M is defined as

$$\mathrm{Ext}^i_{R\mathsf{Y}}(M,N) = \mathrm{H}^i\big(\mathrm{Hom}_R(M,I)\big).$$

Specially, if X is the subcategory of injectives, then $\mathrm{Ext}^i_{R\mathsf{Y}}(M,N)$ is actually the cohomology group $\mathrm{Ext}^i_R(M,N)$.

Dually, one has the definition of X*-projective dimension*, $\mathsf{X}\text{-}\operatorname{pd}_R(M)$, of an R-module M. Also, for R-modules M and N with $P \xrightarrow{\simeq} M$ a proper X-resolution, the i-th *relative* X*-cohomology* of M with coefficients in N is defined as

$$\operatorname{Ext}^i_{\mathsf{X}R}(M,N) = \mathrm{H}^i\big(\operatorname{Hom}_R(P,N)\big).$$

The next result can be found in [5, Lemma 2.4].

Lemma 2. *Let M be an R-module. Then for each $n \geq 0$, the following statements are equivalent.*

(i) $\mathsf{X}\text{-}\operatorname{pd}_R(M) \leq n$.
(ii) $\operatorname{Ext}^i_{\mathsf{X}R}(M,-) = 0$ *for all $i > n$.*
(iii) *For each proper* X*-resolution $P \xrightarrow{\simeq} M$,* $\operatorname{Coker}(P_{n+1} \to P_n)$ *is in* X.

Dually, one has the following result.

Lemma 3. *Let N be an R-module. Then for each $n \geq 0$, the following statements are equivalent.*

(i) $\mathsf{Y}\text{-}\operatorname{id}_R(N) \leq n$.
(ii) $\operatorname{Ext}^i_{R\mathsf{Y}}(-,N) = 0$ *for all $i > n$.*
(iii) *For each proper* Y*-coresolution $N \xrightarrow{\simeq} I$,* $\operatorname{Ker}(I^n \to I^{n+1})$ *is in* Y.

(Co)Syzygies. A proper Y-coresolution $N \xrightarrow{\simeq} I$ of an R-module N is called *special* if each $\mathrm{Z}^i(I) = \operatorname{Ker}(I^i \to I^{i+1})$ is in ${}^{\perp}\mathsf{Y}$ for $i \geq 1$. We let $\Theta^i_{\mathsf{Y}}N$ denote the kernel $\mathrm{Z}^i(I)$ for some special proper Y-coresolution $N \xrightarrow{\simeq} I$; it is always called the i-th Y*-cosyzygy* of N.

Dually, a proper X-resolution $P \xrightarrow{\simeq} M$ of an R-module M is called *special* if each $\mathrm{C}_i(P) = \operatorname{Coker}(P_{i+1} \to P_i)$ is in $\mathsf{X}^{\perp}$ for $i \geq 1$. We let $\Omega^{\mathsf{X}}_i M$ denote the cokernel $\mathrm{C}_i(P)$ for some special proper X-resolution $P \xrightarrow{\simeq} M$; it is always called the i-th X*-syzygy* of M.

Remark 4. We always set $\Theta_{\mathsf{Y}}N = \Theta^1_{\mathsf{Y}}N$ and $\Omega^{\mathsf{X}}M = \Omega^{\mathsf{X}}_1 M$. The fact that $\Theta^i_{\mathsf{Y}}N$ is in ${}^{\perp}\mathsf{Y}$ and $\Omega^{\mathsf{X}}_i M$ is in $\mathsf{X}^{\perp}$ for each $i \geq 1$ is used frequently in the paper.

Setup. Throughout this paper, the symbol W denotes a special precovering subcategory of R-Mod, and the symbol V denotes a special preenveloping subcategory of R-Mod. In this case, every R-module has a special proper W-resolution and a special proper V-coresolution.

2. Satellites and completion of cohomology sequences

In this section, we give a description of the completion of cohomology sequences via satellites of functors. Throughout the section, F (resp., G) denotes a contravariant (resp., covariant) additive functor from R-Mod to the abelian group category Ab.

Left satellites. For each R-module M, there is a special V-preenvelope

$$0 \longrightarrow M \longrightarrow I \xrightarrow{\pi} \Theta_{\mathsf{V}}M \longrightarrow 0$$

of M with $I \in \mathsf{V}$ and $\Theta_{\mathsf{V}}M \in {}^{\perp}\mathsf{V}$. Following Cartan and Eilenberg [4], the first *left satellite* of F with respect to V, denoted $\mathrm{S}^{-1}_{\mathsf{V}}\,\mathrm{F}$, is defined as $\mathrm{S}^{-1}_{\mathsf{V}}\,\mathrm{F}(M) = \operatorname{Ker}\mathrm{F}(\pi)$. Then $\mathrm{S}^{-1}_{\mathsf{V}}\,\mathrm{F}$ is a contravariant additive functor from R-Mod to Ab, and it is independent of the choices of special V-preenvelopes.

We set $\mathrm{S}^{-n}_{\mathsf{V}}\,\mathrm{F} = \mathrm{S}^{-1}_{\mathsf{V}}(\mathrm{S}^{-n+1}_{\mathsf{V}}\,\mathrm{F})$ for each $n > 0$, and set $\mathrm{S}^{0}_{\mathsf{V}}\,\mathrm{F} = \mathrm{F}$.

Dually, for each R-module M, there is a special W-precover

$$0 \longrightarrow \Omega^{\mathsf{W}}M \xrightarrow{\epsilon} P \longrightarrow M \longrightarrow 0$$

of M with $P \in \mathsf{W}$ and $\Omega^{\mathsf{W}}M \in \mathsf{W}^{\perp}$. The first *left satellite* of G with respect to W, denoted $\mathrm{S}^{-1}_{\mathsf{W}}\,\mathrm{G}$, is defined as $\mathrm{S}^{-1}_{\mathsf{W}}\,\mathrm{G}(M) = \operatorname{Ker}\mathrm{G}(\epsilon)$. Then $\mathrm{S}^{-1}_{\mathsf{W}}\,\mathrm{G}$ is a covariant additive functor from R-Mod to Ab, and it is independent of the choices of special W-precovers.

We set $\mathrm{S}^{-n}_{\mathsf{W}}\,\mathrm{G} = \mathrm{S}^{-1}_{\mathsf{W}}(\mathrm{S}^{-n+1}_{\mathsf{W}}\,\mathrm{G})$ for each $n > 0$, and set $\mathrm{S}^{0}_{\mathsf{W}}\,\mathrm{G} = \mathrm{G}$.

Remark 5. Since $\mathrm{S}_{\mathsf{V}}^{-n}\mathrm{F}(I) = 0$ for each $I \in \mathsf{V}$ and any $n > 0$, there is a natural isomorphism

$$\mathrm{S}_{\mathsf{V}}^{-n}\mathrm{F}(M) \cong \mathrm{S}_{\mathsf{V}}^{-n+k}\mathrm{F}(\Theta_{\mathsf{V}}^{k}M)$$

for $n > k \geq 0$. Similarly, since $\mathrm{S}_{\mathsf{W}}^{-n}\mathrm{G}(P) = 0$ for each $P \in \mathsf{W}$ and all $n > 0$, there is a natural isomorphism

$$\mathrm{S}_{\mathsf{W}}^{-n}\mathrm{G}(M) \cong \mathrm{S}_{\mathsf{W}}^{-n+k}\mathrm{G}(\Omega_{k}^{\mathsf{W}}M)$$

for $n > k \geq 0$.

The functor F is called *half* V*-exact* if for each short exact sequence $0 \to A' \to A \to A'' \to 0$ of R-modules with $A'' \in {}^{\perp}\mathsf{V}$, the sequence $\mathrm{F}(A'') \to \mathrm{F}(A) \to \mathrm{F}(A')$ is exact. The functor G is called *half* W*-exact* if for each short exact sequence $0 \to A' \to A \to A'' \to 0$ of R-modules with $A' \in \mathsf{W}^{\perp}$, the sequence $\mathrm{G}(A') \to \mathrm{G}(A) \to \mathrm{G}(A'')$ is exact.

Lemma 6. *The following statements hold.*

(a) *If* F *is half* V*-exact, then so is* $\mathrm{S}_{\mathsf{V}}^{-i}\mathrm{F}(-)$ *for each* $i \geq 1$.
(b) *If* G *is half* W*-exact, then so is* $\mathrm{S}_{\mathsf{W}}^{-i}\mathrm{G}(-)$ *for each* $i \geq 1$.

Proof. We only prove (a); the statement (b) is proved similarly.

It is enough to prove that $\mathrm{S}_{\mathsf{V}}^{-1}\mathrm{F}(-)$ is half V-exact. Let

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be an exact sequence of R-modules with $M'' \in {}^{\perp}\mathsf{V}$. Fix special V-preenvelopes $0 \to M' \to I' \to \Theta_{\mathsf{V}}M' \to 0$ and $0 \to M'' \to I'' \to \Theta_{\mathsf{V}}M'' \to 0$. It follows from [6, Remark 8.2.2] that there is a commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I' & \longrightarrow & I' \oplus I'' & \longrightarrow & I'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Theta_{\mathsf{V}}M' & \longrightarrow & \Theta_{\mathsf{V}}M & \longrightarrow & \Theta_{\mathsf{V}}M'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where the middle column is a special V-preenvelope of M, as $\Theta_{\mathsf{V}}M'$ and $\Theta_{\mathsf{V}}M''$ are in ${}^{\perp}\mathsf{V}$ (see Remark 4). Applying the functor F to the above diagram, one gets the next commutative diagram with columns exact:

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{S}_{\mathsf{V}}^{-1}\mathrm{F}(M'') & \longrightarrow & \mathrm{S}_{\mathsf{V}}^{-1}\mathrm{F}(M) & \longrightarrow & \mathrm{S}_{\mathsf{V}}^{-1}\mathrm{F}(M') \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{F}(\Theta_{\mathsf{V}}M'') & \longrightarrow & \mathrm{F}(\Theta_{\mathsf{V}}M) & \longrightarrow & \mathrm{F}(\Theta_{\mathsf{V}}M') \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{F}(I'') & \longrightarrow & \mathrm{F}(I'') \oplus \mathrm{F}(I') & \longrightarrow & \mathrm{F}(I').
\end{array}$$

The second non-zero row is exact as F is half V-exact, and the third one is exact clearly. Thus by [4, III. Lemma 3.2] the sequence $\mathrm{S}_{\mathsf{V}}^{-1}\mathrm{F}(M'') \to \mathrm{S}_{\mathsf{V}}^{-1}\mathrm{F}(M) \to \mathrm{S}_{\mathsf{V}}^{-1}\mathrm{F}(M')$ is exact. □

The following result is a relative version of [4, Theorem 3.1]. We mention that there is a mistake in the proof of [4, Theorem 3.1], which was discovered and fixed by Flanders in [8]. More precisely, an error occurred while proving the exactness of the sequence $\mathrm{S}_1\mathrm{T}(A'') \to \mathrm{T}(A') \to \mathrm{T}(A)$ in [4, Theorem 3.1]. The authors claimed that the connecting homomorphism $\mathrm{S}_1\mathrm{T}(A'') \to \mathrm{T}(M)$ obtained by the Snake Lemma coincides with the inclusion map $\mathrm{S}_1\mathrm{T}(A'') \hookrightarrow \mathrm{T}(M)$. Flanders analysed the reason in [8] for the inconsistency of the above two homomorphisms; the main reason is that the homomorphisms γ and μ in the commutative diagram in [8, p. 834] are not the same. Furthermore, Flanders proved directly the exactness of the sequence $\mathrm{S}_1\mathrm{T}(A'') \to \mathrm{T}(A') \to \mathrm{T}(A)$ using the diagram chasing. For the convenience of the readers, we provide a detailed proof for the next result.

Proposition 7. *The following statements hold.*

(a) *Let* F *be half* V*-exact. If* $0 \to M' \to M \to M'' \to 0$ *is a short exact sequence of* R*-modules with* $M'' \in {}^{\perp}\mathsf{V}$*, then there is an exact sequence*

$$\cdots \longrightarrow \mathrm{S}_{\mathsf{V}}^{-1}\mathrm{F}(M) \longrightarrow \mathrm{S}_{\mathsf{V}}^{-1}\mathrm{F}(M') \longrightarrow \mathrm{F}(M'') \longrightarrow \mathrm{F}(M) \longrightarrow \mathrm{F}(M').$$

(b) *Let* G *be half* W*-exact. If* $0 \to N' \to N \to N'' \to 0$ *is a short exact sequence of* R*-modules with* $N' \in \mathsf{W}^{\perp}$*, then there is an exact sequence*

$$\cdots \longrightarrow \mathrm{S}_{\mathsf{W}}^{-1}\mathrm{G}(N) \longrightarrow \mathrm{S}_{\mathsf{W}}^{-1}\mathrm{G}(N'') \longrightarrow \mathrm{G}(N') \longrightarrow \mathrm{G}(N) \longrightarrow \mathrm{G}(N'').$$

Proof. We only prove (a); the statement (b) is proved similarly. The proof of the exactness of the sequence $\mathrm{S}_{\mathsf{V}}^{-1}\mathrm{F}(M') \to \mathrm{F}(M'') \to \mathrm{F}(M)$ is similar to the one in [8]. We use a slightly different proof to prove the exactness of the sequence $\mathrm{S}_{\mathsf{V}}^{-1}\mathrm{F}(M) \to \mathrm{S}_{\mathsf{V}}^{-1}\mathrm{F}(M') \to \mathrm{F}(M'')$, which seems more direct to us.

Fix a special V-preenvelope $0 \to M \to I \to \Theta_{\mathsf{V}}M \to 0$, and consider the following pushout diagram:

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M' & \xrightarrow{f} & M & \xrightarrow{g} & M'' & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\tau} & & \\ 0 & \longrightarrow & M' & \xrightarrow{\mu'} & I & \xrightarrow{\upsilon'} & C' & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle\upsilon} & & \downarrow{\scriptstyle\tau'} & & \\ & & & & \Theta_{\mathsf{V}}M & = & \Theta_{\mathsf{V}}M & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0. & & \end{array} \qquad (1)$$

Since M'' and $\Theta_{\mathsf{V}}M$ are in ${}^{\perp}\mathsf{V}$, so is C', which yields that the exact sequence $0 \to M' \to I \to C' \to 0$ is a special V-preenvelope of M'. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \xrightarrow{f} & M & \xrightarrow{g} & M'' & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\tau} & & \\ 0 & \longrightarrow & M' & \xrightarrow{\mu'} & I & \xrightarrow{\upsilon'} & C' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \| & & \downarrow{\scriptstyle\tau'} & & \\ 0 & \longrightarrow & M & \xrightarrow{\mu} & I & \xrightarrow{\upsilon} & \Theta_{\mathsf{V}}M & \longrightarrow & 0. \end{array} \qquad (2)$$

Applying the functor F to the diagram (2), one gets the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{S}_{\mathrm{V}}^{-1}\,\mathrm{F}(M) & \xrightarrow{i} & \mathrm{F}(\Theta_{\mathrm{V}} M) & \xrightarrow{\mathrm{F}(v)} & \mathrm{F}(I) & \\
& \Big\downarrow{\scriptstyle \mathrm{S}_{\mathrm{V}}^{-1}\mathrm{F}(f)} & & \Big\downarrow{\scriptstyle \mathrm{F}(\tau')} & & \Big\| & \\
0 \longrightarrow & \mathrm{S}_{\mathrm{V}}^{-1}\,\mathrm{F}(M') & \xrightarrow{k'} & \mathrm{F}(C') & \xrightarrow{\mathrm{F}(v')} & \mathrm{F}(I) & \quad (3)\\
& \Big\downarrow{\scriptstyle \lambda} & & \Big\downarrow{\scriptstyle \mathrm{F}(\tau)} & & \Big\downarrow{\scriptstyle \mathrm{F}(\mu)} & \\
0 \longrightarrow & \mathrm{Ker}\,\mathrm{F}(g) & \xrightarrow{k} & \mathrm{F}(M'') & \xrightarrow{\mathrm{F}(g)} & \mathrm{F}(M), &
\end{array}$$

where $\mathrm{S}_{\mathrm{V}}^{-1}\,\mathrm{F}(f)$ and λ are obtained by the universal property of kernels. Thus one has a homomorphism $\Upsilon_1 = k\lambda\colon \mathrm{S}_{\mathrm{V}}^{-1}\,\mathrm{F}(M') \to \mathrm{F}(M'')$, whose composition with $\mathrm{F}(g)$ is zero. It follows from the diagram (3) that $\Upsilon_1 \mathrm{S}_{\mathrm{V}}^{-1}\,\mathrm{F}(f) = k\lambda \mathrm{S}_{\mathrm{V}}^{-1}\,\mathrm{F}(f) = \mathrm{F}(\tau)\mathrm{F}(\tau')i = \mathrm{F}(\tau'\tau)i=0$. Thus the sequence

$$\mathrm{S}_{\mathrm{V}}^{-1}\,\mathrm{F}(M) \xrightarrow{\mathrm{S}_{\mathrm{V}}^{-1}\mathrm{F}(f)} \mathrm{S}_{\mathrm{V}}^{-1}\,\mathrm{F}(M') \xrightarrow{\Upsilon_1} \mathrm{F}(M'') \xrightarrow{\mathrm{F}(g)} \mathrm{F}(M)$$

is a complex. In the following we prove that this sequence is actually exact.

Consider the following commutative diagram with all exact columns:

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
\mathrm{S}_{\mathrm{V}}^{-1}\,\mathrm{F}(M) & \xrightarrow{\mathrm{S}_{\mathrm{V}}^{-1}\mathrm{F}(f)} & \mathrm{S}_{\mathrm{V}}^{-1}\,\mathrm{F}(M') & \xrightarrow{\Upsilon_1} & \mathrm{F}(M'') \\
\Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle k'} & & \Big\| \\
\mathrm{F}(\Theta_{\mathrm{V}} M) & \xrightarrow{\mathrm{F}(\tau')} & \mathrm{F}(C') & \xrightarrow{\mathrm{F}(\tau)} & \mathrm{F}(M'') \\
\Big\downarrow{\scriptstyle \mathrm{F}(v)} & & \Big\downarrow{\scriptstyle \mathrm{F}(v')} & & \Big\downarrow \\
\mathrm{F}(I) & = & \mathrm{F}(I) & \longrightarrow & 0.
\end{array}$$

The second non-zero row is exact as the functor F is half V-exact. It follows from [4, III. Lemma 3.2] that the sequence

$$\mathrm{S}_{\mathrm{V}}^{-1}\,\mathrm{F}(M) \xrightarrow{\mathrm{S}_{\mathrm{V}}^{-1}\mathrm{F}(f)} \mathrm{S}_{\mathrm{V}}^{-1}\,\mathrm{F}(M') \xrightarrow{\Upsilon_1} \mathrm{F}(M'')$$

is exact. Next we prove that the sequence

$$\mathrm{S}_{\mathrm{V}}^{-1}\,\mathrm{F}(M') \xrightarrow{\Upsilon_1} \mathrm{F}(M'') \xrightarrow{\mathrm{F}(g)} \mathrm{F}(M)$$

is exact. It suffices to show $\mathrm{Ker}\,\mathrm{F}(g) \subseteq \mathrm{Im}\,\Upsilon_1$. Consider the next pushout diagram:

$$\begin{array}{ccccccccc}
& & 0 & & 0 & & & & \\
& & \Big\downarrow & & \Big\downarrow & & & & \\
0 & \longrightarrow & M' & \xrightarrow{f} & M & \xrightarrow{g} & M'' & \longrightarrow & 0 \\
& & \Big\downarrow{\scriptstyle \mu'} & & \Big\downarrow{\scriptstyle \varphi} & & \Big\| & & \\
0 & \longrightarrow & I & \xrightarrow{\alpha} & I \sqcup_{M'} M & \xrightarrow{\beta} & M'' & \longrightarrow & 0 \qquad (4)\\
& & \Big\downarrow{\scriptstyle v'} & & \Big\downarrow{\scriptstyle \psi} & & & & \\
& & C' & = & C' & & & & \\
& & \Big\downarrow & & \Big\downarrow & & & & \\
& & 0 & & 0. & & & &
\end{array}$$

Define $\delta\colon M'' \to I \sqcup_{M'} M$ by $\delta(m'') = \big[\big(-\mu(m), m\big)\big]_{\mathrm{Im}(-\mu', f)}$, where m is in M such that $g(m) = m''$. Then it is clear that δ is a homomorphism such that $\beta\delta = \mathrm{id}_{M''}$. Moreover, by the diagram (2) one has $\psi\delta(m'') = v'\big(-\mu(m)\big) = -\tau g(m) = -\tau(m'')$, which yields that $\psi\delta = -\tau$. We are now ready to prove $\mathrm{Ker}\,\mathrm{F}(g) \subseteq \mathrm{Im}\,\Upsilon_1$. For each element $a \in \mathrm{F}(M'')$ such that $\mathrm{F}(g)(a) = 0$. By the diagram (4) one has $0=\mathrm{F}(g)(a)=\mathrm{F}(\beta\varphi)(a)=\mathrm{F}(\varphi)\mathrm{F}(\beta)(a)$, it follows that $\mathrm{F}(\beta)(a)\in\operatorname{Ker}\mathrm{F}(\varphi)=\operatorname{Im}\mathrm{F}(\psi)$, as the sequence

$$\mathrm{F}(C')\xrightarrow{\mathrm{F}(\psi)}\mathrm{F}(I\sqcup_{M'}M)\xrightarrow{\mathrm{F}(\varphi)}\mathrm{F}(M)$$

is exact. So there is an element $b\in\mathrm{F}(C')$ such that $\mathrm{F}(\psi)(b)=\mathrm{F}(\beta)(a)$. Furthermore, one has $\mathrm{F}(\delta)\mathrm{F}(\psi)(b)=\mathrm{F}(\delta)\mathrm{F}(\beta)(a)$, which yields $a=-\mathrm{F}(\tau)(b)$ as $\beta\delta=\mathrm{id}_{M''}$ and $\psi\delta=-\tau$. Since $\mathrm{F}(v')(b)=\mathrm{F}(\psi\alpha)(b)=\mathrm{F}(\alpha)\mathrm{F}(\psi)(b)=\mathrm{F}(\alpha)\mathrm{F}(\beta)(a)=0$, one has $b\in\operatorname{Ker}\mathrm{F}(v')=\operatorname{Im}k'$ by diagram (3), it follows that there is an element $c\in\mathrm{S}_{\mathsf{V}}^{-1}\mathrm{F}(M')$ satisfying $b=k'(c)$. Consequently, $a=-\mathrm{F}(\tau)(b)=-\mathrm{F}(\tau)\big(k'(c)\big)=-\Upsilon_1(c)\in\operatorname{Im}\Upsilon_1$, so one obtains $\operatorname{Ker}\mathrm{F}(g)\subseteq\operatorname{Im}\Upsilon_1$, as desired. Hence one gets an exact sequence

$$\mathrm{S}_{\mathsf{V}}^{-1}\mathrm{F}(M)\xrightarrow{\mathrm{S}_{\mathsf{V}}^{-1}\mathrm{F}(f)}\mathrm{S}_{\mathsf{V}}^{-1}\mathrm{F}(M')\xrightarrow{\Upsilon_1}\mathrm{F}(M'')\xrightarrow{\mathrm{F}(g)}\mathrm{F}(M),$$

which yields that the sequence

$$\mathrm{S}_{\mathsf{V}}^{-1}\mathrm{F}(M)\xrightarrow{\mathrm{S}_{\mathsf{V}}^{-1}\mathrm{F}(f)}\mathrm{S}_{\mathsf{V}}^{-1}\mathrm{F}(M')\xrightarrow{\Upsilon_1}\mathrm{F}(M'')\xrightarrow{\mathrm{F}(g)}\mathrm{F}(M)\xrightarrow{\mathrm{F}(f)}\mathrm{F}(M')$$

is exact as F is half V-exact. We notice that $\mathrm{S}_{\mathsf{V}}^{-i}\mathrm{F}(-)$ is half V-exact for each $i\geq 1$ by Lemma 6, then one gets the exact sequence in the statement. □

Connected (cohomology) sequences. A family $\mathrm{F}^*=\{\mathrm{F}^n\mid n\in\mathbb{Z}\}$ of contravariant additive functors from R-Mod to Ab is called a V-*connected sequence* if for each short exact sequence $0\to A'\to A\to A''\to 0$ of R-modules with $A''\in{}^{\perp}\mathsf{V}$, the sequence

$$\cdots\longrightarrow\mathrm{F}^n(A'')\longrightarrow\mathrm{F}^n(A)\longrightarrow\mathrm{F}^n(A')\xrightarrow{\delta}\mathrm{F}^{n+1}(A'')\longrightarrow\cdots\tag{5}$$

of abelian groups is a complex. If furthermore the sequence (5) is exact, then F^* is called a V-*cohomology sequence*.

Dually, one has the definition of W-*connected/cohomology* sequences.

Mislin and Nucinkis introduced the concepts of P-completions and I-completions of cohomology sequence of functors in [11] and [12], respectively. In the following we introduce the relative versions of these concepts.

Definition 8. *Let* $\mathrm{F}^*=\{\mathrm{F}^n\mid n\in\mathbb{Z}\}$ *be a* V*-cohomology sequence of contravariant functors. The* V-completion *of* F^* *is a* V*-cohomology sequence* $\check{\mathrm{F}}^*=\{\check{\mathrm{F}}^n\mid n\in\mathbb{Z}\}$ *of contravariant functors together with a morphism* $\tau\colon\mathrm{F}^*\to\check{\mathrm{F}}^*$ *satisfying the following conditions:*

- $\check{\mathrm{F}}^n(C)=0$ *for each R-module* $C\in\mathsf{V}$ *and all* $n\in\mathbb{Z}$*;*
- *if* $\mathrm{T}^*=\{\mathrm{T}^n\mid n\in\mathbb{Z}\}$ *is a* V*-cohomology sequence of contravariant functors satisfying* $\mathrm{T}^n(C)=0$ *for each* $C\in\mathsf{V}$ *and all* $n\in\mathbb{Z}$*, and if* $v\colon\mathrm{F}^*\to\mathrm{T}^*$ *is a morphism, then there exists a unique morphism* $\sigma\colon\check{\mathrm{F}}^*\to\mathrm{T}^*$ *such that* $\sigma\tau=v$*.*

Similarly, for a W*-cohomology sequence* $\mathrm{G}^*=\{\mathrm{G}^n\mid n\in\mathbb{Z}\}$ *of covariant functors one has the definition of* W-completion $\varsigma\colon\mathrm{G}^*\to\widehat{\mathrm{G}}^*$.

The next lemma is proved similarly as in [12, Proposition 2.3]. For the sake of readability, we provide its proof here.

Lemma 9. *Let* $\mathrm{F}^{\leq 0}=\{\mathrm{F}^n\mid n\leq 0\}$ *and* $\mathrm{T}^{\leq 0}=\{\mathrm{T}^n\mid n\leq 0\}$ *be* V*-connected sequences of contravariant functors and* $\phi^0\colon\mathrm{F}^0\to\mathrm{T}^0$ *a natural transformation. If* $\mathrm{T}^{\leq 0}$ *is a* V*-cohomology sequence and satisfies* $\mathrm{T}^{-m}(C)=0$ *for each R-module* $C\in\mathsf{V}$ *and all* $m>0$*, then the following statements hold:*

(a) ϕ^0 *extends uniquely to* $\phi^{\leq 0}\colon\mathrm{F}^{\leq 0}\to\mathrm{T}^{\leq 0}$ *and* $\phi^{\leq 0}$ *factors uniquely through the canonical morphism* $\mathrm{F}^{\leq 0}\to\mathrm{S}_{\mathsf{V}}^{\leq 0}\mathrm{F}^0$*;*

(b) *if* F^0 *is half* V*-exact and* ϕ^0 *is an equivalence, then the induced morphism* $\mathrm{S}_{\mathsf{V}}^{\leq 0}\mathrm{F}^0\to\mathrm{T}^{\leq 0}$ *is an equivalence.*

Proof. (a). We proceed by induction on n. Suppose that natural transformations $\phi^q\colon \mathrm{F}^q \to \mathrm{T}^q$ are already defined for $n < q \leq 0$, and commute with the connecting homomorphisms. Let M be an R-module and $0 \to M \to I \to \Theta_{\mathsf{V}} M \to 0$ a special V-preenvelope. Then one gets the following commutative diagram:

$$\begin{array}{ccccccc} & & \mathrm{F}^n(M) & \longrightarrow & \mathrm{F}^{n+1}(\Theta_{\mathsf{V}} M) & \longrightarrow & \mathrm{F}^{n+1}(I) \\ & & \Big\downarrow{\scriptstyle \phi^n_M} & & \Big\downarrow & & \Big\downarrow \\ 0 & \longrightarrow & \mathrm{T}^n(M) & \longrightarrow & \mathrm{T}^{n+1}(\Theta_{\mathsf{V}} M) & \longrightarrow & \mathrm{T}^{n+1}(I), \end{array}$$

where the top row is a complex and the bottom row is exact and ϕ^n_M is obtained by the universal property of kernels. Moreover, it is easy to check that $\phi^n\colon \mathrm{F}^n \to \mathrm{T}^n$ is a natural transformation.

Now we prove that ϕ^n commutes with the connecting homomorphisms. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of R-modules with $M'' \in {}^{\perp}\mathsf{V}$ and $0 \to M' \to I' \to \Theta_{\mathsf{V}} M' \to 0$ a special V-preenvelope. Consider the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \Big\| & & \Big\downarrow & & \Big\downarrow & & \\ 0 & \longrightarrow & M' & \longrightarrow & I' & \longrightarrow & \Theta_{\mathsf{V}} M' & \longrightarrow & 0. \end{array}$$

This yields a commutative diagram:

$$\begin{array}{ccccc} \mathrm{F}^n(M') & \longrightarrow & \mathrm{F}^{n+1}(\Theta_{\mathsf{V}} M') & \longrightarrow & \mathrm{F}^{n+1}(M'') \\ \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\ \mathrm{T}^n(M') & \longrightarrow & \mathrm{T}^{n+1}(\Theta_{\mathsf{V}} M') & \longrightarrow & \mathrm{T}^{n+1}(M''). \end{array}$$

Hence the desired commutativity relation follows.

In the following we prove that $\phi^{\leq 0}\colon \mathrm{F}^{\leq 0} \to \mathrm{T}^{\leq 0}$ factors uniquely through the canonical morphism $\delta^{\leq 0}\colon \mathrm{F}^{\leq 0} \to \mathrm{S}^{\leq 0}_{\mathsf{V}}\mathrm{F}^0$. It suffices to show that $\phi^{-1}\colon \mathrm{F}^{-1} \to \mathrm{T}^{-1}$ factors uniquely through the canonical morphism $\delta^{-1}\colon \mathrm{F}^{-1} \to \mathrm{S}^{-1}_{\mathsf{V}}\mathrm{F}^0$. For each R-module M and a special V-preenvelope $0 \to M \to I \to \Theta_{\mathsf{V}} M \to 0$, one has the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{S}^{-1}_{\mathsf{V}}\mathrm{F}^0(M) & \longrightarrow & \mathrm{F}^0(\Theta_{\mathsf{V}} M) & \longrightarrow & \mathrm{F}^0(I) \\ & & \Big\downarrow{\scriptstyle \lambda^{-1}_M} & & \Big\downarrow & & \Big\downarrow \\ 0 & \longrightarrow & \mathrm{T}^{-1}(M) & \longrightarrow & \mathrm{T}^0(\Theta_{\mathsf{V}} M) & \longrightarrow & \mathrm{T}^0(I), \end{array}$$

where λ^{-1}_M is obtained by the universal property of kernels. Similarly, it is easy to check that $\lambda^{-1}\colon \mathrm{F}^{-1} \to \mathrm{S}^{-1}_{\mathsf{V}}\mathrm{F}^0$ is a natural transformation and satisfies $\varphi^{-1} = \lambda^{-1}\delta^{-1}$. Finally, the conclusion follows by induction.

(b). It follows directly from Proposition 7 and the Five Lemma in view of the special V-preenvelope $0 \to M \to I \to \Theta_{\mathsf{V}} M \to 0$ of an R-module M. □

Dually, we have the following result that can be found in [11, Theorem 1.1].

Lemma 10. *Let* $\mathrm{G}^{\leq 0} = \{\mathrm{G}^n \mid n \leq 0\}$ *and* $\mathrm{U}^{\leq 0} = \{\mathrm{U}^n \mid n \leq 0\}$ *be* W*-connected sequences of covariant functors and* $\varphi^0\colon \mathrm{G}^0 \to \mathrm{U}^0$ *a natural transformation. If* $\mathrm{G}^{\leq 0}$ *is a* W*-cohomology sequence and satisfies* $\mathrm{G}^{-m}(D) = 0$ *for each* R*-module* $D \in \mathsf{W}$ *and all* $m > 0$*, then the following statements hold:*

(a) φ^0 *extends uniquely to* $\varphi^{\leq 0}\colon \mathrm{G}^{\leq 0} \to \mathrm{U}^{\leq 0}$ *and* $\varphi^{\leq 0}$ *factors uniquely through the canonical morphism* $\mathrm{G}^{\leq 0} \to \mathrm{S}^{\leq 0}_{\mathsf{W}}\mathrm{G}^0$*;*

(b) *if* G^0 *is half* W*-exact and* φ^0 *is an equivalence, then the induced morphism* $\mathrm{S}^{\leq 0}_{\mathsf{W}}\mathrm{G}^0 \to \mathrm{U}^{\leq 0}$ *is an equivalence.*

Remark 11 (Construction). We fix $n \in \mathbb{Z}$. Let $\mathrm{F}^* = \{\mathrm{F}^n \mid n \in \mathbb{Z}\}$ be a V-connected sequence of contravariant functors and M an R-module. For each $k \geq 1$, the exact sequence $0 \to \Theta_{\mathsf{V}}^k M \to I^k \to \Theta_{\mathsf{V}}^{k+1} M \to 0$ yields a complex

$$\mathrm{F}^{n+k}(\Theta_{\mathsf{V}}^k M) \xrightarrow{\delta} \mathrm{F}^{n+k+1}(\Theta_{\mathsf{V}}^{k+1} M) \longrightarrow \mathrm{F}^{n+k+1}(I^k).$$

The connecting homomorphism δ induces a homomorphism from $\mathrm{F}^{n+k}(\Theta_{\mathsf{V}}^k M)$ to the kernel $\mathrm{S}_{\mathsf{V}}^{-1}\mathrm{F}^{n+k+1}(\Theta_{\mathsf{V}}^k M) \cong \mathrm{S}_{\mathsf{V}}^{-(k+1)}\mathrm{F}^{n+k+1}(M)$; see Remark 5. Composed with the natural embedding from $\mathrm{S}_{\mathsf{V}}^{-1}\mathrm{F}^{n+k}(\Theta_{\mathsf{V}}^{k-1} M) \cong \mathrm{S}_{\mathsf{V}}^{-k}\mathrm{F}^{n+k}(M)$ to $\mathrm{F}^{n+k}(\Theta_{\mathsf{V}}^k M)$, it yields a homomorphism

$$\delta\colon \mathrm{S}_{\mathsf{V}}^{-k}\mathrm{F}^{n+k}(M) \longrightarrow \mathrm{S}_{\mathsf{V}}^{-(k+1)}\mathrm{F}^{n+k+1}(M).$$

Similarly, for a W-connected sequence of covariant functors $\mathrm{G}^* = \{\mathrm{G}^n \mid n \in \mathbb{Z}\}$, one has a homomorphism

$$\partial\colon \mathrm{S}_{\mathsf{W}}^{-k}\mathrm{G}^{n+k}(M) \longrightarrow \mathrm{S}_{\mathsf{W}}^{-(k+1)}\mathrm{G}^{n+k+1}(M)$$

for each R-module M and any $k \geq 1$.

The following result is proved similarly as in [11, Theorem 2.2] (see also [12, Theorem 2.5]).

Theorem 12. *The following statements hold.*

(a) *Each* V*-cohomology sequence* $\mathrm{F}^* = \{\mathrm{F}^n \mid n \in \mathbb{Z}\}$ *of contravariant functors admits a unique* V*-completion* $\check{\mathrm{F}}^* = \{\check{\mathrm{F}}^n \mid n \in \mathbb{Z}\}$ *with*

$$\check{\mathrm{F}}^n(M) = \operatorname{colim}_i \mathrm{S}_{\mathsf{V}}^{-i}\mathrm{F}^{n+i}(M)$$

for each R*-module* M*.*

(b) *Each* W*-cohomology sequence* $\mathrm{G}^* = \{\mathrm{G}^n \mid n \in \mathbb{Z}\}$ *of covariant functors admits a unique* W*-completion* $\widehat{\mathrm{G}}^* = \{\widehat{\mathrm{G}}^n \mid n \in \mathbb{Z}\}$ *with*

$$\widehat{\mathrm{G}}^n(N) = \operatorname{colim}_j \mathrm{S}_{\mathsf{W}}^{-j}\mathrm{G}^{n+j}(N)$$

for each R*-module* N*.*

Here, the homomorphisms in the direct systems are provided in Remark 11.

Proof. We only prove (a); the statement (b) is proved similarly.

Since F^* is a V-cohomology sequence of contravariant functors, one gets a V-cohomology sequence $\mathrm{S}_{\mathsf{V}}^{\leq 0}\mathrm{F}^n$ by Proposition 7 for each $n \in \mathbb{Z}$, and it can be extended to a V-cohomology sequence $\mathrm{F}^*\langle n\rangle$ by setting

$$\mathrm{F}^j\langle n\rangle = \begin{cases} \mathrm{S}_{\mathsf{V}}^{j-n}\mathrm{F}^n, & \text{if } j < n,\\ \mathrm{F}^n, & \text{if } j \geq n.\end{cases}$$

Then the identity morphism $\mathrm{F}^n \to \mathrm{F}^n$ induces a unique morphism $\mathrm{F}^{\leq n} \to \mathrm{S}_{\mathsf{V}}^{\leq 0}\mathrm{F}^n$ and we extend it further to $\tau_n^*\colon \mathrm{F}^* \to \mathrm{F}^*\langle n\rangle$ with $\tau_n^j = \mathrm{id}_{\mathrm{F}^j}$ for all $j \geq n$. Similarly, for all $m \geq n$, the identity morphism $\mathrm{F}^m \to \mathrm{F}^m$ extends uniquely to a morphism $\tau_{n,m}^*\colon \mathrm{F}^*\langle n\rangle \to \mathrm{F}^*\langle m\rangle$ with $\tau_{n,m}^j = \mathrm{id}_{\mathrm{F}^j}$ for all $j \geq m$. Hence we now define

$$\check{\mathrm{F}}^* = \operatorname{colim}\{\mathrm{F}^*\langle n\rangle \mid \tau_{n,m}^*\}.$$

Since $\tau_{n,m}^*\tau_n^* = \tau_m^*$ for $m \geq n$, there is a natural morphism

$$\tau^* = \operatorname{colim} \tau_n^*\colon \mathrm{F}^* \longrightarrow \check{\mathrm{F}}^*.$$

The exactness of colimits implies that $\check{\mathrm{F}}^*$ is a V-cohomology sequence. So for each R-module M, one has

$$\check{\mathrm{F}}^j(M) = \operatorname{colim}_i \mathrm{S}_{\mathsf{V}}^{-i}\mathrm{F}^{i+j}(M), \quad i \geq 0$$

such that $\check{\mathrm{F}}^j(C) = 0$ for each R-module $C \in \mathsf{V}$ and all $j \in \mathbb{Z}$, as $\mathrm{S}_{\mathsf{V}}^{-i}\mathrm{F}^{i+j}(C) = 0$ for $i > 0$. Next, we show the universal property of τ^*. Let $\mathrm{F}^* \to \mathrm{T}^*$ be any morphism with T^* a V-cohomological sequence of contravariant functors and $\mathrm{T}^n(C) = 0$ for each R-module $C \in \mathsf{V}$ and any $n \in \mathbb{Z}$. Then each $\mathrm{F}^n \to \mathrm{T}^n$ extends uniquely to $\mathrm{S}_{\mathsf{V}}^{\leq 0}\mathrm{F}^n \to \mathrm{S}_{\mathsf{V}}^{\leq 0}\mathrm{T}^n$, and one has $\mathrm{S}_{\mathsf{V}}^{\leq 0}\mathrm{T}^n \cong \mathrm{T}^{\leq n}$ for the identity morphism $\mathrm{T}^n \to \mathrm{T}^n$ by Lemma 9. Consequently, one gets a unique morphism $\check{\mathrm{F}}^* \to \mathrm{T}^*$ and the morphism $\mathrm{F}^* \to \mathrm{T}^*$ factors uniquely through τ^*. The uniqueness of the V-completion is immediately obtained by its definition. □

The next corollary is immediate by the uniqueness of completion.

Corollary 13. *The following statements hold.*

(a) *If* F^* *is a* V*-cohomology sequence of contravariant functors satisfying* $\mathrm{F}^n(C) = 0$ *for each* $C \in \mathsf{V}$ *and all* $n \in \mathbb{Z}$*, then there is a natural isomorphism* $\mathrm{F}^* \cong \check{\mathrm{F}}^*$.

(b) *If* G^* *is a* W*-cohomology sequence of covariant functors satisfying* $\mathrm{G}^n(D) = 0$ *for each* $D \in \mathsf{W}$ *and all* $n \in \mathbb{Z}$*, then there is a natural isomorphism* $\mathrm{G}^* \cong \widehat{\mathrm{G}}^*$.

Remark 14. Let M and N be R-modules, and let $\mathrm{F}^* = \mathrm{Ext}_R^*(-,N)$ and $\mathrm{G}^* = \mathrm{Ext}_R^*(M,-)$. It follows from Theorem 12 that the V-completion of F^* and W-completion of G^* are

$$\check{\mathrm{F}}^* = \{\check{\mathrm{F}}^n = \mathrm{colim}_i\, \mathrm{S}_{\mathsf{V}}^{-i}\, \mathrm{Ext}_R^{n+i}(-,N) \mid n \in \mathbb{Z}\}$$

and

$$\widehat{\mathrm{G}}^* = \{\widehat{\mathrm{G}}^n = \mathrm{colim}_j\, \mathrm{S}_{\mathsf{W}}^{-j}\, \mathrm{Ext}_R^{n+j}(M,-) \mid n \in \mathbb{Z}\},$$

respectively. In [10, Definition 3.4], we defined a contravariant additive functor $\widetilde{\mathrm{Ext}}_{\mathsf{V}}^n(-,N)$ as

$$\widetilde{\mathrm{Ext}}_{\mathsf{V}}^n(M,N) = \mathrm{colim}_i\, \overline{\mathrm{Hom}}_R^{\mathsf{V}}(\Theta_{\mathsf{V}}^i M, \Theta_{\mathsf{V}}^{i+n} N)$$

for each R-module M; it is actually the n-th *complete cohomology* of M and N with respect to V. We also defined in [10, Definition 3.7] a covariant additive functor $\widehat{\mathrm{Ext}}_{\mathsf{W}}^n(M,-)$ as

$$\widehat{\mathrm{Ext}}_{\mathsf{W}}^n(M,N) = \mathrm{colim}_i\, \underline{\mathrm{Hom}}_R^{\mathsf{W}}(\Omega_{i+n}^{\mathsf{W}} M, \Omega_i^{\mathsf{W}} N)$$

for each R-module N; it is actually the n-th *complete cohomology* of M and N with respect to W. If $\mathrm{Ext}_R^{\geqslant 1}({}^{\perp}\mathsf{V},\mathsf{V}) = 0 = \mathrm{Ext}_R^{\geqslant 1}(\mathsf{W},\mathsf{W}^{\perp})$, then it follows from [10, Proposition 3.12] that the functors $\widetilde{\mathrm{Ext}}_{\mathsf{V}}^*(-,N)$ and $\widehat{\mathrm{Ext}}_{\mathsf{W}}^*(M,-)$ are actually the V-completion of $\mathrm{Ext}_R^*(-,N)$ and the W-completion of $\mathrm{Ext}_R^*(M,-)$, respectively.

3. Balancedness of relative complete cohomology

In this section, we study the relation between the complete cohomology groups $\widetilde{\mathrm{Ext}}_{\mathsf{V}}^n(M,N)$ and $\widehat{\mathrm{Ext}}_{\mathsf{W}}^n(M,N)$ given in Remark 14.

Balanced pair. Following [5], a pair (X,Y) of subcategories of R-Mod is called a *balanced pair* if the following conditions hold:

- X is precovering and Y is preenveloping;
- for each R-module M, there is a proper X-resolution $P \to M$ such that it is $\mathrm{Hom}_R(-,\mathsf{Y})$-exact;
- for each R-module N, there is a proper Y-coresolution $N \to I$ such that it is $\mathrm{Hom}_R(\mathsf{X},-)$-exact.

Remark 15. If (X,Y) is a balanced pair in R-Mod, then by [5, Lemma 2.1] there is a natural isomorphism $\mathrm{Ext}_{\mathsf{X}R}^n(M,N) \cong \mathrm{Ext}_{R\mathsf{Y}}^n(M,N)$ for all R-modules M and N, and each $n \geq 0$.

Lemma 16. *Let* (W,V) *be a balanced pair. Then for all* R*-modules* M *and* N *the following statements hold.*

(a) *The family* $\widetilde{\mathrm{Ext}}_{\mathsf{V}}^*(M,-)$ *is a* W*-cohomology sequence of covariant functors.*

(b) *The family* $\widehat{\mathrm{Ext}}_{\mathsf{W}}^*(-,N)$ *is a* V*-cohomology sequence of contravariant functors.*

Proof. We only prove (a); the statement (b) is proved similarly.

Let $0 \to N' \to N \to N'' \to 0$ be a short exact sequence of R-modules with $N' \in \mathsf{W}^{\perp}$. Then it is $\mathrm{Hom}_R(\mathsf{W},-)$-exact. Since (W,V) is a balanced pair, by [5, Proposition 2.2] the above sequence is also $\mathrm{Hom}_R(-,\mathsf{V})$-exact. Thus by [10, Proposition 4.7] the sequence

$$\cdots \longrightarrow \overset{\smile}{\mathrm{Ext}}{}^n_{\mathsf{V}}(M,N') \longrightarrow \overset{\smile}{\mathrm{Ext}}{}^n_{\mathsf{V}}(M,N) \longrightarrow \overset{\smile}{\mathrm{Ext}}{}^n_{\mathsf{V}}(M,N'') \longrightarrow \overset{\smile}{\mathrm{Ext}}{}^{n+1}_{\mathsf{V}}(M,N') \longrightarrow \cdots$$

is exact. This yields that $\overset{\smile}{\mathrm{Ext}}{}^*_{\mathsf{V}}(M,-)$ is a W-cohomology sequence. □

Let M and N be R-modules. For the functor $\mathrm{F} = \mathrm{Ext}^i_R(-,N)$, the value of the left satellite functor $\mathrm{S}^{-n}_{\mathsf{V}}\mathrm{F}$ at M, $\mathrm{S}^{-n}_{\mathsf{V}}\mathrm{F}(M)$, is always denoted $\mathrm{S}^{-n}_{\mathsf{V}}\mathrm{Ext}^i_R(M,N)$. For the functor $\mathrm{G} = \mathrm{Ext}^i_R(M,-)$, the value of the left satellite functor $\mathrm{S}^{-n}_{\mathsf{W}}\mathrm{G}$ at N, $\mathrm{S}^{-n}_{\mathsf{W}}\mathrm{G}(N)$, is always denoted $\mathrm{S}^{-n}_{\mathsf{W}}\mathrm{Ext}^i_R(M,N)$. Thus $\mathrm{S}^{-j}_{\mathsf{W}}\mathrm{S}^{-i}_{\mathsf{V}}\mathrm{Ext}^n_R(M,N)$ denotes $\mathrm{S}^{-j}_{\mathsf{W}}\big(\mathrm{S}^{-i}_{\mathsf{V}}\mathrm{Ext}^n_R(M,-)\big)(N)$ and $\mathrm{S}^{-i}_{\mathsf{V}}\mathrm{S}^{-j}_{\mathsf{W}}\mathrm{Ext}^n_R(M,N)$ denotes $\mathrm{S}^{-i}_{\mathsf{V}}\big(\mathrm{S}^{-j}_{\mathsf{W}}\mathrm{Ext}^n_R(-,N)\big)(M)$.

The next result is a relative version of [4, III. Theorem 7.1].

Lemma 17. *For all R-modules M and N, and each $i,j,n \geq 0$, there is a natural isomorphism*

$$\mathrm{S}^{-j}_{\mathsf{W}}\mathrm{S}^{-i}_{\mathsf{V}}\mathrm{Ext}^n_R(M,N) \cong \mathrm{S}^{-i}_{\mathsf{V}}\mathrm{S}^{-j}_{\mathsf{W}}\mathrm{Ext}^n_R(M,N).$$

Proof. It is enough to prove $\mathrm{S}^{-1}_{\mathsf{W}}\mathrm{S}^{-1}_{\mathsf{V}}\mathrm{Ext}^n_R(M,N) \cong \mathrm{S}^{-1}_{\mathsf{V}}\mathrm{S}^{-1}_{\mathsf{W}}\mathrm{Ext}^n_R(M,N)$ for each $n \geq 0$. Let $0 \to M \to I \to \Theta_{\mathsf{V}}M \to 0$ be a special V-preenvelope of M and $0 \to \Omega^{\mathsf{W}}N \to J \to N \to 0$ a special W-precover of N. Then one obtains the following two commutative diagrams with exact rows and columns:

$$\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \downarrow & & & & \\
 & & \mathrm{S}^{-1}_{\mathsf{V}}\mathrm{S}^{-1}_{\mathsf{W}}\mathrm{Ext}^n_R(M,N) & & & & \\
 & & \downarrow & & & & \\
0 & \longrightarrow & \mathrm{S}^{-1}_{\mathsf{W}}\mathrm{Ext}^n_R(\Theta_{\mathsf{V}}M,N) & \longrightarrow & \mathrm{Ext}^n_R(\Theta_{\mathsf{V}}M,\Omega^{\mathsf{W}}N) & \longrightarrow & \mathrm{Ext}^n_R(\Theta_{\mathsf{V}}M,J) \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{S}^{-1}_{\mathsf{W}}\mathrm{Ext}^n_R(I,N) & \longrightarrow & \mathrm{Ext}^n_R(I,\Omega^{\mathsf{W}}N), & &
\end{array} \tag{6}$$

$$\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \downarrow & & & & \\
 & & \mathrm{S}^{-1}_{\mathsf{W}}\mathrm{S}^{-1}_{\mathsf{V}}\mathrm{Ext}^n_R(M,N) & & & & \\
 & & \downarrow & & & & \\
0 & \longrightarrow & \mathrm{S}^{-1}_{\mathsf{V}}\mathrm{Ext}^n_R(M,\Omega^{\mathsf{W}}N) & \longrightarrow & \mathrm{Ext}^n_R(\Theta_{\mathsf{V}}M,\Omega^{\mathsf{W}}N) & \longrightarrow & \mathrm{Ext}^n_R(I,\Omega^{\mathsf{W}}N) \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{S}^{-1}_{\mathsf{V}}\mathrm{Ext}^n_R(M,J) & \longrightarrow & \mathrm{Ext}^n_R(\Theta_{\mathsf{V}}M,J). & &
\end{array} \tag{7}$$

The diagrams (6) and (7) imply that both the sequences

$$0 \longrightarrow \mathrm{S}^{-1}_{\mathsf{V}}\mathrm{S}^{-1}_{\mathsf{W}}\mathrm{Ext}^n_R(M,N) \longrightarrow \mathrm{Ext}^n_R(\Theta_{\mathsf{V}}M,\Omega^{\mathsf{W}}N) \longrightarrow \mathrm{Ext}^n_R(I,\Omega^{\mathsf{W}}N) \oplus \mathrm{Ext}^n_R(\Theta_{\mathsf{V}}M,J)$$

and

$$0 \longrightarrow \mathrm{S}^{-1}_{\mathsf{W}}\mathrm{S}^{-1}_{\mathsf{V}}\mathrm{Ext}^n_R(M,N) \longrightarrow \mathrm{Ext}^n_R(\Theta_{\mathsf{V}}M,\Omega^{\mathsf{W}}N) \longrightarrow \mathrm{Ext}^n_R(I,\Omega^{\mathsf{W}}N) \oplus \mathrm{Ext}^n_R(\Theta_{\mathsf{V}}M,J)$$

are exact. Thus one gets the following commutative diagram with exact rows:

$$\begin{array}{ccccc}
0 \to \mathsf{S}_{\mathsf{V}}^{-1}\,\mathsf{S}_{\mathsf{W}}^{-1}\operatorname{Ext}_R^n(M,N) & \to & \operatorname{Ext}_R^n(\Theta_{\mathsf{V}} M, \Omega^{\mathsf{W}} N) & \to & \operatorname{Ext}_R^n(I, \Omega^{\mathsf{W}} N) \oplus \operatorname{Ext}_R^n(\Theta_{\mathsf{V}} M, J) \\
\Big\downarrow \phi & & \Big\| & & \Big\| \\
0 \to \mathsf{S}_{\mathsf{W}}^{-1}\,\mathsf{S}_{\mathsf{V}}^{-1}\operatorname{Ext}_R^n(M,N) & \to & \operatorname{Ext}_R^n(\Theta_{\mathsf{V}} M, \Omega^{\mathsf{W}} N) & \to & \operatorname{Ext}_R^n(I, \Omega^{\mathsf{W}} N) \oplus \operatorname{Ext}_R^n(\Theta_{\mathsf{V}} M, J),
\end{array}$$

where ϕ is obtained by the universal property of kernels, moreover ϕ is an isomorphism by the Five Lemma, as desired. □

It is easy to see that $\{\operatorname{colim}_j \mathsf{S}_{\mathsf{W}}^{-j}\operatorname{Ext}_R^{n+j}(-,N) \mid n \in \mathbb{Z}\}$ is a V-connected sequence of contravariant functors, and $\{\operatorname{colim}_i \mathsf{S}_{\mathsf{V}}^{-i}\operatorname{Ext}_R^{n+i}(M,-) \mid n \in \mathbb{Z}\}$ is a W-connected sequence of covariant functors, where the morphisms in the direct systems are provided by those δ and ∂ in Remark 11.

Lemma 18. *Let M and N be R-modules. For each $n \in \mathbb{Z}$, there exists a natural isomorphism*

$$\operatorname{colim}_j \mathsf{S}_{\mathsf{W}}^{-j} \operatorname{colim}_i \mathsf{S}_{\mathsf{V}}^{-i} \operatorname{Ext}_R^{n+i+j}(M,N) \cong \operatorname{colim}_i \mathsf{S}_{\mathsf{V}}^{-i} \operatorname{colim}_j \mathsf{S}_{\mathsf{W}}^{-j} \operatorname{Ext}_R^{n+i+j}(M,N).$$

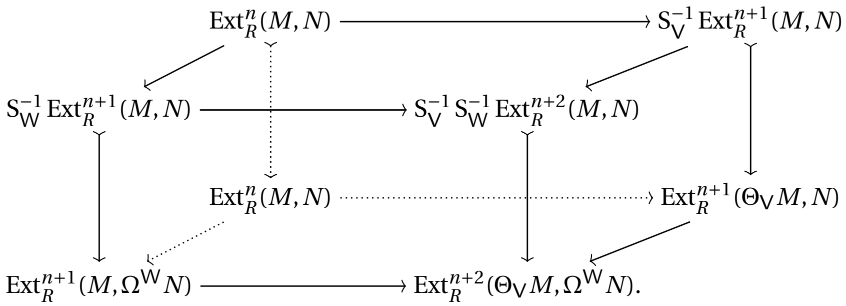
Proof. By Remark 11 and Lemma 17 we have the following diagram:

$$\begin{array}{ccccccc}
\operatorname{Ext}_R^n(M,N) & \xrightarrow{\delta} & \mathsf{S}_{\mathsf{V}}^{-1}\operatorname{Ext}_R^{n+1}(M,N) & \xrightarrow{\delta} & \mathsf{S}_{\mathsf{V}}^{-2}\operatorname{Ext}_R^{n+2}(M,N) & \longrightarrow & \cdots \\
\big\downarrow \partial & & \big\downarrow \partial & & \big\downarrow \partial & & \\
\mathsf{S}_{\mathsf{W}}^{-1}\operatorname{Ext}_R^{n+1}(M,N) & \xrightarrow{\delta} & \mathsf{S}_{\mathsf{V}}^{-1}\mathsf{S}_{\mathsf{W}}^{-1}\operatorname{Ext}_R^{n+2}(M,N) & \xrightarrow{\delta} & \mathsf{S}_{\mathsf{V}}^{-2}\mathsf{S}_{\mathsf{W}}^{-1}\operatorname{Ext}_R^{n+3}(M,N) & \longrightarrow & \cdots \\
\big\downarrow \partial & & \big\downarrow \partial & & \big\downarrow \partial & & \\
\mathsf{S}_{\mathsf{W}}^{-2}\operatorname{Ext}_R^{n+2}(M,N) & \xrightarrow{\delta} & \mathsf{S}_{\mathsf{V}}^{-1}\mathsf{S}_{\mathsf{W}}^{-2}\operatorname{Ext}_R^{n+3}(M,N) & \xrightarrow{\delta} & \mathsf{S}_{\mathsf{V}}^{-2}\mathsf{S}_{\mathsf{W}}^{-2}\operatorname{Ext}_R^{n+4}(M,N) & \longrightarrow & \cdots \\
\big\downarrow & & \big\downarrow & & \big\downarrow & & \\
\vdots & & \vdots & & \vdots & &
\end{array} \tag{8}$$

Claim. *Each of the squares in the above diagram is anticommutative.*

Proof of the claim. We only prove the anticommutativity for the square in the upper left corner; one may prove similarly for the others.

From the constructions of δ and ∂ in Remark 11 and the definition of left satellite functors, one gets the next 3D-diagram with side squares all commutative, as the connected homomorphisms are natural.



In the above diagram, the bottom square is anticommutative; see Rotman [13, Theorem 11.24]. Thus the top square is anticommutative, as all the vertical homomorphisms are natural embeddings. This finishes the proof of the claim.

Thus the diagram (8) can be rewritten as the next commutative diagram:

$$\begin{array}{ccccc}
\operatorname{Ext}_R^n(M,N) & \xrightarrow{\delta\delta} & \mathrm{S}_{\mathsf{V}}^{-2}\operatorname{Ext}_R^{n+2}(M,N) & \longrightarrow & \cdots \\
\big\downarrow{\scriptstyle\partial\partial} & & \big\downarrow{\scriptstyle\partial\partial} & & \\
\mathrm{S}_{\mathsf{W}}^{-2}\operatorname{Ext}_R^{n+2}(M,N) & \xrightarrow{\delta\delta} & \mathrm{S}_{\mathsf{V}}^{-2}\mathrm{S}_{\mathsf{W}}^{-2}\operatorname{Ext}_R^{n+4}(M,N) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \\
\vdots & & \vdots & &
\end{array}$$

Now the desired isomorphism in the statement follows. □

Assume that (W,V) is a balanced pair and $\operatorname{Ext}_R^{\geqslant 1}({}^{\perp}\mathsf{V},\mathsf{V}) = 0 = \operatorname{Ext}_R^{\geqslant 1}(\mathsf{W},\mathsf{W}^{\perp})$. By [10, Proposition 3.12] one has

$$\widehat{\operatorname{Ext}}_{\mathsf{W}}^n(M,N) \cong \operatorname{colim}_j \mathrm{S}_{\mathsf{W}}^{-j}\operatorname{Ext}_R^{n+j}(M,N)$$

for all R-modules M and N and each $n \in \mathbb{Z}$. From Lemma 16, the family

$$\mathrm{F}(N)^* = \left\{\mathrm{F}(N)^n = \widehat{\operatorname{Ext}}_{\mathsf{W}}^n(-,N) \,\middle|\, n \in \mathbb{Z}\right\}$$

is a V-cohomology sequence of contravariant functors. Hence by Theorem 12 it admits a V-completion $\widetilde{\mathrm{F}(N)}^*$ with

$$\widetilde{\mathrm{F}(N)}^n(M) = \operatorname{colim}_i \mathrm{S}_{\mathsf{V}}^{-i}\mathrm{F}(N)^{n+i}(M) = \operatorname{colim}_i \mathrm{S}_{\mathsf{V}}^{-i}\operatorname{colim}_j \mathrm{S}_{\mathsf{W}}^{-j}\operatorname{Ext}_R^{n+i+j}(M,N).$$

Similarly, by [10, Proposition 3.12] one has

$$\widetilde{\operatorname{Ext}}_{\mathsf{V}}^n(M,N) \cong \operatorname{colim}_i \mathrm{S}_{\mathsf{V}}^{-i}\operatorname{Ext}_R^{n+i}(M,N)$$

for all R-modules M and N and each $n \in \mathbb{Z}$. From Lemma 16, the family

$$\mathrm{G}(M)^* = \left\{\mathrm{G}(M)^n = \widetilde{\operatorname{Ext}}_{\mathsf{V}}^n(M,-) \,\middle|\, n \in \mathbb{Z}\right\}$$

is a W-cohomology sequence of covariant functors, and so by Theorem 12 it admits a W-completion $\widehat{\mathrm{G}(M)}^*$ with

$$\widehat{\mathrm{G}(M)}^n(N) = \operatorname{colim}_j \mathrm{S}_{\mathsf{W}}^{-j}\mathrm{G}(M)^{n+j}(N) = \operatorname{colim}_j \mathrm{S}_{\mathsf{W}}^{-j}\operatorname{colim}_i \mathrm{S}_{\mathsf{V}}^{-i}\operatorname{Ext}_R^{n+j+i}(M,N).$$

Therefore, it follows from Lemma 18 that there is a natural isomorphism

$$\widetilde{\mathrm{F}(N)}^n(M) \cong \widehat{\mathrm{G}(M)}^n(N) \tag{9}$$

for all R-modules M and N, and each $n \in \mathbb{Z}$. Combining the vanishing results in [10, Theorems 4.9 and 4.11], we give a balanced result for complete cohomology groups $\widetilde{\operatorname{Ext}}_{\mathsf{V}}^n(M,N)$ and $\widehat{\operatorname{Ext}}_{\mathsf{W}}^n(M,N)$.

Theorem 19. *Let* (W,V) *be a balanced pair. Assume that* W *and* V *are closed under direct summands and* $\operatorname{Ext}_R^{\geqslant 1}(\mathsf{W},\mathsf{W}^{\perp}) = 0 = \operatorname{Ext}_R^{\geqslant 1}({}^{\perp}\mathsf{V},\mathsf{V})$. *Then the following statements are equivalent.*

(i) *All R-modules in V have finite W-projective dimension and all R-modules in W have finite V-injective dimension.*

(ii) *For all R-modules M and N and each $n \in \mathbb{Z}$, there is a natural isomorphism*

$$\widetilde{\operatorname{Ext}}_{\mathsf{V}}^n(M,N) \cong \widehat{\operatorname{Ext}}_{\mathsf{W}}^n(M,N).$$

(iii) *For each R-module M in V or W, there is a natural isomorphism*

$$\widetilde{\operatorname{Ext}}_{\mathsf{V}}^0(M,M) \cong \widehat{\operatorname{Ext}}_{\mathsf{W}}^0(M,M).$$

Proof. The implication (ii) $\Longrightarrow$ (iii) is clear.

(i) $\Longrightarrow$ (ii). For each R-module $C \in \mathsf{V}$, $D \in \mathsf{W}$ and $n \in \mathbb{Z}$, one has

$$\mathrm{F}(N)^n(C) = \widehat{\mathrm{Ext}}^n_{\mathsf{W}}(C,N) = 0$$

by [10, Theorem 4.11], and $\underline{\mathrm{G}(M)^n(D)} = \underline{\breve{\mathrm{Ext}}^n_{\mathsf{V}}(M,D)} = 0$ by [10, Theorem 4.9]. So it follows from Corollary 13 that $\mathrm{F}(N)^* \cong \widetilde{\mathrm{F}(N)}^*$ and $\mathrm{G}(M)^* \cong \widehat{\mathrm{G}(M)}^*$. Thus by the isomorphism (9), one has $\breve{\mathrm{Ext}}^n_{\mathsf{V}}(M,N) \cong \widehat{\mathrm{Ext}}^n_{\mathsf{W}}(M,N)$ for all R-modules M and N, and each $n \in \mathbb{Z}$.

(iii) $\Longrightarrow$ (i). We let $C \in \mathsf{V}$ and $D \in \mathsf{W}$. Then one has

$$\breve{\mathrm{Ext}}^0_{\mathsf{V}}(D,D) \cong \widehat{\mathrm{Ext}}^0_{\mathsf{W}}(D,D) = 0$$

by [10, Theorem 4.11], and $\widehat{\mathrm{Ext}}^0_{\mathsf{W}}(C,C) \cong \breve{\mathrm{Ext}}^0_{\mathsf{V}}(C,C) = 0$ by [10, Theorem 4.9]. Thus $\mathsf{V}\text{-id}_R(D) < \infty$ and $\mathsf{W}\text{-pd}_R(C) < \infty$ again by [10, Theorems 4.9 and 4.11]. □

4. An application to Tate cohomology

In this section, we prove a balanced result for Tate cohomology introduced in [15] relying on the complete cohomology discussed before. We set

$$\mathsf{W}\text{-pd}(\mathsf{V}) = \sup\{\mathsf{W}\text{-pd}_R(C) \mid C \in \mathsf{V}\} \qquad \text{and} \qquad \mathsf{V}\text{-id}(\mathsf{W}) = \sup\{\mathsf{V}\text{-id}_R(D) \mid D \in \mathsf{W}\}.$$

Lemma 20. *Let* (W,V) *be a balanced pair. Then the following statements hold.*

(a) *If all R-modules in* V *have finite* W*-projective dimension, then there is an inequality* $\mathsf{W}\text{-pd}(\mathsf{V}) \leq \mathsf{V}\text{-id}(\mathsf{W})$.

(b) *If all R-modules in* W *have finite* V*-injective dimension, then there is an inequality* $\mathsf{V}\text{-id}(\mathsf{W}) \leq \mathsf{W}\text{-pd}(\mathsf{V})$.

Proof. We only prove (a); the statement (b) is proved similarly.

Assume that $\mathsf{V}\text{-id}(\mathsf{W}) = d < \infty$. Let C be in V, and let $\mathsf{W}\text{-pd}_R(C) = n < \infty$ by assumption. Thus there exists an R-module M such that $\mathrm{Ext}^n_{\mathsf{W}R}(C,M) \neq 0$ by Lemma 2. Let $0 \to \Omega^{\mathsf{W}} M \to P \to M \to 0$ be a special W-precover of M with $P \in \mathsf{W}$ and $\Omega^{\mathsf{W}} M \in \mathsf{W}^{\perp}$. Then one gets an exact sequence

$$\mathrm{Ext}^n_{\mathsf{W}R}(C,P) \longrightarrow \mathrm{Ext}^n_{\mathsf{W}R}(C,M) \longrightarrow \mathrm{Ext}^{n+1}_{\mathsf{W}R}(C,\Omega^{\mathsf{W}} M)$$

of abelian groups by Sather-Wagstaff, Sharif and White [14, Lemma 1.15(a)]. Notice that $\mathrm{Ext}^{n+1}_{\mathsf{W}R}(C,\Omega^{\mathsf{W}} M) = 0$ by Lemma 2 and $\mathrm{Ext}^n_{\mathsf{W}R}(C,M) \neq 0$, so one gets $\mathrm{Ext}^n_{\mathsf{W}R}(C,P) \neq 0$. Since (W,V) is a balanced pair, one has $\mathrm{Ext}^n_{R\mathsf{V}}(C,P) \cong \mathrm{Ext}^n_{\mathsf{W}R}(C,P) \neq 0$. It follows from Lemma 3 that $n \leq d$, as $\mathsf{V}\text{-id}_R(P) \leq d$. Thus one has $\mathsf{W}\text{-pd}_R(C) \leq d$, which yields $\mathsf{W}\text{-pd}(\mathsf{V}) \leq d$. □

Tate (co)resolutions. Recall from [15] that a complex T of R-modules in V is *totally* V*-acyclic* if it is acyclic and the complexes $\mathrm{Hom}_R(C,T)$ and $\mathrm{Hom}_R(T,C)$ are acyclic for each R-module $C \in \mathsf{V}$. Let N be an R-module. A *Tate* V*-coresolution* of N is a diagram $N \xrightarrow{\simeq} I \xrightarrow{\alpha} T$ wherein T is a totally V-acyclic complex of R-modules in V, $N \xrightarrow{\simeq} I$ is a proper V-coresolution of N, and α^n is an isomorphism for $n \gg 0$.

Dually, one has the definitions of a *totally* W*-acyclic* complex H and a *Tate* W*-resolution* $H \xrightarrow{\gamma} P \xrightarrow{\simeq} M$ of M.

Tate cohomology. Let M be an R-module admitting a Tate W-resolution $H \longrightarrow P \xrightarrow{\simeq} M$. Following [15, Definition 4.1], for each R-module N and each $n \in \mathbb{Z}$, the n-th *Tate cohomology* of M with coefficients in N is defined as

$$\widehat{\mathrm{Ext}}^n_{\mathsf{W}R}(M,N) = \mathrm{H}^n\big(\mathrm{Hom}_R(H,N)\big).$$

Let N be an R-module admitting a Tate V-coresolution $N \xrightarrow{\simeq} I \longrightarrow T$. Then for each R-module M, the n-th *Tate cohomology* of N with coefficients in M is defined as

$$\widehat{\mathrm{Ext}}^n_{R\mathsf{V}}(M,N) = \mathrm{H}^n\big(\mathrm{Hom}_R(M,T)\big).$$

Remark 21. Assume that W and V are closed under direct summands. Then by [10, Proposition 4.15] the Tate cohomology group $\widehat{\mathrm{Ext}}^n_{\mathsf{W}R}(M,N)$ is actually the complete cohomology group $\widehat{\mathrm{Ext}}^n_{\mathsf{W}}(M,N)$ whenever R-module M has a Tate W-resolution. Also by [10, Proposition 4.14], the Tate cohomology group $\widehat{\mathrm{Ext}}^n_{R\mathsf{V}}(M,N)$ is actually the complete cohomology group $\widetilde{\mathrm{Ext}}^n_{\mathsf{V}}(M,N)$ whenever R-module N has a Tate V-coresolution.

Theorem 22. *Let* (W,V) *be a balanced pair. Assume that* W *and* V *are closed under direct summands and* $\mathrm{Ext}^{\geqslant 1}_R(\mathsf{W},\mathsf{W}^{\perp})=0=\mathrm{Ext}^{\geqslant 1}_R({}^{\perp}\mathsf{V},\mathsf{V})$. *Then the following statements are equivalent.*

(i) *All R-modules in V have finite W-projective dimension and all R-modules in W have finite V-injective dimension.*
(ii) $\mathsf{W}\text{-pd}(\mathsf{V})=\mathsf{V}\text{-id}(\mathsf{W})<\infty$.
(iii) *Each R-module has a Tate V-coresolution.*
(iv) *Each R-module has a Tate W-resolution.*

In this case, for all R-modules M and N and each $n\in\mathbb{Z}$, there is a natural isomorphism $\widehat{\mathrm{Ext}}^n_{\mathsf{W}R}(M,N)\cong\widehat{\mathrm{Ext}}^n_{R\mathsf{V}}(M,N)$.

Proof. Since an arbitrary direct sum of R-modules in W is also in W, the invariant V-id(W) is finite if and only if all R-modules in W have finite V-injective dimension. So the statements (i) and (ii) are equivalent by Lemma 20.

(ii) $\Longrightarrow$ (iii). Assume that $\mathsf{W}\text{-pd}(\mathsf{V})=\mathsf{V}\text{-id}(\mathsf{W})=n<\infty$. Let M be an R-module with $M\xrightarrow{\simeq}I$ a proper V-coresolution of M. Fix a proper W-resolution $P\xrightarrow{\simeq}M$ of M. Then from [5, Proposition 2.2] the exact sequence

$$\cdots\longrightarrow P_1\longrightarrow P_0\longrightarrow M\longrightarrow 0$$

is $\mathrm{Hom}_R(-,\mathsf{V})$-exact, as (W,V) is a balanced pair. So one gets an exact sequence

$$\cdots\longrightarrow \Theta^n_{\mathsf{V}}P_1\longrightarrow \Theta^n_{\mathsf{V}}P_0\longrightarrow \Theta^n_{\mathsf{V}}M\longrightarrow 0$$

with each $\Theta^n_{\mathsf{V}}P_i\in\mathsf{V}$, which is $\mathrm{Hom}_R(-,\mathsf{V})$-exact. Assembling the above sequence and the $\mathrm{Hom}_R(-,\mathsf{V})$-exact exact sequence

$$0\longrightarrow \Theta^n_{\mathsf{V}}M\longrightarrow I^n\longrightarrow I^{n+1}\longrightarrow\cdots$$

with each $I^n\in\mathsf{V}$, one gets an acyclic complex T of R-modules in V

$$T=\cdots\longrightarrow \Theta^n_{\mathsf{V}}P_1\longrightarrow \Theta^n_{\mathsf{V}}P_0\longrightarrow I^n\longrightarrow I^{n+1}\longrightarrow\cdots,$$

which is $\mathrm{Hom}_R(-,\mathsf{V})$-exact. Next we prove that the sequence T is also $\mathrm{Hom}_R(\mathsf{V},-)$-exact. To this end, let $C\in\mathsf{V}$. Then one has $\mathsf{W}\text{-pd}_R(C)\leq n$. Let $\mathrm{C}^i(T)$ denote the kernel of the homomorphism from T^i to T^{i+1} for each $i\in\mathbb{Z}$. Note that T is $\mathrm{Hom}_R(-,\mathsf{V})$-exact and each $T_i\in\mathsf{V}$. One gets

$$\begin{aligned}\mathrm{Ext}^1_{R\mathsf{V}}(C,\mathrm{C}^i(T))&\cong\mathrm{Ext}^{n+1}_{R\mathsf{V}}\big(C,\mathrm{C}^{i-n}(T)\big)\\&\cong\mathrm{Ext}^{n+1}_{\mathsf{W}R}\big(C,\mathrm{C}^{i-n}(T)\big)\\&=0,\end{aligned}$$

where the first isomorphism holds by [14, Lemma 1.15(d)], the second one follows from [5, Lemma 2.1], and the equality holds by Lemma 2 as $\mathsf{W}\text{-pd}_R(C)\leq n$. Thus the complex T is $\mathrm{Hom}_R(\mathsf{V},-)$-exact, and hence there is a homomorphism $\alpha\colon I\to T$. Then one gets a Tate V-coresolution $M\xrightarrow{\simeq}I\xrightarrow{\alpha}T$.

(iii) $\Longrightarrow$ (i). For each R-module $D\in\mathsf{W}$, there is a Tate V-coresolution $D\xrightarrow{\simeq}I\longrightarrow T$ by assumption. T is $\mathrm{Hom}_R(-,\mathsf{V})$-exact, so it is $\mathrm{Hom}_R(\mathsf{W},-)$-exact by [5, Proposition 2.2]. Thus from [10, Proposition 4.14], one has

$$\widetilde{\mathrm{Ext}}^n_{\mathsf{V}}(D,D)\cong\mathrm{H}^n\big(\mathrm{Hom}_R(D,T)\big)=0,$$

and so $\mathsf{V}\text{-id}_R(D)<\infty$ by [10, Theorem 4.9].

Next, for each R-module $C \in \mathsf{V}$, we prove $\mathsf{W}\text{-pd}_R(C) < \infty$. Let M be an R-module. By assumption there is a Tate V-coresolution $M \xrightarrow{\simeq} I' \xrightarrow{\mu} T'$ of M. Then there exists an integer $n > 0$ such that μ^j is an isomorphism for each $j \geq n$. Thus for each $i > n$, one has

$$\begin{aligned}\operatorname{Ext}^i_{\mathsf{W}R}(C,M) &\cong \operatorname{Ext}^i_{R\mathsf{V}}(C,M)\\ &= \mathrm{H}^i\big(\operatorname{Hom}_R(C,I')\big)\\ &\cong \mathrm{H}^i\big(\operatorname{Hom}_R(C,T')\big)\\ &= 0,\end{aligned}$$

where the first isomorphism holds as (W,V) is a balanced pair. Let

$$i_M = \inf\big\{n \,\big|\, \operatorname{Ext}^i_{\mathsf{W}R}(C,M) = 0 \text{ for all } i > n\big\}.$$

We claim that there is an integer s such that $i_M \leq s$ for each R-module M. Actually, it suffices to show that for any sequence $\{M_t\}_{t\in\mathbb{N}}$ of R-modules there exists such an integer s. Set $L = \prod_{t=0}^{\infty} M_t$ and let $s = i_L$. Then one has $i_{M_t} \leq s$ for each $t \in \mathbb{N}$ as

$$\big\{n \,\big|\, \operatorname{Ext}^i_{\mathsf{W}R}(C,L) = 0 \text{ for all } i > n\big\} \subseteq \big\{n \,\big|\, \operatorname{Ext}^i_{\mathsf{W}R}(C,M_t) = 0 \text{ for all } i > n\big\}.$$

Thus one gets that $\operatorname{Ext}^{s+l}_{\mathsf{W}R}(C,M) = 0$ for each R-module M and all $l > 0$, and so $\mathsf{W}\text{-pd}_R(C) \leq s < \infty$ by Lemma 2.

The implications (ii) $\Longrightarrow$ (iv) and (iv) $\Longrightarrow$ (i) are proved similarly.

Finally, for all R-modules M and N, and each $n \in \mathbb{Z}$, one has

$$\begin{aligned}\widehat{\operatorname{Ext}}^n_{\mathsf{W}R}(M,N) &\cong \widehat{\operatorname{Ext}}^n_{\mathsf{W}}(M,N)\\ &\cong \widetilde{\operatorname{Ext}}^n_{\mathsf{V}}(M,N)\\ &\cong \widehat{\operatorname{Ext}}^n_{R\mathsf{V}}(M,N),\end{aligned}$$

where the first isomorphism follows from [10, Proposition 4.15], the second one holds by Theorem 19, and the last one holds by [10, Proposition 4.14]. □

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