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Irrationality of the length spectrum

Irrationalité du spectre des longueurs

George Peterzil [a] **and Guy Sapire** [a]

[a] Einstein Institute of Mathematics, The Hebrew University of Jerusalem, 91904 Jerusalem, Israel
E-mails: george.peterzil@mail.huji.ac.il, guy.sapire@mail.huji.ac.il

Abstract. It is a classical result of Dal'Bo that the length spectrum of a non-elementary Fuchsian group is non-arithmetic, namely, it generates a dense additive subgroup of $\mathbb{R}$. In this note we provide an elementary proof of an extension of this theorem: a non-elementary Fuchsian group contains two elements whose lengths are linearly independent over $\mathbb{Q}$, reproving a result of Prasad and Rapinchuk [9].

Résumé. C'est un résultat classique de Dal'Bo que le spectre des longueurs d'un groupe fuchsien non élémentaire est non arithmétique, c'est-à-dire qu'il génère un sous-groupe additif dense de $\mathbb{R}$. Dans cette note, nous fournissons une preuve élémentaire d'une extension de ce théorème : un groupe fuchsien non élémentaire contient deux éléments dont les longueurs sont linéairement indépendantes sur $\mathbb{Q}$, reproduisant un résultat de Prasad et Rapinchuk [9] dans un cas particulier.

Keywords. Length spectrum, Fuchsian group, trace field.

Mots-clés. Spectre de longueur, groupe fuchsien, corps de traces.

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1. Introduction

Let Γ be a non-elementary Fuchsian group, that is, a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$ which does not contain a cyclic subgroup of finite index.

Definition 1. *We say that $\gamma \in \Gamma \setminus \{e\}$ is* hyperbolic *whenever it is diagonalizable over $\mathbb{R}$. Given a hyperbolic $\gamma \in \mathrm{PSL}_2(\mathbb{R})$, we define the* translation length of γ, *denoted $\ell(\gamma)$, to be $2\log(|\lambda|)$, where λ is the eigenvalue of γ with $|\lambda| \geq 1$. We define $L(\Gamma)$ to be the set of translation lengths of hyperbolic elements in Γ.*

The group $\mathrm{PSL}_2(\mathbb{R})$ is the group of orientation-preserving isometries of the hyperbolic plane. Under the classical correspondence between (conjugacy classes of) hyperbolic $\gamma \in \Gamma$ and closed geodesics on the orbifold $\mathbb{H}/\Gamma$, the set $L(\Gamma)$ can also be characterized as the set of lengths of closed geodesics on $\mathbb{H}/\Gamma$, where the latter is equipped with the hyperbolic metric.

The following is a well-known theorem of Dal'Bo.

Theorem 2 (Non-arithmeticity of the length spectrum [3]). *Let $\Gamma \leq \mathrm{PSL}_2(\mathbb{R})$ be a non-elementary Fuchsian group. Then $L(\Gamma)$ generates a dense additive subgroup of $\mathbb{R}$.*

Non-arithmeticity has been widely applied in homogeneous dynamics, for example in proving the topological mixing of the geodesic flow and the density of horocycles originating from a horocyclic limit point of Γ.

This result has been generalized to related notions of spectra in other settings as well, see e.g. [1,4,5]. We make use of some basic results on radicals in finitely-generated field extensions of $\mathbb{Q}$ to give an elementary proof of the following result.

Theorem 3. *Let Γ be a non-elementary Fuchsian group. Then $L(\Gamma) \subseteq \mathbb{R}$ contains two elements which are linearly independent over $\mathbb{Q}$.*

This theorem is a special case of a result by Prasad and Rapinchuk (see [9, Remark 1]), which was recently generalized in [2]. In fact, it is shown in [2, Proposition 1.8] that $L(\Gamma)$ contains an infinite set of lengths which is linearly independent over $\mathbb{Q}$. Both theorems rely on deep results in number theory.

It is worth noting that the results in Section 3 can be rephrased and proved in the arithmetic language of heights, in particular using the Northcott property of Moriwaki's height (see [7, Section 4], and in general [10]).

2. Proof of the theorem

We first recall a fundamental notion in the theory of discrete groups.

Definition 4. *Let Γ be a matrix group. The* trace field of Γ *is defined to be*

$$k\Gamma = \mathbb{Q}\big(\{\mathrm{tr}(\gamma) : \gamma \in \Gamma\}\big)$$

We start with the following.

Lemma 5. *The trace field of a finitely-generated matrix group Γ is finitely generated over $\mathbb{Q}$.*

Proof. If $\Gamma = \langle \gamma_1, \ldots, \gamma_n \rangle$ then the trace field is contained in the extension of $\mathbb{Q}$ generated by the entries of $\gamma_1, \ldots, \gamma_n$, which is finitely-generated. □

Our final ingredient will be the following proposition, which follows directly from Lemma 5 and Proposition 10, which we will prove later.

Proposition 6. *Let $\Gamma \leq \mathrm{PSL}_2(\mathbb{R})$ be a finitely-generated Fuchsian group, $k\Gamma$ its trace field, $r \in k\Gamma$ positive with $r \neq 1$, $K/k\Gamma$ a finitely generated extension of $k\Gamma$. Then there are only finitely many rational $0 < q < 1$ such that r^q is at most quadratic over K.*

Proof of Theorem 3. Since every non-elementary Γ contains a non-elementary finitely generated subgroup, we will assume that Γ is finitely generated. Assume towards a contradiction that $L(\Gamma) \subseteq \ell_0 \mathbb{Q}$ for some $\ell_0 \in L(\Gamma)$, and for every $\gamma \in \Gamma$, let $q_\gamma \in \mathbb{Q}$ be such that $\ell(\gamma) = q_\gamma \ell_0$. Let $K = k\Gamma(e^{\frac{\ell_0}{2}})$ be the field generated by $e^{\frac{\ell_0}{2}}$ over $k\Gamma$. We claim that for any $\gamma \in \Gamma$ we have that $e^{\frac{\{q_\gamma\}\ell_0}{2}}$ is at most quadratic over K, where $\{q_\gamma\} \in [0,1)$ is the fractional part of q_γ. First recall that for any $\gamma \in \Gamma$, by [6, Lemma 12.1.2] we have

$$\left|\mathrm{tr}(\gamma)\right| = e^{\frac{\ell(\gamma)}{2}} + e^{-\frac{\ell(\gamma)}{2}}$$

This means that $x = e^{\frac{\ell(\gamma)}{2}}$ satisfies $x + x^{-1} \in k\Gamma$, so there is some $\lambda \in k\Gamma$ such that x satisfies $x^2 - x\lambda + 1 = 0$, hence $e^{\frac{\ell(\gamma)}{2}} = e^{\frac{q_\gamma \ell_0}{2}}$ is at most quadratic over $k\Gamma$ and thus also over K. This gives us that $e^{\frac{\{q_\gamma\}\ell_0}{2}} = e^{\frac{q_\gamma \ell_0}{2}} (e^{\frac{\ell_0}{2}})^{-\lfloor q_\gamma \rfloor}$ is at most quadratic over K, where $\lfloor q_\gamma \rfloor$ denotes the integer part of q_γ. By Proposition 6 with $r = e^{\frac{\ell_0}{2}}$, there is a bound $n \in \mathbb{N}$ on the denominators of such q_γ, hence $L(\Gamma) \subseteq \frac{1}{n!}\ell_0 \mathbb{Z}$ and in particular $\langle L(\Gamma) \rangle \subseteq \mathbb{R}$ is a discrete subgroup, but this is a contradiction to Theorem 2. □

Remark 7. It is worth noting that a similar proof works for a general non-elementary group $\Gamma \leq \mathrm{SO}(n,1)^+$. To see this, write $k'\Gamma$ for the field generated by the entries of a (finite) generating set of Γ. Then for every hyperbolic $\gamma \in \Gamma$ with translation length ℓ, we have that e^{ℓ} is a root of the characteristic polynomial of γ, which is of degree $n+1$ over $k'\Gamma$. The rest of the proof follows the same lines, using the general non-arithmeticity result proved in [5].

3. Radicals in finitely generated extensions of $\mathbb{Q}$

Proposition 8. *Let $K \subseteq \mathbb{R}$ be a finitely generated extension of $\mathbb{Q}$, $r \in K$ positive with $r \neq 1$. Then there are only finitely many $n \in \mathbb{N}$ such that $r^{\frac{1}{n}} \in K$.*

Proof. Write $L = \mathbb{Q}(r)$ and denote by $\widetilde{L}$ the relative algebraic closure of L in K. Note that $\widetilde{L}$ is a finite extension of L since it is algebraic over L and a subfield of K, which is finitely generated. If r is algebraic, then $\widetilde{L}$ is a number field (i.e. a finite extension of $\mathbb{Q}$). Write $\mathcal{O}_{\widetilde{L}}$ for the ring of integers of $\widetilde{L}$, that is, the integral closure of $\mathbb{Z}$ in $\widetilde{L}$, and suppose first $r \in \mathcal{O}_{\widetilde{L}}^{\times}$. Since r is integral and $\mathcal{O}_{\widetilde{L}}$ is integrally closed in $\widetilde{L}$, any element of the form $r^{\frac{1}{n}}$ is in $\mathcal{O}_{\widetilde{L}}^{\times}$. By Dirichlet's unit theorem [8, Theorem 7.4] this group is free up to roots of unity. Because $r > 0$ and is different from 1 it is not a root of unity, so it can only have finitely many roots in $\mathcal{O}_{\widetilde{L}}$. If $r \notin \mathcal{O}_{\widetilde{L}}^{\times}$, recall that by the unique factorization of fractional ideals [8, Corollary 3.9], there exist prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_t \subseteq \mathcal{O}_K$ and $e_1, \dots, e_t \in \mathbb{Z} \setminus \{0\}$ such that

$$(r) = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_t^{e_t}$$

so for every $k \in \mathbb{N}$, if a k-th root of r is in K we get

$$(r^{1/k})^k = (r) = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_t^{e_t}$$

and so from uniqueness necessarily $k \mid e_i$ for every $i \leq t$, meaning the only roots of r that could be in K are d-th roots for d divisors of $\gcd(e_1, \dots, e_t)$.

Otherwise r is transcendental. Because r is a free generator of L, the degree of $r^{\frac{1}{n}}$ over L is n and so for every $n > [\widetilde{L} : L]$ necessarily $r^{\frac{1}{n}} \notin K$. □

The following basic lemma will be of use.

Lemma 9. *Let $K \subseteq \mathbb{R}$ be a field, $t, n \in \mathbb{N}$ and $\alpha \in \mathbb{R}^{\times}$ such that $\alpha^n \in K$ and $\left[K(\alpha) : K\right] = t$, then $\alpha^t \in K$.*

Proof. The minimal polynomial $p(x)$ of α over K is of degree t. Since $x^n - \alpha^n = \prod_{i=1}^{n}(x - \zeta_n^k \alpha)$ (for ζ_n a fixed primitive root of unity) has coefficients in K and has a root α, we must have $p(x) \mid x^n - \alpha^n$, hence there are $1 \leq k_1, \dots, k_t \leq n$ such that $p(x) = \prod_{i=1}^{t}(x - \zeta_n^{k_i} \alpha)$, and in particular the constant coefficient of $p(x)$ is $(-1)^t \zeta_n^{k_1 + \dots + k_t} \alpha^t \in K$. Since $K \subseteq \mathbb{R}$ and $\alpha^t \in \mathbb{R}$, $(-1)^t \zeta_n^{k_1 + \dots + k_t}$ is a real root of unity hence equal to ± 1, so the constant coefficient of $p(x)$ is $\pm \alpha^t \in K$ and so $\alpha^t \in K$. □

Using this, we prove the following.

Proposition 10. *Let $K \subseteq \mathbb{R}$ be a finitely generated extension of $\mathbb{Q}$, $r \in K$ positive, $r \neq 1$ and $t \in \mathbb{N}$. Then there are only finitely many $n \in \mathbb{N}$ such that $\left[K(r^{\frac{1}{n}}) : K\right] = t$. Moreover, there are only finitely many rational $0 < q < 1$ such that r^q is of degree at most t over K.*

Proof. Suppose $n \in \mathbb{N}$ satisfies $\left[K(r^{\frac{1}{n}}) : K\right] = t$. By Lemma 9 for $\alpha = r^{\frac{1}{n}}$ we get $r^{\frac{t}{n}} \in K$ and by Proposition 8 there are only finitely such n. Let n_0 be minimal such that for all $n \geq n_0$ we have that $r^{\frac{1}{n}}$ is of degree strictly greater than t over K. Assume towards a contradiction that there is some rational $0 < q = \frac{m}{n} < 1$ such that $r^q \in K$, with $\gcd(m, n) = 1$. By Bézout's identity, there are integers $x, y \in \mathbb{Z}$ such that $mx + ny = 1$, so in particular $r^{\frac{1}{n}} = (r^q)^x r^y \in K(r^q)$, so $\left[K(r^q) : K\right] \geq \left[K(r^{\frac{1}{n}}) : K\right] > t$, arriving at a contradiction. □

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Declaration of interests

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