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Asymptotic optimality of the edge finite element approximation of the time-harmonic Maxwell's equations

Optimalité asymptotique pour l'approximation par éléments finis de Nédélec des équations de Maxwell en régime harmonique

Théophile Chaumont-Frelet [a] **and Alexandre Ern** [b, c]

[a] Inria Univ. Lille and Laboratoire Paul Painlevé, 59655 Villeneuve-d'Ascq, France

[b] CERMICS, ENPC, Institut Polytechnique de Paris, 77455 Marne-la-Vallée cedex 2, France

[c] Inria, 48 rue Barrault, 75647 Paris, France

E-mails: theophile.chaumont@inria.fr, alexandre.ern@enpc.fr

Abstract. We analyze the conforming approximation of the time-harmonic Maxwell's equations using Nédélec (edge) finite elements. We prove that the approximation is asymptotically optimal, i.e., the approximation error in the energy norm is bounded by the best-approximation error times a constant that tends to one as the mesh is refined and/or the polynomial degree is increased. Moreover, under the same conditions on the mesh and/or the polynomial degree, we establish discrete inf-sup stability with a constant that corresponds to the continuous constant up to a factor of two at most. Our proofs apply under minimal regularity assumptions on the exact solution, so that general domains, material coefficients, and right-hand sides are allowed.

Résumé. On analyse l'approximation conforme des équations de Maxwell en régime harmonique par des éléments finis de Nédélec. On montre l'optimalité asymptotique de l'approximation, *i.e.*, l'erreur d'approximation en norme d'énergie est bornée par l'erreur de la meilleure approximation multipliée par une constante qui tend vers un quand le pas du maillage tend vers zéro ou le degré polynomial tend vers l'infini. De plus, sous ces mêmes hypothèses sur le maillage et le degré polynomial, on établit une condition de stabilité inf-sup avec une constante qui vaut au plus deux fois la valeur de la constante de stabilité du problème continu. Les preuves s'appliquent sous des hypothèses de régularité minimale sur la solution, ce qui permet de considérer une large classe de domaines, propriétés matériaux et termes sources.

Keywords. Electromagnetics, finite element methods, Maxwell's equations, duality argument, asymptotic optimality.

Mots-clés. Électromagnétisme, méthodes d'éléments finis, équations de Maxwell, argument de dualité, optimalité asymptotique.

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1. Introduction

This work analyzes the conforming approximation of the time-harmonic Maxwell's equations using Nédélec (edge) finite elements. In this section, we present the model problem, outline the main challenges in its finite element approximation, and discuss the present contributions in view of the existing literature.

1.1. *Setting*

Let $D \subset \mathbb{R}^d$, $d = 3$, be an open, bounded, connected, Lipschitz polyhedron with boundary ∂D and outward unit normal $\boldsymbol{n}_D$. For the present purposes, a polyhedron can be understood as a finite union of tetrahedra; see, e.g., [22, Section 5.2]. We do not make any simplifying assumption on the topology of D. We use boldface fonts for vectors, vector fields, and functional spaces composed of such fields. More details on the notation are given in Section 2.

Given a positive real number $\omega > 0$ representing a frequency and a source term $\boldsymbol{J}: D \to \mathbb{R}^3$, and focusing for simplicity on homogeneous Dirichlet boundary conditions (a.k.a. perfect electric conductor boundary conditions), the model problem consists in finding $\boldsymbol{E}: D \to \mathbb{R}^3$ such that

$$-\omega^2 \boldsymbol{\epsilon} \boldsymbol{E} + \nabla \times (\boldsymbol{\mu}^{-1} \nabla \times \boldsymbol{E}) = \boldsymbol{J} \quad \text{in } D, \tag{1a}$$

$$\boldsymbol{E} \times \boldsymbol{n}_D = \boldsymbol{0} \quad \text{on } \partial D, \tag{1b}$$

where $\boldsymbol{\epsilon}$ represents the electric permittivity of the materials contained in D and $\boldsymbol{\mu}$ their magnetic permeability. Both material properties can vary in D and take symmetric positive-definite values with eigenvalues uniformly bounded from above and from below away from zero. We assume that ω is not a resonant frequency, so that (1) is uniquely solvable in $\boldsymbol{H}_0(\mathbf{curl}; D)$ for every $\boldsymbol{J}$ in the topological dual space $\boldsymbol{H}_0(\mathbf{curl}; D)'$. The time-harmonic Maxwell's equations (1) are one of the central models of electrodynamics. Therefore, efficient discretizations are a cornerstone for the computational modelling of electromagnetic wave propagation [25,35].

1.2. *Main challenges when approximating* (1)

To highlight the main challenges associated with the finite element approximation of (1), let us first briefly discuss the Helmholtz problem that arises when considering polarized electromagnetic waves. In this case, a two-dimensional domain $\widehat{D} \subset \mathbb{R}^2$ is considered, and the component of the electric field normal to $\widehat{D}$, $\widehat{E}: \widehat{D} \to \mathbb{R}$, satisfies

$$-\omega^2 \widehat{\epsilon} \widehat{E} - \nabla \cdot (\widehat{\boldsymbol{\mu}}^{-1} \nabla \widehat{E}) = \widehat{J} \quad \text{in } \widehat{D}, \tag{2a}$$

$$\widehat{E} = 0 \quad \text{on } \partial \widehat{D}, \tag{2b}$$

where $\widehat{J}: \widehat{D} \to \mathbb{R}$ is the component of $\boldsymbol{J}$ normal to $\widehat{D}$, $\widehat{\epsilon}$ the normal-normal component of $\boldsymbol{\epsilon}$, and $\widehat{\boldsymbol{\mu}}$ the cofactor matrix of the tangent-tangent sub-matrix of $\boldsymbol{\mu}$. The Helmholtz problem also arises in other contexts, such as (three-dimensional) acoustic wave propagation.

The Laplace operator in (2) is coercive over $H_0^1(\widehat{D})$, and the compact embedding $H_0^1(\widehat{D}) \hookrightarrow L^2(\widehat{D})$ ensures that the negative term in (2a) is a compact perturbation. As a result, at the continuous level, (2) falls into the framework of the Fredholm alternative, and the Helmholtz problem is well-posed in $H_0^1(\widehat{D})$ if and only if ω is not a resonant frequency. Compactness also has direct implications at the discrete level. The most common manifestation is probably the celebrated Aubin–Nitsche duality argument. Specifically, setting $\omega = 0$ and considering conforming Lagrange finite elements for simplicity, it is well-known that the finite element approximation, $\widehat{E}_h$, converges to $\widehat{E}$ faster in the $L^2(\widehat{D})$-norm than in the $H_0^1(\widehat{D})$-norm.

When $\omega > 0$, this observation can be leveraged into a technique often called Schatz's argument [39]. The key idea is that the negative $L^2(\widehat{D})$-term in (2) becomes negligible on sufficiently fine meshes, leaving only a coercive term and thus enabling the derivation of a Céa-like lemma. More specifically, considering the Lagrange finite element space $\widehat{V}_h \subset H^1_0(\widehat{D})$ with mesh size h and polynomial degree $k \geq 1$, one can show that if the mesh is sufficiently refined and/or the polynomial degree is sufficiently increased, the finite element approximation, $\widehat{E}_h$, is uniquely defined and satisfies the following error bound:

$$(1-\gamma)\|\!|\widehat{E}-\widehat{E}_h|\!\|\leq \min_{\widehat{v}_h\in\widehat{V}_h}\|\!|\widehat{E}-\widehat{v}_h|\!\|, \tag{3}$$

where the natural energy norm for (2) is

$$\|\!|\widehat{w}|\!\|^2 := \omega^2\|\widehat{\epsilon}^{\frac{1}{2}}\,\widehat{w}\|^2_{L^2(\widehat{D})}+\|\widehat{\boldsymbol{\mu}}^{-\frac{1}{2}}\nabla\widehat{w}\|^2_{L^2(\widehat{D})}, \tag{4}$$

and where the approximation factor γ satisfies $\lim_{h/k\to 0}\gamma = 0$ (the notation means that the limit is taken by sending the ratio h/k to zero, which can, in particular, be realized by sending h to zero with k fixed or k to infinity with h fixed). Crucially, (3) implies that the finite element approximation is *asymptotically optimal*. The original argument of Schatz in [39] for conforming finite elements requires some extra smoothness on the solution, and it has been extended in a number of ways. Of particular importance is the seminal work [30] which tracks the dependence of γ on the frequency ω, the mesh size h and the polynomial degree k. This has later been extended to non-smooth domains and varying coefficients [10,28,31].

The challenges encountered in the Helmholtz problem (2) are also present in the time-harmonic Maxwell's equations (1), with one key additional difficulty: the lack of compactness of the embedding $\boldsymbol{H}_0(\mathbf{curl};D)\hookrightarrow \boldsymbol{L}^2(D)$. This is remedied by working in the subspace $\boldsymbol{H}_0(\mathbf{curl};D)\cap\boldsymbol{H}(\mathrm{div};\boldsymbol{\epsilon};D)$, with $\boldsymbol{H}(\mathrm{div};\boldsymbol{\epsilon};D) := \{\boldsymbol{v}\in\boldsymbol{L}^2(D);\ \nabla\cdot(\boldsymbol{\epsilon}\boldsymbol{v})\in L^2(D)\}$, which compactly embeds into $\boldsymbol{L}^2(D)$ [3,16,41]. Therefore, a crucial ingredient in the analysis of any finite element approximation to (1) is to derive some suitable control on the divergence of the discrete solution. This is discussed in the context of Lagrange finite elements, e.g., in [6,8,17,18]. However, a somewhat more popular approach to approximate the time-harmonic Maxwell's equations in a conforming setting hinges on Nédélec finite elements [37,38]. This is the approach considered herein, and we now discuss our main contributions to this topic.

1.3. *Main contributions*

The Nédélec finite element approximation to (1) is $\boldsymbol{H}_0(\mathbf{curl};D)$-conforming, but only weakly $\boldsymbol{H}(\mathrm{div};\boldsymbol{\epsilon};D)$-conforming. The lack of $\boldsymbol{H}(\mathrm{div};\boldsymbol{\epsilon};D)$-conformity must be taken into account in the error analysis. Early contributions to the topic are based on the concept of collective compactness. A seminal work in this regard is [27] for lowest-order Nédélec elements, and extensions to higher-order elements have been carried out [4,36]. We also refer the reader to [7,9], and to [35, Section 7.3] for an overview.

Later on, duality proofs in the spirit of Schatz were proposed. For Maxwell's equations, the Aubin–Nitsche trick cannot be applied directly, so that an intermediate step involving a commuting interpolation operator is added to the proof. This argument seems to date back to the work of Girault [24] using canonical interpolation operators, and has been used for Maxwell's equations in [34,42]. As pointed out in [24, Remark 3.1], one drawback of using the canonical interpolation operators is the extra regularity requirement on the exact solution. This limitation was lifted following the development of commuting quasi-interpolation operators working under minimal regularity assumptions (see [2,14,15,40] and [22, Chapters 22-23]). The application to Maxwell's equations can be found in [11,21], see also [23, Chapter 44].

The *asymptotic optimality* of the Nédélec finite element approximation to the time-harmonic Maxwell's equations with impedance boundary conditions has been shown very recently in [33, Theorem 9.7]. Therein, the domain boundary is assumed to be smooth and connected, and constant material properties are considered. In particular, this allows for a frequency-explicit analysis, i.e., the error estimate holds true under an explicit smallness condition on the mesh size and the polynomial degree in terms of the frequency ω (up to undetermined numerical factors). In the present work, we establish a similar result, namely an error estimate of the form

$$(1-\gamma)|||\boldsymbol{E}-\boldsymbol{E}_h||| \leq \min_{\boldsymbol{v}_h\in\boldsymbol{V}_h^{\mathrm{c}}} |||\boldsymbol{E}-\boldsymbol{v}_h|||, \tag{5}$$

where $\boldsymbol{V}_h^{\mathrm{c}}$ is the Nédélec finite element space with mesh size h and polynomial degree $k \geq 0$, the natural energy norm is

$$|||\boldsymbol{w}|||^2 := \omega^2\|\boldsymbol{\epsilon}^{\frac{1}{2}}\boldsymbol{w}\|^2_{\boldsymbol{L}^2(D)} + \|\boldsymbol{\mu}^{-\frac{1}{2}}\nabla\times\boldsymbol{w}\|^2_{\boldsymbol{L}^2(D)}, \tag{6}$$

and the approximation factor γ again satisfies $\lim_{h/k\to 0}\gamma = 0$. Herein, we consider perfect electric conductor boundary conditions. The main novelty is that our proofs are valid irrespective of the topology of the domain Ω and apply under a minimal regularity assumption on the exact solution, so that general domains, material coefficients and right-hand sides are allowed. For brevity, we do not perform a frequency-explicit analysis, although we observe that the dependence of γ on key parameters can be traced following [12,32,33].

1.4. *Outline*

The paper is organized as follows. In Section 2, we briefly present the continuous setting, and in Section 3, we do the same for the discrete setting. In Section 4, we present the error and stability analysis. The main results in this section are Theorems 9 and 11. Finally, in Section 5, we establish bounds on the approximation and divergence conformity factors introduced in the analysis, proving that these quantities tend to zero as the mesh is refined. Incidentally, we notice that the results established in Section 4 hold more generally when working on a generic subspace of $\boldsymbol{H}_0(\mathbf{curl}; D)$ which is not necessarily constructed using finite elements. The finite element structure is used in Section 5.

The main motivation for presenting our main results using approximation factors is that these factors encapsulate the frequency dependency in the constants appearing in the (stability and) error estimate(s). Notice, in particular, that general situations for the material properties, such as those discussed at the end of Section 5, are particularly challenging as it is not possible to infer explicit frequency-dependent decay rates. Employing approximation factors also allows us to present more clearly the main ideas related to Schatz's duality argument in the context of Maxwell's equations, as shown in Section 4.

2. Continuous setting

In this section, we briefly recall the functional setting for the time-harmonic Maxwell's equations, formulate the model problem and examine its inf-sup stability.

2.1. *Functional spaces*

We use standard notation for Lebesgue and Sobolev spaces. To alleviate the notation, the inner product and associated norm in the spaces $L^2(D)$ and $\boldsymbol{L}^2(D)$ are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. The material properties $\boldsymbol{\epsilon}$ and $\boldsymbol{\nu} := \boldsymbol{\mu}^{-1}$ are measurable functions that take symmetric positive-definite values in D with eigenvalues uniformly bounded from above and from below away from zero. It is convenient to introduce the inner product and associated norm weighted by either $\boldsymbol{\epsilon}$ or $\boldsymbol{\nu}$, leading to the notation $(\cdot,\cdot)_{\boldsymbol{\epsilon}}$, $\|\cdot\|_{\boldsymbol{\epsilon}}$, $(\cdot,\cdot)_{\boldsymbol{\nu}}$ and $\|\cdot\|_{\boldsymbol{\nu}}$. Whenever no confusion can arise, we use the symbol $^\perp$ to denote orthogonality with respect to the inner product $(\cdot,\cdot)_{\boldsymbol{\epsilon}}$. Moreover, all the projection operators denoted using the symbol $\boldsymbol{\Pi}$ are meant to be orthogonal with respect to this inner product; we say that the projections are $\boldsymbol{L}^2_{\boldsymbol{\epsilon}}$-orthogonal.

We consider the Hilbert Sobolev spaces

$$\boldsymbol{H}(\mathbf{curl};D) := \left\{\boldsymbol{v}\in\boldsymbol{L}^2(D)\;\middle|\;\nabla\times\boldsymbol{v}\in\boldsymbol{L}^2(D)\right\}, \tag{7a}$$

$$\boldsymbol{H}(\mathbf{curl}=\mathbf{0};D) := \left\{\boldsymbol{v}\in\boldsymbol{H}(\mathbf{curl};D)\;\middle|\;\nabla\times\boldsymbol{v}=\mathbf{0}\right\}, \tag{7b}$$

$$\boldsymbol{H}_0(\mathbf{curl};D) := \left\{\boldsymbol{v}\in\boldsymbol{H}(\mathbf{curl};D)\;\middle|\;\gamma^{\mathrm{c}}_{\partial D}(\boldsymbol{v})=\mathbf{0}\right\}, \tag{7c}$$

$$\boldsymbol{H}_0(\mathbf{curl}=\mathbf{0};D) := \left\{\boldsymbol{v}\in\boldsymbol{H}_0(\mathbf{curl};D)\;\middle|\;\nabla_0\times\boldsymbol{v}=\mathbf{0}\right\}, \tag{7d}$$

where the tangential trace operator $\gamma^{\mathrm{c}}_{\partial D}\colon \boldsymbol{H}(\mathbf{curl};D)\to\boldsymbol{H}^{-\frac{1}{2}}(\partial D)$ is the extension by density of the tangent trace operator such that $\gamma^{\mathrm{c}}_{\partial D}(\boldsymbol{v}) = \boldsymbol{v}|_{\partial D}\times\boldsymbol{n}_D$ for smooth fields. The subscript $_0$ indicates the curl operator acting on fields respecting homogeneous Dirichlet conditions. Notice that $\nabla\times$ and $\nabla_0\times$ are adjoint to each other, i.e., $(\nabla_0\times\boldsymbol{v},\boldsymbol{w}) = (\boldsymbol{v},\nabla\times\boldsymbol{w})$ for all $(\boldsymbol{v},\boldsymbol{w})\in\boldsymbol{H}_0(\mathbf{curl};D)\times\boldsymbol{H}(\mathbf{curl};D)$.

We consider the subspace

$$\boldsymbol{X}^{\mathrm{c}}_0 := \boldsymbol{H}_0(\mathbf{curl};D)\cap\boldsymbol{H}_0(\mathbf{curl}=\mathbf{0};D)^\perp, \tag{8}$$

and we introduce the $\boldsymbol{L}^2_{\boldsymbol{\epsilon}}$-orthogonal projection

$$\boldsymbol{\Pi}^{\mathrm{c}}_0\colon \boldsymbol{L}^2(D)\longrightarrow\boldsymbol{H}_0(\mathbf{curl}=\mathbf{0};D). \tag{9}$$

Since $\nabla H^1_0(D)\subset\boldsymbol{H}_0(\mathbf{curl}=\mathbf{0};D)$, any field $\boldsymbol{\xi}\in\boldsymbol{X}^{\mathrm{c}}_0$ is such that $\nabla\cdot(\boldsymbol{\epsilon}\boldsymbol{\xi})=0$ in D. Hence, $\boldsymbol{X}^{\mathrm{c}}_0$ compactly embeds into $\boldsymbol{L}^2(D)$ [41].

Remark 1 (Topology of D). Recalling that orthogonality is meant with respect to the inner product $(\cdot,\cdot)_{\boldsymbol{\epsilon}}$, we have $\boldsymbol{H}_0(\mathbf{curl}=\mathbf{0};D)^\perp\subset\left\{\boldsymbol{v}\in\boldsymbol{L}^2(D),\ \nabla\cdot(\boldsymbol{\epsilon}\boldsymbol{v})=0\right\}$ with equality if and only if ∂D is connected (see, e.g., [1, Proposition 3.18] for the connection between the first de Rham cohomology group and the connectedness of the boundary in the case of vector fields with zero tangential component).

2.2. *Model problem*

We focus for simplicity on homogeneous Dirichlet boundary conditions and consider the functional space

$$\boldsymbol{V}^{\mathrm{c}}_0 := \boldsymbol{H}_0(\mathbf{curl};D). \tag{10}$$

Given a positive real number $\omega>0$ and a source term $\boldsymbol{J}\in(\boldsymbol{V}^{\mathrm{c}}_0)'$ (the topological dual space of $\boldsymbol{V}^{\mathrm{c}}_0$), the model problem amounts to finding $\boldsymbol{E}\in\boldsymbol{V}^{\mathrm{c}}_0$ such that

$$b(\boldsymbol{E},\boldsymbol{w}) = \langle\boldsymbol{J},\boldsymbol{w}\rangle\quad\forall\,\boldsymbol{w}\in\boldsymbol{V}^{\mathrm{c}}_0, \tag{11}$$

with the bilinear form defined on $\boldsymbol{V}^{\mathrm{c}}_0\times\boldsymbol{V}^{\mathrm{c}}_0$ such that

$$b(\boldsymbol{v},\boldsymbol{w}) := -\omega^2(\boldsymbol{v},\boldsymbol{w})_{\boldsymbol{\epsilon}} + (\nabla_0\times\boldsymbol{v},\nabla_0\times\boldsymbol{w})_{\boldsymbol{\nu}}, \tag{12}$$

and where the brackets on the right-hand side of (11) denote the duality product between $(\boldsymbol{V}^{\mathrm{c}}_0)'$ and $\boldsymbol{V}^{\mathrm{c}}_0$. In what follows, we assume that ω^2 is not an eigenvalue of the $\nabla\times(\boldsymbol{\nu}\nabla_0\times\cdot)$ operator in D. As a result, the model problem (11) is well-posed. We equip the space $\boldsymbol{H}(\mathbf{curl};D)$ and its subspaces defined in (7) with the energy norm defined in (6), and observe that the bilinear form b satisfies $\left|b(\boldsymbol{v},\boldsymbol{w})\right|\le |||\boldsymbol{v}|||\,|||\boldsymbol{w}|||$.

2.3. *Inf-sup stability*

For all $\boldsymbol{g} \in \boldsymbol{L}^2(D)$, let $\boldsymbol{v_g} \in \boldsymbol{V}_0^{\mathrm{c}}$ denote the unique solution to (11) with right-hand side $(\boldsymbol{g}, \boldsymbol{w})_{\boldsymbol{\epsilon}}$, i.e., $b(\boldsymbol{v_g}, \boldsymbol{w}) = (\boldsymbol{g}, \boldsymbol{w})_{\boldsymbol{\epsilon}}$ for all $\boldsymbol{w} \in \boldsymbol{V}_0^{\mathrm{c}}$. We introduce the (nondimensional) stability constant

$$\beta_{\mathrm{st}} := \sup_{\substack{\boldsymbol{g} \in \boldsymbol{H}_0(\mathbf{curl}=\mathbf{0};D)^{\perp} \\ \|\boldsymbol{g}\|_{\boldsymbol{\epsilon}}=1}} \omega |||\boldsymbol{v_g}|||. \tag{13}$$

Lemma 2 (Inf-sup stability). *The following holds:*

$$\frac{1}{1+2\beta_{\mathrm{st}}} \le \inf_{\substack{\boldsymbol{v} \in \boldsymbol{V}_0^{\mathrm{c}} \\ |||\boldsymbol{v}|||=1}} \sup_{\substack{\boldsymbol{w} \in \boldsymbol{V}_0^{\mathrm{c}} \\ |||\boldsymbol{w}|||=1}} \left| b(\boldsymbol{v}, \boldsymbol{w}) \right| \le \frac{1}{\beta_{\mathrm{st}}}. \tag{14}$$

Proof.

(1) Lower bound. Let $\boldsymbol{v} \in \boldsymbol{V}_0^{\mathrm{c}}$ and let us set $\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{v}_\Pi$ with $\boldsymbol{v}_0 := (I - \boldsymbol{\Pi}_0^{\mathrm{c}})(\boldsymbol{v})$ and $\boldsymbol{v}_\Pi := \boldsymbol{\Pi}_0^{\mathrm{c}}(\boldsymbol{v})$. Let $\boldsymbol{\xi}_0 \in \boldsymbol{V}_0^{\mathrm{c}}$ be the adjoint solution such that $b(\boldsymbol{w}, \boldsymbol{\xi}_0) = \omega^2 (\boldsymbol{w}, \boldsymbol{v}_0)_{\boldsymbol{\epsilon}}$ for all $\boldsymbol{w} \in \boldsymbol{V}_0^{\mathrm{c}}$. Since $\boldsymbol{v}_0 \in \boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}; D)^{\perp}$ by construction, we have

$$|||\boldsymbol{\xi}_0||| \le \beta_{\mathrm{st}} \omega \|\boldsymbol{v}_0\|_{\boldsymbol{\epsilon}} \le \beta_{\mathrm{st}} |||\boldsymbol{v}_0|||, \tag{15}$$

owing to the symmetry of b and the definition of β_{st} for the first bound, and the definition of the $|||\cdot|||$-norm for the second bound. Moreover, taking the test function $\boldsymbol{w} := \boldsymbol{v} \in \boldsymbol{V}_0^{\mathrm{c}}$ in the adjoint problem, we infer that

$$b(\boldsymbol{v}, \boldsymbol{\xi}_0) = \omega^2 (\boldsymbol{v}, \boldsymbol{v}_0)_{\boldsymbol{\epsilon}} = \omega^2 \|\boldsymbol{v}_0\|_{\boldsymbol{\epsilon}}^2,$$

since $\boldsymbol{v}_0$ and $\boldsymbol{v}_\Pi$ are $\boldsymbol{L}^2_{\boldsymbol{\epsilon}}$-orthogonal. In addition, invoking the symmetry of b and since $\boldsymbol{v}_\Pi$ is curl-free, we have

$$\begin{aligned} b(\boldsymbol{v}, \boldsymbol{v}_0 - \boldsymbol{v}_\Pi) &= b(\boldsymbol{v}_0 + \boldsymbol{v}_\Pi, \boldsymbol{v}_0 - \boldsymbol{v}_\Pi) \\ &= b(\boldsymbol{v}_0, \boldsymbol{v}_0) - b(\boldsymbol{v}_\Pi, \boldsymbol{v}_\Pi) \\ &= |||\boldsymbol{v}_0|||^2 - 2\omega^2 \|\boldsymbol{v}_0\|_{\boldsymbol{\epsilon}}^2 + \omega^2 \|\boldsymbol{v}_\Pi\|_{\boldsymbol{\epsilon}}^2 \\ &= |||\boldsymbol{v}|||^2 - 2\omega^2 \|\boldsymbol{v}_0\|_{\boldsymbol{\epsilon}}^2. \end{aligned}$$

Combining the above two identities proves that

$$b(\boldsymbol{v}, \boldsymbol{v}_0 + 2\boldsymbol{\xi}_0 - \boldsymbol{v}_\Pi) = |||\boldsymbol{v}|||^2.$$

Finally, owing to (15), we have

$$\begin{aligned} |||\boldsymbol{v}_0 + 2\boldsymbol{\xi}_0 - \boldsymbol{v}_\Pi|||^2 &= |||\boldsymbol{v}_0 + 2\boldsymbol{\xi}_0|||^2 + \omega^2 \|\boldsymbol{v}_\Pi\|_{\boldsymbol{\epsilon}}^2 \\ &\le (1+2\beta_{\mathrm{st}})^2 |||\boldsymbol{v}_0|||^2 + \omega^2 \|\boldsymbol{v}_\Pi\|_{\boldsymbol{\epsilon}}^2 \\ &\le (1+2\beta_{\mathrm{st}})^2 |||\boldsymbol{v}|||^2, \end{aligned}$$

since $|||\boldsymbol{v}_0|||^2 + \omega^2 \|\boldsymbol{v}_\Pi\|_{\boldsymbol{\epsilon}}^2 = |||\boldsymbol{v}|||^2$. This proves the lower bound.

(2) Upper bound. For all $\boldsymbol{\phi} \in (\boldsymbol{V}_0^{\mathrm{c}})'$, let $\boldsymbol{v}_{\boldsymbol{\phi}}$ denote the unique solution to (11) with right-hand side $\langle \boldsymbol{\phi}, \boldsymbol{w} \rangle$. Let α denote the inf-sup constant in (14). Then $\alpha > 0$ since (11) is well posed, and we have $\alpha^{-1} = \sup_{\boldsymbol{\phi} \in (\boldsymbol{V}_0^{\mathrm{c}})'} \frac{|||\boldsymbol{v}_{\boldsymbol{\phi}}|||}{\|\boldsymbol{\phi}\|_{(\boldsymbol{V}_0^{\mathrm{c}})'}}$ (see, e.g., [23, Lemma C.51]). We consider the linear forms $\boldsymbol{\phi_g} \in (\boldsymbol{V}_0^{\mathrm{c}})'$ such that $\langle \boldsymbol{\phi_g}, \boldsymbol{w} \rangle := (\boldsymbol{g}, \boldsymbol{w})_{\boldsymbol{\epsilon}}$ for all $\boldsymbol{w} \in \boldsymbol{V}_0^{\mathrm{c}}$ for some function $\boldsymbol{g} \in \boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}; D)^{\perp}$. Owing to the Cauchy–Schwarz inequality and the definition of the $|||\cdot|||$-norm, we have

$$\|\boldsymbol{\phi_g}\|_{(\boldsymbol{V}_0^{\mathrm{c}})'} = \sup_{\boldsymbol{w} \in \boldsymbol{V}_0^{\mathrm{c}}} \frac{\left|\langle \boldsymbol{\phi_g}, \boldsymbol{w} \rangle\right|}{|||\boldsymbol{w}|||} \le \sup_{\boldsymbol{w} \in \boldsymbol{V}_0^{\mathrm{c}}} \frac{\|\boldsymbol{g}\|_{\boldsymbol{\epsilon}} \|\boldsymbol{w}\|_{\boldsymbol{\epsilon}}}{|||\boldsymbol{w}|||} \le \omega^{-1} \|\boldsymbol{g}\|_{\boldsymbol{\epsilon}}.$$

Restricting the supremum defining α^{-1} to the above linear forms, we infer that

$$\alpha^{-1} \geq \sup_{\boldsymbol{g}\in \boldsymbol{H}_0(\mathbf{curl}=\mathbf{0};D)^{\perp}} \frac{|||\boldsymbol{v_g}|||}{\omega^{-1}\|\boldsymbol{g}\|_{\boldsymbol{\epsilon}}} = \beta_{\rm st}.$$

This proves the upper bound. □

Remark 3 (Inf-sup condition). The stability constant $\beta_{\rm st}$ is bounded since we are assuming that ω is not a resonant frequency. This constant is expected to degenerate as ω approaches a resonant frequency since $\beta_{\rm st}$ essentially scales as ω times the reciprocal of the distance of ω to the spectrum of the $\boldsymbol{\epsilon}^{-1}\nabla\times(\boldsymbol{\nu}\nabla_0\times\cdot)$ operator (see [12] and [22, Theorem 35.13]). Even if ω is not a resonant frequency, the stability constant $\beta_{\rm st}$ is expected to be large. Interestingly, (14) means that, up to at most a factor of two, the inf-sup constant of the bilinear form b can be estimated as $(1+2\beta_{\rm st})^{-1}$. For a study of the Fredholmness of discrete operators in the different context of electric field integral equations, we refer the reader to [13].

3. Discrete setting

In this section, we present the setting to formulate the discrete problem, and we introduce two (nondimensional) quantities to be used in the error and stability analysis presented in the next section.

3.1. *Mesh and finite element spaces*

Let $\mathcal{T}_h$ be an affine simplicial mesh covering D exactly. A generic mesh cell is denoted K, its diameter h_K and its outward unit normal $\boldsymbol{n}_K$. Let $k \geq 0$ be the polynomial degree. Let $\mathbb{P}_{k,d}$ be the space composed of d-variate polynomials of total degree at most k and set $\boldsymbol{\mathbb{P}}_{k,d} := [\mathbb{P}_{k,d}]^d$. Let $\boldsymbol{\mathbb{N}}_{k,d}$ be the space composed of the d-variate Nédélec polynomials of order k of the first kind (recall that $\boldsymbol{\mathbb{P}}_{k,d} \subsetneq \boldsymbol{\mathbb{N}}_{k,d} \subsetneq \boldsymbol{\mathbb{P}}_{k+1,d}$). We consider the discrete subspace:

$$\boldsymbol{V}^{\rm c}_{h0} := \left\{\boldsymbol{v}_h \in \boldsymbol{V}^{\rm c}_0 \;\middle|\; \boldsymbol{v}_h|_K \in \boldsymbol{\mathbb{N}}_{k,d},\ \forall\, K \in \mathcal{T}_h\right\}. \tag{16}$$

Moreover, we let

$$\boldsymbol{V}^{\rm c}_{h0}(\mathbf{curl}=\mathbf{0}) := \left\{\boldsymbol{v}_h \in \boldsymbol{V}^{\rm c}_{h0} \;\middle|\; \nabla_0\times\boldsymbol{v}_h = \mathbf{0}\right\}. \tag{17}$$

The $\boldsymbol{L}^2_{\boldsymbol{\epsilon}}$-orthogonal projection

$$\boldsymbol{\Pi}^{\rm c}_{h0} \colon \boldsymbol{L}^2(D) \longrightarrow \boldsymbol{V}^{\rm c}_{h0}(\mathbf{curl}=\mathbf{0}) \tag{18}$$

plays a key role in what follows. We notice that $\boldsymbol{\Pi}^{\rm c}_{h0} \circ \boldsymbol{\Pi}^{\rm c}_{0} = \boldsymbol{\Pi}^{\rm c}_{h0}$ since $\boldsymbol{V}^{\rm c}_{h0}(\mathbf{curl}=\mathbf{0}) \subset \boldsymbol{H}_0(\mathbf{curl}=\mathbf{0};D)$. We introduce the subspace

$$\boldsymbol{X}^{\rm c}_{h0} := \boldsymbol{V}^{\rm c}_{h0} \cap \boldsymbol{V}^{\rm c}_{h0}(\mathbf{curl}=\mathbf{0})^{\perp}, \tag{19}$$

which is composed of fields $\boldsymbol{v}_h \in \boldsymbol{V}^{\rm c}_{h0}$ such that $\boldsymbol{\Pi}^{\rm c}_{h0}(\boldsymbol{v}_h) = \mathbf{0}$. Loosely speaking, discrete fields in $\boldsymbol{X}^{\rm c}_{h0}$ are discretely divergence-free (and satisfy a finite number of additional constraints when ∂D is not connected).

3.2. *Discrete problem*

The discrete problem reads as follows: find $\boldsymbol{E}_h \in \boldsymbol{V}^{\rm c}_{h0}$ such that

$$b(\boldsymbol{E}_h, \boldsymbol{w}_h) = \langle \boldsymbol{J}, \boldsymbol{w}_h\rangle \quad \forall\, \boldsymbol{w}_h \in \boldsymbol{V}^{\rm c}_{h0}. \tag{20}$$

The well-posedness of (20) is established in Section 4.

3.3. *Approximation and divergence conformity factors*

In this section, we define two (nondimensional) quantities, the approximation factor and the divergence conformity factor. Both factors are used in the error and inf-sup stability analysis. We prove in Section 5 that both factors tend to zero as the mesh size tends to zero or the polynomial degree is increased.

For all $\boldsymbol{\theta} \in \boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}; D)^{\perp}$, we consider the adjoint problem consisting of finding $\boldsymbol{\zeta_\theta} \in \boldsymbol{V}_0^{\rm c}$ such that

$$b(\boldsymbol{w}, \boldsymbol{\zeta_\theta}) = (\boldsymbol{w}, \boldsymbol{\theta})_{\boldsymbol{\epsilon}} \quad \forall\, \boldsymbol{w} \in \boldsymbol{V}_0^{\rm c}. \tag{21}$$

Taking any test function $\boldsymbol{w} \in \boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}; D) \subset \boldsymbol{V}_0^{\rm c}$ shows that

$$\omega^2 (\boldsymbol{w}, \boldsymbol{\zeta_\theta})_{\boldsymbol{\epsilon}} = b(\boldsymbol{w}, \boldsymbol{\zeta_\theta}) = (\boldsymbol{w}, \boldsymbol{\theta})_{\boldsymbol{\epsilon}} = 0, \tag{22}$$

where the first equality follows from $\nabla_0 \times \boldsymbol{w} = \mathbf{0}$, the second from the definition of the adjoint solution, and the third from the assumption $\boldsymbol{\theta} \in \boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}; D)^{\perp}$. Since $\boldsymbol{w}$ is arbitrary in $\boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}; D)$, this proves that $\boldsymbol{\zeta_\theta} \in \boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}; D)^{\perp}$. Thus, $\boldsymbol{\zeta_\theta} \in \boldsymbol{X}_0^{\rm c}$, and owing to the compact embedding $\boldsymbol{X}_0^{\rm c} \hookrightarrow \boldsymbol{L}^2(D)$, it is reasonable to expect that the (nondimensional) approximation factor

$$\gamma_{\rm ap} := \sup_{\substack{\boldsymbol{\theta} \in \boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}; D)^{\perp} \\ \|\boldsymbol{\theta}\|_{\boldsymbol{\epsilon}} = 1}} \min_{\boldsymbol{v}_h^{\rm c} \in \boldsymbol{V}_{h0}^{\rm c}} \omega |\!|\!| \boldsymbol{\zeta_\theta} - \boldsymbol{v}_h^{\rm c} |\!|\!|, \tag{23}$$

exhibits some decay rate as the mesh size tends to zero and/or the polynomial degree is increased.

The second useful quantity is the (nondimensional) divergence conformity factor

$$\gamma_{\rm dv} := \sup_{\substack{\boldsymbol{v}_h \in \boldsymbol{X}_{h0}^{\rm c} \\ \|\nabla_0 \times \boldsymbol{v}_h\|_{\boldsymbol{\nu}} = 1}} \omega \left\| \Pi_0^{\rm c}(\boldsymbol{v}_h) \right\|_{\boldsymbol{\epsilon}}. \tag{24}$$

Loosely speaking, the adopted notation reminds us that $\gamma_{\rm dv}$ essentially measures how much a discrete field that is discretely divergence-free is not pointwise divergence-free. It is again reasonable to expect, under the same conditions as above, that $\gamma_{\rm dv}$ tends to zero.

4. Error and stability analysis

In this section, we analyze the conforming approximation (20) of the model problem (11). We first establish an error estimate under a smallness condition on the mesh size. As usual in duality arguments, we first assume that a discrete solution exists, and then observe that the error estimate implies the uniqueness, and thus also the existence, of the discrete solution. Finally, we establish an inf-sup condition under a similar assumption on the mesh size.

4.1. *Error decomposition and preliminary bounds*

Let us define the bilinear form b^+ on $\boldsymbol{V}_0^{\rm c} \times \boldsymbol{V}_0^{\rm c}$ such that

$$b^+(\boldsymbol{v}, \boldsymbol{w}) := \omega^2 (\boldsymbol{v}, \boldsymbol{w})_{\boldsymbol{\epsilon}} + (\nabla_0 \times \boldsymbol{v}, \nabla_0 \times \boldsymbol{w})_{\boldsymbol{\nu}}. \tag{25}$$

We define the best-approximation operator $\mathscr{B}_{h0}^{\rm c} : \boldsymbol{V}_0^{\rm c} \to \boldsymbol{V}_{h0}^{\rm c}$ as follows: for all $\boldsymbol{v} \in \boldsymbol{V}_0^{\rm c}$, $\mathscr{B}_{h0}^{\rm c}(\boldsymbol{v}) \in \boldsymbol{V}_{h0}^{\rm c}$ is such that

$$b^+\big(\boldsymbol{v} - \mathscr{B}_{h0}^{\rm c}(\boldsymbol{v}), \boldsymbol{w}_h\big) = 0 \quad \forall\, \boldsymbol{w}_h \in \boldsymbol{V}_{h0}^{\rm c}. \tag{26}$$

Lemma 4 (Properties of $\mathscr{B}_{h0}^{\rm c}$). *The best-approximation operator $\mathscr{B}_{h0}^{\rm c}$ defined in* (26) *enjoys the following two properties:*

$$|\!|\!| \mathscr{B}_{h0}^{\rm c}(\boldsymbol{v}) |\!|\!| \le |\!|\!| \boldsymbol{v} |\!|\!|, \qquad \forall\, \boldsymbol{v} \in \boldsymbol{V}_0^{\rm c}, \tag{27a}$$

$$\mathscr{B}_{h0}^{\rm c}(\boldsymbol{v}) \in \boldsymbol{V}_{h0}^{\rm c}(\mathbf{curl} = \mathbf{0})^{\perp}, \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}_{h0}^{\rm c}(\mathbf{curl} = \mathbf{0})^{\perp}. \tag{27b}$$

Proof. (27a) follows from the fact that the bilinear form b^+ is the inner product associated with the $|||\cdot|||$-norm. To prove (27b), take any $\boldsymbol{w}_h \in \boldsymbol{V}^{\rm c}_{h0}(\mathbf{curl} = \mathbf{0})$ in (26) and observe that $\omega^2\big(\boldsymbol{v} - \mathscr{B}^{\rm c}_{h0}(\boldsymbol{v}), \boldsymbol{w}_h\big)_{\boldsymbol{\epsilon}} = b^+\big(\boldsymbol{v} - \mathscr{B}^{\rm c}_{h0}(\boldsymbol{v}), \boldsymbol{w}_h\big) = 0$. □

Assume that $\boldsymbol{E}_h$ solves (20). We define the approximation error and the best-approximation error as follows:

$$\boldsymbol{e} := \boldsymbol{E} - \boldsymbol{E}_h, \qquad \boldsymbol{\eta} := \boldsymbol{E} - \mathscr{B}^{\rm c}_{h0}(\boldsymbol{E}). \tag{28}$$

We consider the error decomposition $\boldsymbol{e} = \boldsymbol{\theta}_0 + \boldsymbol{\theta}_\Pi$, with

$$\boldsymbol{\theta}_0 := (I - \boldsymbol{\Pi}^{\rm c}_0)(\boldsymbol{e}) \in \boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}; D)^\perp, \qquad \boldsymbol{\theta}_\Pi := \boldsymbol{\Pi}^{\rm c}_0(\boldsymbol{e}) \in \boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}; D). \tag{29}$$

The motivation behind this decomposition is that $\omega\|\boldsymbol{\theta}_0\|_{\boldsymbol{\epsilon}}$ can be bounded by a duality argument, whereas $\omega\|\boldsymbol{\theta}_\Pi\|_{\boldsymbol{\epsilon}}$ represents a divergence conformity error. The main steps of the error analysis are as follows: (i) bound on the $L^2_{\boldsymbol{\epsilon}}$-norm of the (loosely speaking) divergence-free part of the error (Lemma 5); (ii) bound on the $L^2_{\boldsymbol{\epsilon}}$-norm of the curl-free part of the error (Lemma 6); (iii) bound on the curl of the error (Lemma 8).

Lemma 5 (Bound on $\boldsymbol{\theta}_0$). *We have*

$$\omega\|\boldsymbol{\theta}_0\|_{\boldsymbol{\epsilon}} \le \gamma_{\rm ap}\, |||\boldsymbol{e}|||, \tag{30}$$

with the approximation factor $\gamma_{\rm ap}$ defined in (23).

Proof. We consider the adjoint problem (21) with data $\boldsymbol{\theta} := \boldsymbol{\theta}_0$. Notice that $\boldsymbol{\theta} \in \boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}; D)^\perp$ by construction. Let $\boldsymbol{\zeta}_{\boldsymbol{\theta}} \in \boldsymbol{X}^{\rm c}_0$ denote the unique adjoint solution to this problem. Since $\boldsymbol{e} \in \boldsymbol{V}^{\rm c}_0$, we have

$$b(\boldsymbol{e}, \boldsymbol{\zeta}_{\boldsymbol{\theta}}) = (\boldsymbol{e}, \boldsymbol{\theta}_0)_{\boldsymbol{\epsilon}}.$$

Using Galerkin's orthogonality together with $(\boldsymbol{\theta}_0, \boldsymbol{\theta}_\Pi)_{\boldsymbol{\epsilon}} = 0$, and multiplying by ω^2 gives

$$\omega^2\|\boldsymbol{\theta}_0\|^2_{\boldsymbol{\epsilon}} = \omega^2(\boldsymbol{e}, \boldsymbol{\theta}_0)_{\boldsymbol{\epsilon}} = \omega^2 b(\boldsymbol{e}, \boldsymbol{\zeta}_{\boldsymbol{\theta}} - \boldsymbol{v}_h) \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}^{\rm c}_{h0}.$$

Owing to the boundedness of the bilinear form b, we infer that

$$\omega^2\|\boldsymbol{\theta}_0\|^2_{\boldsymbol{\epsilon}} \le \omega^2 |||\boldsymbol{e}|||\; |||\boldsymbol{\zeta}_{\boldsymbol{\theta}} - \boldsymbol{v}_h|||.$$

Invoking the approximation factor $\gamma_{\rm ap}$ gives

$$\omega^2\|\boldsymbol{\theta}_0\|^2_{\boldsymbol{\epsilon}} \le |||\boldsymbol{e}|||\, \gamma_{\rm ap}\omega\|\boldsymbol{\theta}_0\|_{\boldsymbol{\epsilon}},$$

since $\boldsymbol{v}_h$ is arbitrary. This proves the claim. □

Lemma 6 (Bound on $\boldsymbol{\theta}_\Pi$). *We have*

$$(1 - \gamma_{\rm dv})\omega\|\boldsymbol{\theta}_\Pi\|_{\boldsymbol{\epsilon}} \le \omega\big\|\boldsymbol{\Pi}^{\rm c}_0(\boldsymbol{\eta})\big\|_{\boldsymbol{\epsilon}} + \gamma_{\rm dv}|||\boldsymbol{\theta}_0|||. \tag{31}$$

Proof. We observe that $\boldsymbol{\Pi}^{\rm c}_{h0}(\boldsymbol{\theta}_\Pi) = \boldsymbol{\Pi}^{\rm c}_{h0}\big(\boldsymbol{\Pi}^{\rm c}_0(\boldsymbol{e})\big) = \boldsymbol{\Pi}^{\rm c}_{h0}(\boldsymbol{e})$ since $\boldsymbol{\Pi}^{\rm c}_{h0} \circ \boldsymbol{\Pi}^{\rm c}_0 = \boldsymbol{\Pi}^{\rm c}_{h0}$. Moreover, Galerkin's orthogonality implies that $\omega^2(\boldsymbol{e}, \boldsymbol{w}_h)_{\boldsymbol{\epsilon}} = -b(\boldsymbol{e}, \boldsymbol{w}_h) = 0$ for all $\boldsymbol{w}_h \in \boldsymbol{V}^{\rm c}_{h0}(\mathbf{curl} = \mathbf{0})$. Hence, $\boldsymbol{\Pi}^{\rm c}_{h0}(\boldsymbol{e}) = \mathbf{0}$, and this proves that

$$\boldsymbol{\theta}_\Pi \in \boldsymbol{V}^{\rm c}_{h0}(\mathbf{curl} = \mathbf{0})^\perp. \tag{32}$$

To bound $\boldsymbol{\theta}_\Pi$, we write

$$\|\boldsymbol{\theta}_\Pi\|^2_{\boldsymbol{\epsilon}} = \big(\boldsymbol{\theta}_\Pi, \mathscr{B}^{\rm c}_{h0}(\boldsymbol{\theta}_\Pi)\big)_{\boldsymbol{\epsilon}} + \big(\boldsymbol{\theta}_\Pi, \boldsymbol{\theta}_\Pi - \mathscr{B}^{\rm c}_{h0}(\boldsymbol{\theta}_\Pi)\big)_{\boldsymbol{\epsilon}} =: \Theta_1 + \Theta_2,$$

and estimate the two terms on the right-hand side. Since $\boldsymbol{\theta}_\Pi = \boldsymbol{\Pi}^{\rm c}_0(\boldsymbol{\theta}_\Pi)$ and $\boldsymbol{\Pi}^{\rm c}_0$ is self-adjoint for the inner product $(\cdot,\cdot)_{\boldsymbol{\epsilon}}$, we obtain

$$\begin{aligned}
\Theta_1 &= \big(\boldsymbol{\theta}_\Pi, \mathscr{B}^{\rm c}_{h0}(\boldsymbol{\theta}_\Pi)\big)_\epsilon \\
&= \big(\boldsymbol{\theta}_\Pi, \boldsymbol{\Pi}^{\rm c}_0\big(\mathscr{B}^{\rm c}_{h0}(\boldsymbol{\theta}_\Pi)\big)\big)_{\boldsymbol{\epsilon}} \\
&\le \|\boldsymbol{\theta}_\Pi\|_{\boldsymbol{\epsilon}} \big\|\boldsymbol{\Pi}^{\rm c}_0\big(\mathscr{B}^{\rm c}_{h0}(\boldsymbol{\theta}_\Pi)\big)\big\|_{\boldsymbol{\epsilon}} \\
&\le \|\boldsymbol{\theta}_\Pi\|_{\boldsymbol{\epsilon}}\, \gamma_{\rm dv}\omega^{-1}\big\|\nabla_0 \times \mathscr{B}^{\rm c}_{h0}(\boldsymbol{\theta}_\Pi)\big\|_{\boldsymbol{\nu}},
\end{aligned}$$

where we used the Cauchy–Schwarz inequality in the third line and the divergence conformity factor defined in (24) in the fourth line (recall that $\mathscr{B}^{\mathrm{c}}_{h0}(\boldsymbol{\theta}_\Pi) \in \boldsymbol{V}^{\mathrm{c}}_{h0}$ by construction and observe that $\mathscr{B}^{\mathrm{c}}_{h0}(\boldsymbol{\theta}_\Pi) \in \boldsymbol{V}^{\mathrm{c}}_{h0}(\mathbf{curl}=\mathbf{0})^\perp$ owing to (27b) and (32)). Since

$$\left\|\nabla_0\times\mathscr{B}^{\mathrm{c}}_{h0}(\boldsymbol{\theta}_\Pi)\right\|_{\boldsymbol{\nu}} \leq \left|\!\left|\!\left|\mathscr{B}^{\mathrm{c}}_{h0}(\boldsymbol{\theta}_\Pi)\right|\!\right|\!\right| \leq |\!|\!|\boldsymbol{\theta}_\Pi|\!|\!| = \omega\|\boldsymbol{\theta}_\Pi\|_{\boldsymbol{\epsilon}},$$

owing to the definition of the $|\!|\!|\cdot|\!|\!|$-norm and the stability property (27a) of $\mathscr{B}^{\mathrm{c}}_{h0}$, we infer that

$$|\Theta_1| \leq \gamma_{\mathrm{dv}}\|\boldsymbol{\theta}_\Pi\|^2_{\boldsymbol{\epsilon}}.$$

Furthermore, recalling that $\boldsymbol{\theta}_\Pi = \boldsymbol{E}-\boldsymbol{E}_h-\boldsymbol{\theta}_0$, we infer that $(I-\mathscr{B}^{\mathrm{c}}_{h0})(\boldsymbol{\theta}_\Pi) = \boldsymbol{\eta}-(I-\mathscr{B}^{\mathrm{c}}_{h0})(\boldsymbol{\theta}_0)$ by definition of $\boldsymbol{\eta}$ and since $(I-\mathscr{B}^{\mathrm{c}}_{h0})(\boldsymbol{E}_h) = \mathbf{0}$. Thus, we obtain

$$\Theta_2 = (\boldsymbol{\theta}_\Pi,\boldsymbol{\eta})_{\boldsymbol{\epsilon}} - \left(\boldsymbol{\theta}_\Pi,\boldsymbol{\theta}_0-\mathscr{B}^{\mathrm{c}}_{h0}(\boldsymbol{\theta}_0)\right)_{\boldsymbol{\epsilon}} =: \Theta_{2,1}-\Theta_{2,2}.$$

The Cauchy–Schwarz inequality gives

$$|\Theta_{2,1}| = \left|(\boldsymbol{\theta}_\Pi,\boldsymbol{\eta})_{\boldsymbol{\epsilon}}\right| = \left|\left(\boldsymbol{\theta}_\Pi,\boldsymbol{\Pi}^{\mathrm{c}}_0(\boldsymbol{\eta})\right)_{\boldsymbol{\epsilon}}\right| \leq \|\boldsymbol{\theta}_\Pi\|_{\boldsymbol{\epsilon}}\left\|\boldsymbol{\Pi}^{\mathrm{c}}_0(\boldsymbol{\eta})\right\|_{\boldsymbol{\epsilon}}.$$

Concerning $\Theta_{2,2}$, we have

$$\begin{aligned}
|\Theta_{2,2}| &= \left|\left(\boldsymbol{\theta}_\Pi,\boldsymbol{\theta}_0-\mathscr{B}^{\mathrm{c}}_{h0}(\boldsymbol{\theta}_0)\right)_{\boldsymbol{\epsilon}}\right| \\
&= \left|\left(\boldsymbol{\theta}_\Pi,\mathscr{B}^{\mathrm{c}}_{h0}(\boldsymbol{\theta}_0)\right)_{\boldsymbol{\epsilon}}\right| \\
&= \left|\left(\boldsymbol{\theta}_\Pi,\boldsymbol{\Pi}^{\mathrm{c}}_0\left(\mathscr{B}^{\mathrm{c}}_{h0}(\boldsymbol{\theta}_0)\right)\right)_{\boldsymbol{\epsilon}}\right| \\
&\leq \|\boldsymbol{\theta}_\Pi\|_{\boldsymbol{\epsilon}}\left\|\boldsymbol{\Pi}^{\mathrm{c}}_0\left(\mathscr{B}^{\mathrm{c}}_{h0}(\boldsymbol{\theta}_0)\right)\right\|_{\epsilon} \\
&\leq \|\boldsymbol{\theta}_\Pi\|_{\boldsymbol{\epsilon}}\,\gamma_{\mathrm{dv}}\omega^{-1}\left\|\nabla_0\times\mathscr{B}^{\mathrm{c}}_{h0}(\boldsymbol{\theta}_0)\right\|_{\boldsymbol{\nu}} \\
&\leq \|\boldsymbol{\theta}_\Pi\|_{\boldsymbol{\epsilon}}\,\gamma_{\mathrm{dv}}\omega^{-1}|\!|\!|\boldsymbol{\theta}_0|\!|\!|.
\end{aligned}$$

Notice that we can again use the divergence conformity factor γ_{dv} since $\boldsymbol{\theta}_0 \in \boldsymbol{H}_0(\mathbf{curl}=\mathbf{0};D)^\perp$ implies by (27b) that $\mathscr{B}^{\mathrm{c}}_{h0}(\boldsymbol{\theta}_0) \in \boldsymbol{V}^{\mathrm{c}}_{h0}(\mathbf{curl}=\mathbf{0})^\perp$. Putting the above bounds on $\Theta_{2,1}$ and $\Theta_{2,2}$ together gives

$$|\Theta_2| \leq \|\boldsymbol{\theta}_\Pi\|_{\boldsymbol{\epsilon}}\,\omega^{-1}\left(\omega\left\|\boldsymbol{\Pi}^{\mathrm{c}}_0(\boldsymbol{\eta})\right\|_{\boldsymbol{\epsilon}}+\gamma_{\mathrm{dv}}|\!|\!|\boldsymbol{\theta}_0|\!|\!|\right).$$

The estimate (31) follows by combining the above bounds on Θ_1 and Θ_2. □

Remark 7 (Lemma 6). Obviously, the estimate (31) is meaningful only if $\gamma_{\mathrm{dv}} < 1$, which holds true if the mesh size is small enough and/or the polynomial degree is large enough; see Section 5 for further insight.

Lemma 8 (Bound on $\nabla_0\times\boldsymbol{\theta}_0$). *We have*

$$\|\nabla_0\times\boldsymbol{\theta}_0\|^2_{\boldsymbol{\nu}} \leq \left|\!\left|\!\left|(I-\boldsymbol{\Pi}^{\mathrm{c}}_0)(\boldsymbol{\eta})\right|\!\right|\!\right|^2 + (2\gamma_{\mathrm{dv}}+3\gamma^2_{\mathrm{ap}})|\!|\!|\boldsymbol{e}|\!|\!|^2 + 2\gamma_{\mathrm{dv}}\omega^2\|\boldsymbol{\theta}_\Pi\|^2_{\boldsymbol{\epsilon}}. \tag{33}$$

Proof. Recalling that $\boldsymbol{\theta}_0 = (I-\boldsymbol{\Pi}^{\mathrm{c}}_0)(\boldsymbol{e})$, a straightforward calculation shows that

$$\begin{aligned}
b(\boldsymbol{\theta}_0,\boldsymbol{\theta}_0) &= b\left(\boldsymbol{\theta}_0,(I-\boldsymbol{\Pi}^{\mathrm{c}}_0)(\boldsymbol{e})\right) \\
&= b\left(\boldsymbol{\theta}_0,(I-\boldsymbol{\Pi}^{\mathrm{c}}_0)(\boldsymbol{\eta})\right) - b\left(\boldsymbol{\theta}_0,(I-\boldsymbol{\Pi}^{\mathrm{c}}_0)\left(\boldsymbol{E}_h-\mathscr{B}^{\mathrm{c}}_{h0}(\boldsymbol{E})\right)\right) \\
&= b\left(\boldsymbol{\theta}_0,(I-\boldsymbol{\Pi}^{\mathrm{c}}_0)(\boldsymbol{\eta})\right) - b\left(\boldsymbol{e},(I-\boldsymbol{\Pi}^{\mathrm{c}}_0)\left(\boldsymbol{E}_h-\mathscr{B}^{\mathrm{c}}_{h0}(\boldsymbol{E})\right)\right) \\
&\leq |\!|\!|\boldsymbol{\theta}_0|\!|\!|\,\left|\!\left|\!\left|(I-\boldsymbol{\Pi}^{\mathrm{c}}_0)(\boldsymbol{\eta})\right|\!\right|\!\right| - b\left(\boldsymbol{e},(I-\boldsymbol{\Pi}^{\mathrm{c}}_0)\left(\boldsymbol{E}_h-\mathscr{B}^{\mathrm{c}}_{h0}(\boldsymbol{E})\right)\right),
\end{aligned}$$

where we used that $b\left(\boldsymbol{\theta}_\Pi,(I-\boldsymbol{\Pi}^{\mathrm{c}}_0)(\cdot)\right) = 0$ on the third line and the boundedness of b on the fourth line. Focusing on the second term on the right-hand side, we notice using Galerkin's orthogonality that

$$\begin{aligned}
-b\left(\boldsymbol{e},(I-\boldsymbol{\Pi}^{\mathrm{c}}_0)\left(\boldsymbol{E}_h-\mathscr{B}^{\mathrm{c}}_{h0}(\boldsymbol{E})\right)\right) &= b\left(\boldsymbol{e},\boldsymbol{\Pi}^{\mathrm{c}}_0\left(\boldsymbol{E}_h-\mathscr{B}^{\mathrm{c}}_{h0}(\boldsymbol{E})\right)\right) \\
&= -\omega^2\left(\boldsymbol{\theta}_\Pi,\boldsymbol{\Pi}^{\mathrm{c}}_0\left(\boldsymbol{E}_h-\mathscr{B}^{\mathrm{c}}_{h0}(\boldsymbol{E})\right)\right)_{\boldsymbol{\epsilon}} \\
&\leq \omega^2\|\boldsymbol{\theta}_\Pi\|_{\boldsymbol{\epsilon}}\left\|\boldsymbol{\Pi}^{\mathrm{c}}_0\left(\boldsymbol{E}_h-\mathscr{B}^{\mathrm{c}}_{h0}(\boldsymbol{E})\right)\right\|_{\boldsymbol{\epsilon}}.
\end{aligned}$$

For all $\boldsymbol{v}_h \in \boldsymbol{V}_{h0}^{\mathrm{c}}(\mathbf{curl} = \mathbf{0})$, we have

$$\begin{aligned}\omega^2\big(\boldsymbol{E}_h - \mathscr{B}_{h0}^{\mathrm{c}}(\boldsymbol{E}), \boldsymbol{v}_h\big)_{\boldsymbol{\epsilon}} &= -b(\boldsymbol{E}_h, \boldsymbol{v}_h) - b^+\big(\mathscr{B}_{h0}^{\mathrm{c}}(\boldsymbol{E}), \boldsymbol{v}_h\big)\\ &= -b(\boldsymbol{E}, \boldsymbol{v}_h) - b^+(\boldsymbol{E}, \boldsymbol{v}_h)\\ &= \omega^2(\boldsymbol{E}, \boldsymbol{v}_h) - \omega^2(\boldsymbol{E}, \boldsymbol{v}_h)\\ &= 0,\end{aligned}$$

where the first and third equalities follow from the fact that $\boldsymbol{v}_h$ is curl-free, and the second from the definition of $\mathscr{B}_{h0}^{\mathrm{c}}$ and Galerkin's orthogonality. This shows that

$$\boldsymbol{E}_h - \mathscr{B}_{h0}^{\mathrm{c}}(\boldsymbol{E}) \in \boldsymbol{V}_{h0}^{\mathrm{c}}(\mathbf{curl} = \mathbf{0})^{\perp},$$

and, therefore, $\boldsymbol{E}_h - \mathscr{B}_{h0}^{\mathrm{c}}(\boldsymbol{E}) \in \boldsymbol{X}_{h0}^{\mathrm{c}}$. Using the divergence conformity factor γ_{dv} defined in (24) then yields

$$\left|b\big(\boldsymbol{e}, (I - \boldsymbol{\Pi}_0^{\mathrm{c}})\big(\boldsymbol{E}_h - \mathscr{B}_{h0}^{\mathrm{c}}(\boldsymbol{E})\big)\big)\right| \le \gamma_{\mathrm{dv}}\omega\|\boldsymbol{\theta}_\Pi\|_{\boldsymbol{\epsilon}}\left\|\nabla_0\times\big(\boldsymbol{E}_h - \mathscr{B}_{h0}^{\mathrm{c}}(\boldsymbol{E})\big)\right\|_{\boldsymbol{\nu}}.$$

Invoking the triangle inequality and the stability property (27a) of $\mathscr{B}_{h0}^{\mathrm{c}}$ gives

$$\begin{aligned}\left\|\nabla_0\times\big(\boldsymbol{E}_h - \mathscr{B}_{h0}^{\mathrm{c}}(\boldsymbol{E})\big)\right\|_{\boldsymbol{\nu}} &\le \left\|\nabla_0\times(\boldsymbol{E}_h - \boldsymbol{E})\right\|_{\boldsymbol{\nu}} + \left\|\nabla_0\times\big(\boldsymbol{E} - \mathscr{B}_{h0}^{\mathrm{c}}(\boldsymbol{E})\big)\right\|_{\boldsymbol{\nu}}\\ &\le |||\boldsymbol{e}||| + \left|\left|\left|\boldsymbol{E} - \mathscr{B}_{h0}^{\mathrm{c}}(\boldsymbol{E})\right|\right|\right|\\ &\le 2|||\boldsymbol{e}|||,\end{aligned}$$

since $\left|\left|\left|\boldsymbol{E} - \mathscr{B}_{h0}^{\mathrm{c}}(\boldsymbol{E})\right|\right|\right| \le |||\boldsymbol{E} - \boldsymbol{E}_h||| = |||\boldsymbol{e}|||$ by definition of the best-approximation operator $\mathscr{B}_{h0}^{\mathrm{c}}$. Putting everything together gives

$$b(\boldsymbol{\theta}_0, \boldsymbol{\theta}_0) \le |||\boldsymbol{\theta}_0|||\,\left|\left|\left|(I - \boldsymbol{\Pi}_0^{\mathrm{c}})(\boldsymbol{\eta})\right|\right|\right| + 2\gamma_{\mathrm{dv}}\omega\|\boldsymbol{\theta}_\Pi\|_{\boldsymbol{\epsilon}}|||\boldsymbol{e}|||.$$

As a result, we have

$$\begin{aligned}\|\nabla_0\times\boldsymbol{\theta}_0\|_{\boldsymbol{\nu}}^2 &= b(\boldsymbol{\theta}_0, \boldsymbol{\theta}_0) + \omega^2\|\boldsymbol{\theta}_0\|_{\boldsymbol{\epsilon}}^2\\ &\le |||\boldsymbol{\theta}_0|||\,\left|\left|\left|(I - \boldsymbol{\Pi}_0^{\mathrm{c}})(\boldsymbol{\eta})\right|\right|\right| + 2\gamma_{\mathrm{dv}}\omega\|\boldsymbol{\theta}_\Pi\|_{\boldsymbol{\epsilon}}|||\boldsymbol{e}||| + \omega^2\|\boldsymbol{\theta}_0\|_{\boldsymbol{\epsilon}}^2.\end{aligned}$$

Invoking Young's inequality for the first and second terms on the right-hand side and using that $|||\boldsymbol{\theta}_0|||^2 = \|\nabla_0\times\boldsymbol{\theta}_0\|_{\boldsymbol{\nu}}^2 + \omega^2\|\boldsymbol{\theta}_0\|_{\boldsymbol{\epsilon}}^2$, we infer that

$$\|\nabla_0\times\boldsymbol{\theta}_0\|_{\boldsymbol{\nu}}^2 \le \left|\left|\left|(I - \boldsymbol{\Pi}_0^{\mathrm{c}})(\boldsymbol{\eta})\right|\right|\right|^2 + 2\gamma_{\mathrm{dv}}|||\boldsymbol{e}|||^2 + 2\gamma_{\mathrm{dv}}\omega^2\|\boldsymbol{\theta}_\Pi\|_{\boldsymbol{\epsilon}}^2 + 3\omega^2\|\boldsymbol{\theta}_0\|_{\boldsymbol{\epsilon}}^2.$$

The assertion now follows from the bound on $\boldsymbol{\theta}_0$ established in Lemma 5. □

4.2. *Error estimate*

We are now ready to establish our main error estimate.

Theorem 9 (A priori error estimate and discrete well-posedness). *Assume that* $\gamma_{\mathrm{dv}} \le 1$. *The following holds:*

$$(1 - 15\gamma_{\mathrm{dv}} - 4\gamma_{\mathrm{ap}}^2)|||\boldsymbol{E} - \boldsymbol{E}_h||| \le \min_{\boldsymbol{v}_h \in \boldsymbol{V}_{h0}^{\mathrm{c}}} |||\boldsymbol{E} - \boldsymbol{v}_h|||. \tag{34}$$

Consequently, if the mesh size is small enough and/or the polynomial degree is large enough so that $15\gamma_{\mathrm{dv}} + 4\gamma_{\mathrm{ap}}^2 < 1$, *the discrete problem* (20) *is well-posed.*

Proof.

(1) In this first step, we establish some preliminary bounds. Since $|||\boldsymbol{\theta}_0||| \le |||\boldsymbol{e}|||$ (because $|||\boldsymbol{e}|||^2 = |||\boldsymbol{\theta}_0|||^2 + \omega^2\|\boldsymbol{\theta}_\Pi\|_{\boldsymbol{\epsilon}}^2$), the estimate (31) implies that

$$(1 - \gamma_{\mathrm{dv}})\omega\|\boldsymbol{\theta}_\Pi\|_{\boldsymbol{\epsilon}} \le \omega\left\|\boldsymbol{\Pi}_0^{\mathrm{c}}(\boldsymbol{\eta})\right\|_{\boldsymbol{\epsilon}} + \gamma_{\mathrm{dv}}|||\boldsymbol{e}|||. \tag{35}$$

Moreover, we have

$$\omega\left\|\boldsymbol{\Pi}_0^{\mathrm{c}}(\boldsymbol{\eta})\right\|_{\boldsymbol{\epsilon}} \le \omega\|\boldsymbol{\eta}\|_{\boldsymbol{\epsilon}} \le |||\boldsymbol{\eta}||| \le |||\boldsymbol{E} - \boldsymbol{E}_h||| = |||\boldsymbol{e}|||.$$

Squaring (35) (recall that $\gamma_{\rm dv} \leq 1$ by assumption) and using the above bound in the double product, we obtain

$$\begin{aligned}(1-\gamma_{\rm dv})^2\omega^2\|\boldsymbol{\theta}_\Pi\|^2_{\boldsymbol{\epsilon}} &\leq \omega^2\left\|\boldsymbol{\Pi}^{\rm c}_0(\boldsymbol{\eta})\right\|^2_{\boldsymbol{\epsilon}} + (2\gamma_{\rm dv}+\gamma^2_{\rm dv})|||\boldsymbol{e}|||^2 \\ &\leq \omega^2\left\|\boldsymbol{\Pi}^{\rm c}_0(\boldsymbol{\eta})\right\|^2_{\boldsymbol{\epsilon}} + 3\gamma_{\rm dv}|||\boldsymbol{e}|||^2,\end{aligned} \tag{36}$$

where the last bound follows from $\gamma_{\rm dv} \leq 1$. Since $\omega\left\|\boldsymbol{\Pi}^{\rm c}_0(\boldsymbol{\eta})\right\|_{\boldsymbol{\epsilon}} \leq |||\boldsymbol{e}|||$ and $\gamma_{\rm dv} \leq 1$, (36) implies that

$$(1-\gamma_{\rm dv})^2\omega^2\|\boldsymbol{\theta}_\Pi\|^2_{\boldsymbol{\epsilon}} \leq 4|||\boldsymbol{e}|||^2. \tag{37}$$

(2) We are now ready to prove (34). Multiplying the estimate (33) from Lemma 8 by $(1-\gamma_{\rm dv})^2$ (which is ≤ 1) and using (37) gives

$$\begin{aligned}(1-\gamma_{\rm dv})^2\|\nabla_0\times\boldsymbol{\theta}_0\|^2_{\boldsymbol{\nu}} &\leq \left|\left|\left|(I-\boldsymbol{\Pi}^{\rm c}_0)(\boldsymbol{\eta})\right|\right|\right|^2 + (2\gamma_{\rm dv}+3\gamma^2_{\rm ap})|||\boldsymbol{e}|||^2 + 2\gamma_{\rm dv}(1-\gamma_{\rm dv})^2\omega^2\|\boldsymbol{\theta}_\Pi\|^2_{\boldsymbol{\epsilon}} \\ &\leq \left|\left|\left|(I-\boldsymbol{\Pi}^{\rm c}_0)(\boldsymbol{\eta})\right|\right|\right|^2 + (10\gamma_{\rm dv}+3\gamma^2_{\rm ap})|||\boldsymbol{e}|||^2.\end{aligned} \tag{38}$$

Since $|||\boldsymbol{e}|||^2 = \omega^2\|\boldsymbol{\theta}_0\|^2_{\boldsymbol{\epsilon}} + \omega^2\|\boldsymbol{\theta}_\Pi\|^2_{\boldsymbol{\epsilon}} + \|\nabla_0\times\boldsymbol{\theta}_0\|^2_{\boldsymbol{\nu}}$, we infer that

$$\begin{aligned}(1-\gamma_{\rm dv})^2|||\boldsymbol{e}|||^2 &\leq \omega^2\|\boldsymbol{\theta}_0\|^2_{\boldsymbol{\epsilon}} + (1-\gamma_{\rm dv})^2\omega^2\|\boldsymbol{\theta}_\Pi\|^2_{\boldsymbol{\epsilon}} + (1-\gamma_{\rm dv})^2\|\nabla_0\times\boldsymbol{\theta}_0\|^2_{\boldsymbol{\nu}} \\ &\leq \gamma^2_{\rm ap}|||\boldsymbol{e}|||^2 + \omega^2\left\|\boldsymbol{\Pi}^{\rm c}_0(\boldsymbol{\eta})\right\|^2_{\boldsymbol{\epsilon}} + 3\gamma_{\rm dv}|||\boldsymbol{e}|||^2 + \left|\left|\left|(I-\boldsymbol{\Pi}^{\rm c}_0)(\boldsymbol{\eta})\right|\right|\right|^2 + (10\gamma_{\rm dv}+3\gamma^2_{\rm ap})|||\boldsymbol{e}|||^2 \\ &= |||\boldsymbol{\eta}|||^2 + (13\gamma_{\rm dv}+4\gamma^2_{\rm ap})|||\boldsymbol{e}|||^2,\end{aligned} \tag{39}$$

where the second bound follows from Lemma 5, (36), and (38), and the last equality follows from $|||\boldsymbol{\eta}|||^2 = \left|\left|\left|(I-\boldsymbol{\Pi}^{\rm c}_0)(\boldsymbol{\eta})\right|\right|\right|^2 + \omega^2\left\|\boldsymbol{\Pi}^{\rm c}_0(\boldsymbol{\eta})\right\|^2_{\boldsymbol{\epsilon}}$. The error estimate (34) follows by observing that $(1-\gamma_{\rm dv})^2 \geq 1-2\gamma_{\rm dv}$.

(3) If the mesh size is small enough so that $15\gamma_{\rm dv}+4\gamma^2_{\rm ap} < 1$, the error estimate (34) implies the uniqueness of the discrete solution. Existence then follows from the fact that (20) amounts to a square linear system. □

Remark 10 (Asymptotic optimality). Notice that in (34), we have $\gamma_{\rm dv} \to 0$ and $\gamma_{\rm ap} \to 0$ as $h/k \to 0$. Hence, we have

$$|||\boldsymbol{E}-\boldsymbol{E}_h||| \leq \big(1+\theta(h)\big)\min_{\boldsymbol{v}_h\in\boldsymbol{V}^{\rm c}_{h0}} |||\boldsymbol{E}-\boldsymbol{v}_h|||,$$

with $\lim_{h/k\to 0}\theta(h) = 0$.

4.3. *Inf-sup stability*

We are now ready to establish our main stability result.

Theorem 11 (Inf-sup stablity). *We have*

$$\min_{\substack{\boldsymbol{v}_h\in\boldsymbol{V}^{\rm c}_{h0}\\ |||\boldsymbol{v}_h|||=1}} \max_{\substack{\boldsymbol{w}_h\in\boldsymbol{V}^{\rm c}_{h0}\\ |||\boldsymbol{w}_h|||=1}} \left|b(\boldsymbol{v}_h,\boldsymbol{w}_h)\right| \geq \frac{1-2(\gamma^2_{\rm dv}+\gamma_{\rm ap})}{1+2\beta_{\rm st}}. \tag{40}$$

Proof. We adapt to the discrete setting the arguments of the proof of Lemma 2. Let $\boldsymbol{v}_h \in \boldsymbol{V}^{\rm c}_{h0}$ and set $\boldsymbol{v}_h = \boldsymbol{v}_{h0} + \boldsymbol{v}_{h\Pi}$ with $\boldsymbol{v}_{h0} := (I-\boldsymbol{\Pi}^{\rm c}_{h0})(\boldsymbol{v}_h) \in \boldsymbol{X}^{\rm c}_{h0}$ and $\boldsymbol{v}_{h\Pi} := \boldsymbol{\Pi}^{\rm c}_{h0}(\boldsymbol{v}_h) \in \boldsymbol{V}^{\rm c}_{h0}$.

(1) In this first step, we gain control on $\omega\|\boldsymbol{v}_{h0}\|_{\boldsymbol{\epsilon}}$. Since $\boldsymbol{v}_{h0}$ is (loosely speaking) discretely divergence-free, but not pointwise divergence-free, we need to consider a further decomposition of $\boldsymbol{v}_{h0}$. Let us set $\boldsymbol{v}_{h0} = \boldsymbol{\phi}_0 + \boldsymbol{\phi}_\Pi$ with $\boldsymbol{\phi}_0 := (I-\boldsymbol{\Pi}^{\rm c}_0)(\boldsymbol{v}_{h0})$ and $\boldsymbol{\phi}_\Pi := \boldsymbol{\Pi}^{\rm c}_0(\boldsymbol{v}_{h0})$. Notice that

$$\omega\|\boldsymbol{\phi}_0\|_{\boldsymbol{\epsilon}} \leq \omega\|\boldsymbol{v}_{h0}\|_{\boldsymbol{\epsilon}} \leq \omega\|\boldsymbol{v}_h\|_{\boldsymbol{\epsilon}} \leq |||\boldsymbol{v}_h|||.$$

Let $\boldsymbol{\xi}_0 \in \boldsymbol{X}^{\rm c}_0$ be the unique adjoint solution such that $b(\boldsymbol{w},\boldsymbol{\xi}_0) = \omega^2(\boldsymbol{w},\boldsymbol{\phi}_0)_{\boldsymbol{\epsilon}}$ for all $\boldsymbol{w} \in \boldsymbol{V}^{\rm c}_0$ (notice that $\boldsymbol{\phi}_0 \in \boldsymbol{H}_0(\mathbf{curl}=\mathbf{0};D)^\perp$). We have

$$|||\boldsymbol{\xi}_0||| \leq \beta_{\rm st}\omega\|\boldsymbol{\phi}_0\|_{\boldsymbol{\epsilon}} \leq \beta_{\rm st}\omega\|\boldsymbol{v}_{h0}\|_{\boldsymbol{\epsilon}}.$$

Let us set $\boldsymbol{\xi}_{h0} := \mathscr{B}^{\mathrm{c}}_{h0}(\boldsymbol{\xi}_0)$. Then $\boldsymbol{\xi}_{h0} \in \boldsymbol{V}^{\mathrm{c}}_{h0}$ by definition, and $\boldsymbol{\xi}_{h0} \in \boldsymbol{V}^{\mathrm{c}}_{h0}(\mathbf{curl} = \mathbf{0})^{\perp}$ by (27b) since $\boldsymbol{\xi}_0 \in \boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}; D)^{\perp}$. Moreover, owing to (27a), we have $|||\boldsymbol{\xi}_{h0}||| \le |||\boldsymbol{\xi}_0|||$. Using these properties, we infer that

$$\begin{aligned} b(\boldsymbol{v}_h, \boldsymbol{\xi}_{h0}) &= b(\boldsymbol{v}_{h0}, \boldsymbol{\xi}_{h0}) + b(\boldsymbol{v}_{h\Pi}, \boldsymbol{\xi}_{h0}) \\ &= b(\boldsymbol{v}_{h0}, \boldsymbol{\xi}_{h0}) \\ &= b(\boldsymbol{v}_{h0}, \boldsymbol{\xi}_0) + b(\boldsymbol{v}_{h0}, \boldsymbol{\xi}_{h0} - \boldsymbol{\xi}_0) \\ &\ge \omega^2 \|\boldsymbol{\phi}_0\|^2_{\boldsymbol{\epsilon}} - \gamma_{\mathrm{ap}} |||\boldsymbol{v}_h|||^2, \end{aligned}$$

since $b(\boldsymbol{v}_{h0}, \boldsymbol{\xi}_0) = \omega^2 (\boldsymbol{v}_{h0}, \boldsymbol{\phi}_0)_{\boldsymbol{\epsilon}} = \omega^2 \|\boldsymbol{\phi}_0\|^2_{\boldsymbol{\epsilon}}$ and

$$\left| b(\boldsymbol{v}_{h0}, \boldsymbol{\xi}_{h0} - \boldsymbol{\xi}_0) \right| \le |||\boldsymbol{v}_{h0}|||\, |||\boldsymbol{\xi}_{h0} - \boldsymbol{\xi}_0||| \le |||\boldsymbol{v}_{h0}|||\, \gamma_{\mathrm{ap}} \omega \|\boldsymbol{\phi}_0\|_{\boldsymbol{\epsilon}} \le \gamma_{\mathrm{ap}} |||\boldsymbol{v}_h|||^2,$$

owing to the boundedness of the bilinear form b, the above bound on $\omega\|\boldsymbol{\phi}_0\|_{\boldsymbol{\epsilon}}$, and since $|||\boldsymbol{v}_{h0}||| \le |||\boldsymbol{v}_h|||$. Moreover, using the divergence conformity factor, we infer that

$$\omega^2 \|\boldsymbol{\phi}_{\Pi}\|^2_{\boldsymbol{\epsilon}} = \omega^2 \left\| \boldsymbol{\Pi}^{\mathrm{c}}_0(\boldsymbol{v}_{h0}) \right\|^2_{\boldsymbol{\epsilon}} \le \gamma^2_{\mathrm{dv}} \|\nabla_0 \times \boldsymbol{v}_{h0}\|^2_{\boldsymbol{\nu}} \le \gamma^2_{\mathrm{dv}} |||\boldsymbol{v}_h|||^2.$$

Since $\|\boldsymbol{v}_{h0}\|^2_{\boldsymbol{\epsilon}} = \|\boldsymbol{\phi}_0\|^2_{\boldsymbol{\epsilon}} + \|\boldsymbol{\phi}_{\Pi}\|^2_{\boldsymbol{\epsilon}}$, putting everything together gives

$$b(\boldsymbol{v}_h, \boldsymbol{\xi}_{h0}) \ge \omega^2 \|\boldsymbol{v}_{h0}\|^2_{\boldsymbol{\epsilon}} - (\gamma^2_{\mathrm{dv}} + \gamma_{\mathrm{ap}}) |||\boldsymbol{v}_h|||^2. \tag{41}$$

(2) Since $b(\boldsymbol{v}_h, \boldsymbol{v}_{h0} - \boldsymbol{v}_{h\Pi}) = |||\boldsymbol{v}_h|||^2 - 2\omega^2 \|\boldsymbol{v}_{h0}\|^2_{\boldsymbol{\epsilon}}$, using (41) yields

$$b(\boldsymbol{v}_h, \boldsymbol{v}_{h0} + 2\boldsymbol{\xi}_{h0} - \boldsymbol{v}_{h\Pi}) \ge |||\boldsymbol{v}_h|||^2 - 2(\gamma^2_{\mathrm{dv}} + \gamma_{\mathrm{ap}}) |||\boldsymbol{v}_h|||^2.$$

Moreover, using the same arguments as in the proof of Lemma 2 and recalling that $|||\boldsymbol{\xi}_{h0}||| \le |||\boldsymbol{\xi}|||$, we obtain

$$|||\boldsymbol{v}_{h0} + 2\boldsymbol{\xi}_{h0} - \boldsymbol{v}_{h\Pi}|||^2 = |||\boldsymbol{v}_{h0} + 2\boldsymbol{\xi}_{h0}|||^2 + |||\boldsymbol{v}_{h\Pi}|||^2 \le (1 + 2\beta_{\mathrm{st}})^2 |||\boldsymbol{v}_h|||^2.$$

Since $\boldsymbol{v}_{h0} + 2\boldsymbol{\xi}_{h0} - \boldsymbol{v}_{h\Pi} \in \boldsymbol{V}^{\mathrm{c}}_{h0}$, this concludes the proof. □

Remark 12 (Discrete inf-sup constant). Since γ_{ap} and γ_{dv} tend to zero as the mesh size is small enough and/or the polynomial degree is large enough, the discrete inf-sup constant appearing on the left-hand side of (40) tends to $(1 + 2\beta_{\mathrm{st}})^{-1}$. Recall from Remark 3 that this quantity corresponds, up to a factor of two at most, to the inf-sup constant of the bilinear form b in the continuous setting.

5. Bound on approximation and divergence conformity factors

In this section, we bound the two (nondimensional) quantities introduced in Section 3.3 and used in Section 4: the approximation factor γ_{ap} and the divergence conformity factor γ_{dv}. To this purpose, we consider the commuting quasi-interpolation operator $\mathscr{J}^{\mathrm{c}}_{h0} \colon \boldsymbol{L}^2(D) \to \boldsymbol{V}^{\mathrm{c}}_{h0}$ and $\mathscr{J}^{\mathrm{d}}_{h0} \colon \boldsymbol{L}^2(D) \to \boldsymbol{V}^{\mathrm{d}}_{h0}$ (the Raviart–Thomas finite element space of order $k \ge 0$ satisfying zero normal boundary conditions); see [2,14,15,40] and also [22, Chapters 22-23]. Both operators are bounded in $\boldsymbol{L}^2(D)$, they are projections, and they satisfy the commuting property $\nabla \times \left(\mathscr{J}^{\mathrm{c}}_{h0}(\boldsymbol{v})\right) = \mathscr{J}^{\mathrm{d}}_{h0}(\nabla \times \boldsymbol{v})$ for all $\boldsymbol{v} \in \boldsymbol{L}^2(D)$.

For positive real numbers A and B, we abbreviate as $A \lesssim B$ the inequality $A \le CB$ with a generic constant C whose value can change at each occurrence as long as it is independent of the mesh size, the frequency parameter ω, and, whenever relevant, any function involved in the bound. The constant C can depend on the shape-regularity of the mesh and the polynomial degree k as well as on the domain Ω and on the contrast for the coefficients $\boldsymbol{\epsilon}$ and $\boldsymbol{\nu}$. Our main focus here is on the limit as $h \to 0$ with k fixed (see Remark 17 for further insight).

We introduce the notation

$$\varepsilon_{\max} := \max_{\boldsymbol{x} \in D} \max_{\substack{\boldsymbol{u} \in \mathbb{R}^d \\ |\boldsymbol{u}|=1}} \max_{\substack{\boldsymbol{v} \in \mathbb{R}^d \\ |\boldsymbol{v}|=1}} \boldsymbol{\epsilon}(\boldsymbol{x})\boldsymbol{u} \cdot \boldsymbol{v}, \qquad \varepsilon_{\min} := \min_{\boldsymbol{x} \in D} \min_{\substack{\boldsymbol{u} \in \mathbb{R}^d \\ |\boldsymbol{u}|=1}} \boldsymbol{\epsilon}(\boldsymbol{x})\boldsymbol{u} \cdot \boldsymbol{u}, \tag{42}$$

and define $\nu_{\max}$ and $\nu_{\min}$ similarly. Then, $\vartheta_{\min} = \sqrt{\nu_{\min}/\varepsilon_{\max}}$ stands for the minimum wavespeed in the domain.

5.1. *Piecewise smooth coefficients*

For the sake of simplicity, we start by assuming that the coefficients are piecewise smooth in D. Then, the following regularity results from [5,19,26] will be useful: there exists $s \in (0,1]$ such that, for all $\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}; D)$ with $\nabla\cdot(\boldsymbol{\epsilon}\boldsymbol{v}) = 0$ and all $\boldsymbol{w} \in \boldsymbol{H}_0(\mathrm{div} = 0; D)$ with $\boldsymbol{\nu}\boldsymbol{w} \in \boldsymbol{H}(\mathbf{curl}; D)$, we have $\boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{H}^s(D)$ with the estimates

$$|\boldsymbol{v}|_{\boldsymbol{H}^s(D)} \lesssim \ell_D^{1-s} \nu_{\min}^{-\frac{1}{2}} \|\nabla_0 \times \boldsymbol{v}\|_{\boldsymbol{\nu}}, \qquad |\boldsymbol{w}|_{\boldsymbol{H}^s(D)} \lesssim \ell_D^{1-s} \frac{1}{\vartheta_{\min}} \nu_{\min}^{-\frac{1}{2}} \left\|\nabla\times(\boldsymbol{\nu}\boldsymbol{w})\right\|_{\boldsymbol{\epsilon}^{-1}}, \tag{43}$$

where ℓ_D is a characteristic length of D introduced for dimensional consistency, e.g., the diameter of D. If D is convex and $\boldsymbol{\epsilon}$ and $\boldsymbol{\nu}$ are (globally) Lipschitz continuous, we can take $s = 1$.

Lemma 13 (Bound on approximation factor). *Let γ_{ap} be defined in* (23). *The following holds:*

$$\gamma_{\mathrm{ap}} \lesssim (1+\beta_{\mathrm{st}}) \left(\frac{\omega \ell_D}{\vartheta_{\min}}\right)^{1-s} \left(\frac{\omega h}{\vartheta_{\min}}\right)^{s}, \tag{44}$$

with the stability constant β_{st} defined in (13).

Proof. Let $\boldsymbol{\theta} \in \boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}; D)^{\perp}$ and let $\boldsymbol{\zeta}_{\boldsymbol{\theta}} \in \boldsymbol{X}_0^{\mathrm{c}}$ solve the adjoint problem (21). On the one hand, invoking (43) and using the stability constant β_{st}, we infer that

$$|\boldsymbol{\zeta}_{\boldsymbol{\theta}}|_{\boldsymbol{H}^s(D)} \lesssim \ell_D^{1-s} \nu_{\min}^{-\frac{1}{2}} \|\nabla_0 \times \boldsymbol{\zeta}_{\boldsymbol{\theta}}\|_{\boldsymbol{\nu}} \le \ell_D^{1-s} \nu_{\min}^{-\frac{1}{2}} |||\boldsymbol{\zeta}_{\boldsymbol{\theta}}||| \lesssim \beta_{\mathrm{st}} \ell_D^{1-s} \omega^{-1} \nu_{\min}^{-\frac{1}{2}} \|\boldsymbol{\theta}\|_{\boldsymbol{\epsilon}}.$$

Invoking the approximation properties of $\mathscr{J}_{h0}^{\mathrm{c}}$ leads to

$$\begin{aligned}
\omega^2 \left\|\boldsymbol{\zeta}_{\boldsymbol{\theta}} - \mathscr{J}_{h0}^{\mathrm{c}}(\boldsymbol{\zeta}_{\boldsymbol{\theta}})\right\|_{\boldsymbol{\epsilon}} &\le \omega^2 \varepsilon_{\max}^{\frac{1}{2}} \left\|\boldsymbol{\zeta}_{\boldsymbol{\theta}} - \mathscr{J}_{h0}^{\mathrm{c}}(\boldsymbol{\zeta}_{\boldsymbol{\theta}})\right\| \\
&\lesssim \omega^2 h^s \varepsilon_{\max}^{\frac{1}{2}} |\boldsymbol{\zeta}_{\boldsymbol{\theta}}|_{\boldsymbol{H}^s(D)} \\
&\lesssim \beta_{\mathrm{st}} \left(\frac{\omega \ell_D}{\vartheta_{\min}}\right)^{1-s} \left(\frac{\omega h}{\vartheta_{\min}}\right)^{s} \|\boldsymbol{\theta}\|_{\boldsymbol{\epsilon}}.
\end{aligned} \tag{45}$$

On the other hand, we have $\boldsymbol{\epsilon}^{-1}\nabla\times(\boldsymbol{\nu}\nabla_0\times\boldsymbol{\zeta}_{\boldsymbol{\theta}}) = \boldsymbol{\theta} + \omega^2 \boldsymbol{\zeta}_{\boldsymbol{\theta}}$, so that

$$\left\|\nabla\times(\boldsymbol{\nu}\nabla_0\times\boldsymbol{\zeta}_{\boldsymbol{\theta}})\right\|_{\boldsymbol{\epsilon}^{-1}} = \left\|\boldsymbol{\epsilon}^{-1}\nabla\times(\boldsymbol{\nu}\nabla_0\times\boldsymbol{\zeta}_{\boldsymbol{\theta}})\right\|_{\boldsymbol{\epsilon}} \le \|\boldsymbol{\theta}\|_{\boldsymbol{\epsilon}} + \omega^2 \|\boldsymbol{\zeta}_{\boldsymbol{\theta}}\|_{\boldsymbol{\epsilon}} \le (1+\beta_{\mathrm{st}}) \|\boldsymbol{\theta}\|_{\boldsymbol{\epsilon}}.$$

Since $\boldsymbol{w} := \nabla_0\times\boldsymbol{\zeta}_{\boldsymbol{\theta}} \in \boldsymbol{H}_0(\mathrm{div} = 0; D)$ with $\boldsymbol{\nu}\boldsymbol{w} \in \boldsymbol{H}(\mathbf{curl}; D)$, we can again invoke (43), giving

$$|\nabla_0\times\boldsymbol{\zeta}_{\boldsymbol{\theta}}|_{\boldsymbol{H}^s(D)} \lesssim \ell_D^{1-s} \frac{\nu_{\min}^{-\frac{1}{2}}}{\vartheta_{\min}} \left\|\nabla\times(\boldsymbol{\nu}\nabla_0\times\boldsymbol{\zeta}_{\boldsymbol{\theta}})\right\|_{\boldsymbol{\epsilon}^{-1}} \le (1+\beta_{\mathrm{st}}) \ell_D^{1-s} \frac{\nu_{\min}^{-\frac{1}{2}}}{\vartheta_{\min}} \|\boldsymbol{\theta}\|_{\boldsymbol{\epsilon}}.$$

Owing to the commuting property $\nabla_0\times\mathscr{J}_{h0}^{\mathrm{c}}(\cdot) = \mathscr{J}_{h0}^{\mathrm{d}}(\nabla_0\times\cdot)$, we infer that

$$\begin{aligned}
\omega \left\|\nabla_0\times\left(\boldsymbol{\zeta}_{\boldsymbol{\theta}} - \mathscr{J}_{h0}^{\mathrm{c}}(\boldsymbol{\zeta}_{\boldsymbol{\theta}})\right)\right\|_{\boldsymbol{\nu}} &= \omega \left\|\nabla_0\times\boldsymbol{\zeta}_{\boldsymbol{\theta}} - \mathscr{J}_{h0}^{\mathrm{d}}(\nabla_0\times\boldsymbol{\zeta}_{\boldsymbol{\theta}})\right\|_{\boldsymbol{\nu}} \\
&\lesssim \omega \nu_{\max}^{\frac{1}{2}} h^s |\nabla_0\times\boldsymbol{\zeta}_{\boldsymbol{\theta}}|_{\boldsymbol{H}^s(D)} \\
&\lesssim (1+\beta_{\mathrm{st}}) \left(\frac{\omega \ell_D}{\vartheta_{\min}}\right)^{1-s} \left(\frac{\omega h}{\vartheta_{\min}}\right)^{s} \|\boldsymbol{\theta}\|_{\boldsymbol{\epsilon}}.
\end{aligned} \tag{46}$$

(Recall that the ratio $\nu_{\max}/\nu_{\min}$ can be hidden in the generic constant C.) The conclusion follows from (45) and (46). □

Lemma 14 (Bound on divergence conformity factor). *Let γ_{dv} be defined in* (24). *The following holds:*

$$\gamma_{\mathrm{dv}} \lesssim \left(\frac{\omega \ell_D}{\vartheta_{\min}}\right)^{1-s} \left(\frac{\omega h}{\vartheta_{\min}}\right)^{s}. \tag{47}$$

Proof.

(1) Let $\boldsymbol{v}_h \in \boldsymbol{X}^{\mathrm{c}}_{h0} = \boldsymbol{V}^{\mathrm{c}}_{h0} \cap \boldsymbol{V}^{\mathrm{c}}_{h0}(\mathbf{curl} = \mathbf{0})^{\perp}$. Let us write

$$\boldsymbol{v}_h = \boldsymbol{w} + \boldsymbol{\Pi}^{\mathrm{c}}_0(\boldsymbol{v}_h),$$

with $\boldsymbol{w} := (I - \boldsymbol{\Pi}^{\mathrm{c}}_0)(\boldsymbol{v}_h)$. By construction, $\boldsymbol{w} \in \boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}; D)^{\perp}$, and we have $\boldsymbol{w} \in \boldsymbol{H}_0(\mathbf{curl}; D)$ since $\boldsymbol{v}_h \in \boldsymbol{V}^{\mathrm{c}}_{h0} \subset \boldsymbol{H}_0(\mathbf{curl}; D)$; hence, $\boldsymbol{w} \in \boldsymbol{X}^{\mathrm{c}}_0$. Invoking (43) and observing that $\nabla_0 \times \boldsymbol{w} = \nabla_0 \times \boldsymbol{v}_h$, we infer that

$$|\boldsymbol{w}|_{\boldsymbol{H}^s(D)} \lesssim \ell_D^{1-s} \nu_{\min}^{-\frac{1}{2}} \|\nabla_0 \times \boldsymbol{v}_h\|_{\boldsymbol{\nu}}.$$

Moreover, we have $\boldsymbol{\Pi}^{\mathrm{c}}_{h0}\big(\boldsymbol{\Pi}^{\mathrm{c}}_0(\boldsymbol{v}_h)\big) = \boldsymbol{\Pi}^{\mathrm{c}}_{h0}(\boldsymbol{v}_h) = \mathbf{0}$ since $\boldsymbol{\Pi}^{\mathrm{c}}_{h0} \circ \boldsymbol{\Pi}^{\mathrm{c}}_0 = \boldsymbol{\Pi}^{\mathrm{c}}_{h0}$ and $\boldsymbol{v}_h \in \boldsymbol{V}^{\mathrm{c}}_{h0}(\mathbf{curl} = \mathbf{0})^{\perp}$. Hence, $\boldsymbol{\Pi}^{\mathrm{c}}_0(\boldsymbol{v}_h) \in \boldsymbol{V}^{\mathrm{c}}_{h0}(\mathbf{curl} = \mathbf{0})^{\perp}$ as well.

(2) Recall the commuting quasi-interpolation operator $\mathscr{J}^{\mathrm{c}}_{h0} : \boldsymbol{L}^2(D) \to \boldsymbol{V}^{\mathrm{c}}_{h0}$. Since $\mathscr{J}^{\mathrm{c}}_{h0}$ leaves $\boldsymbol{V}^{\mathrm{c}}_{h0}$ pointwise invariant, we have $(I - \mathscr{J}^{\mathrm{c}}_{h0})(\boldsymbol{v}_h) = \mathbf{0}$, so that

$$(I - \mathscr{J}^{\mathrm{c}}_{h0})\big(\boldsymbol{\Pi}^{\mathrm{c}}_0(\boldsymbol{v}_h)\big) = -(I - \mathscr{J}^{\mathrm{c}}_{h0})(\boldsymbol{w}).$$

Moreover, since $\boldsymbol{\Pi}^{\mathrm{c}}_0(\boldsymbol{v}_h)$ is curl-free by construction, the commuting property of $\mathscr{J}^{\mathrm{c}}_{h0}$ implies that

$$\mathscr{J}^{\mathrm{c}}_{h0}\big(\boldsymbol{\Pi}^{\mathrm{c}}_0(\boldsymbol{v}_h)\big) \in \boldsymbol{V}^{\mathrm{c}}_{h0}(\mathbf{curl} = \mathbf{0}).$$

Since $\boldsymbol{\Pi}^{\mathrm{c}}_0(\boldsymbol{v}_h) \in \boldsymbol{V}^{\mathrm{c}}_{h0}(\mathbf{curl} = \mathbf{0})^{\perp}$, we infer that

$$\begin{aligned}
\big\|\boldsymbol{\Pi}^{\mathrm{c}}_0(\boldsymbol{v}_h)\big\|^2_{\boldsymbol{\epsilon}} &= \big(\boldsymbol{\Pi}^{\mathrm{c}}_0(\boldsymbol{v}_h), \boldsymbol{\Pi}^{\mathrm{c}}_0(\boldsymbol{v}_h)\big)_{\epsilon} \\
&= \big(\boldsymbol{\Pi}^{\mathrm{c}}_0(\boldsymbol{v}_h), \boldsymbol{\Pi}^{\mathrm{c}}_0(\boldsymbol{v}_h) - \mathscr{J}^{\mathrm{c}}_{h0}\big(\boldsymbol{\Pi}^{\mathrm{c}}_0(\boldsymbol{v}_h)\big)\big)_{\boldsymbol{\epsilon}} \\
&= -\big(\boldsymbol{\Pi}^{\mathrm{c}}_0(\boldsymbol{v}_h), \boldsymbol{w} - \mathscr{J}^{\mathrm{c}}_{h0}(\boldsymbol{w})\big)_{\boldsymbol{\epsilon}}.
\end{aligned}$$

The Cauchy–Schwarz inequality together with the approximation properties of $\mathscr{J}^{\mathrm{c}}_{h0}$ gives

$$\big\|\boldsymbol{\Pi}^{\mathrm{c}}_0(\boldsymbol{v}_h)\big\|_{\boldsymbol{\epsilon}} \lesssim h^s \varepsilon_{\max}^{\frac{1}{2}} |\boldsymbol{w}|_{\boldsymbol{H}^s(D)},$$

and we conclude using the above bound on $|\boldsymbol{w}|_{\boldsymbol{H}^s(D)}$. □

Remark 15 (Bound on γ_{dv}). The above proof can be rewritten as the following statement:

$$\gamma_{\mathrm{dv}} \leq \gamma_{\mathscr{J}} := \sup_{\substack{\boldsymbol{w} \in \boldsymbol{X}^{\mathrm{c}}_0 \\ \|\nabla_0 \times \boldsymbol{w}\|_{\boldsymbol{\nu}} = 1}} \omega \big\|\boldsymbol{w} - \mathscr{J}^{\mathrm{c}}_{h0}(\boldsymbol{w})\big\|_{\boldsymbol{\epsilon}} \lesssim \left(\frac{\omega \ell_D}{\vartheta_{\min}}\right)^{1-s} \left(\frac{\omega h}{\vartheta_{\min}}\right)^{s}. \tag{48}$$

This shows that γ_{dv} is bounded by an approximation factor on $\boldsymbol{X}^{\mathrm{c}}_0$ using the commuting quasi-interpolation operator $\mathscr{J}^{\mathrm{c}}_{h0}$. Notice that only the rightmost bound uses (43).

Remark 16 (Convex domain). For a convex domain D, the factors are bounded as

$$\gamma_{\mathrm{ap}} \lesssim (1 + \beta_{\mathrm{st}}) \frac{\omega h}{\vartheta_{\min}}, \qquad \gamma_{\mathrm{dv}} \lesssim \frac{\omega h}{\vartheta_{\min}}.$$

The quantity $(\omega h)/\vartheta_{\min}$ is inversly proportional to the (minimal) number of mesh elements per wavelength. It is therefore reasonable to assume that $\gamma_{\mathrm{dv}} \lesssim 1$. We also see that γ_{ap} is typically not bounded for all frequencies assuming a constant number of elements per wavelength, since β_{st} can be large. This is the standard manifestation of dispersion errors, also known in this context as pollution effect. This is completely standard, and also happens in the (simpler) case of Helmholtz problems. It is interesting to notice that the constraint that γ_{dv} is small, which is specific to Maxwell's equations, is less restrictive than the constraint that γ_{ap} is small, which is common to Maxwell and Helmholtz equations.

Remark 17 (Reduced dispersion for high-order elements). When the domain D and the coefficients are smooth, it is shown in [12] that

$$\gamma_{\mathrm{ap}} \lesssim \frac{\omega h}{\vartheta_{\min}} + (1+\beta_{\mathrm{st}})\left(\frac{\omega h}{\vartheta_{\min}}\right)^{k+1},$$

so that γ_{ap} is small if

$$\frac{\omega h}{\vartheta_{\min}} \lesssim \beta_{\mathrm{st}}^{-\frac{1}{k+1}}. \tag{49}$$

For large frequencies (or frequencies close to resonant frequencies), β_{st} becomes large, so that the number of elements per wavelength needs to be increased. Nevertheless, (49) expresses that the required increase is less important for higher order elements, which corresponds to numerical observations. It is also expected that such a result remains true for general domains and piecewise smooth coefficients if the mesh is suitably refined locally, but this claim has only been established for two-dimensional problems in [10]. We finally refer the reader to [32] where stronger results explicit in the polynomial degree k are established, but under stronger assumptions on the domain.

Remark 18 (k convergence). When $s=1$, we can consider the interpolation operators from [29], instead of the quasi-interpolation operators $\mathscr{J}_{h0}^{\mathrm{c}}$ and $\mathscr{J}_{h0}^{\mathrm{d}}$ considered above, thereby showing that γ_{ap} and γ_{dv} tend to zero (and optimally so) as k is increased (and h is fixed). More generally, a similar strategy can be employed as long as a piecewise version of the regularity shift in (43) holds true with $s > \frac{1}{2}$. In such a case, we can use, e.g., the interpolation operators from [20].

5.2. *General coefficients*

Here, we consider general coefficients $\boldsymbol{\epsilon}$ and $\boldsymbol{\nu}$ for which (43) may not hold for any $s>0$.

Lemma 19 (Convergence of approximation factor). *We have* $\gamma_{\mathrm{ap}} \to 0$ *as* $h \to 0$.

Proof. Let us set

$$\mathscr{B}_{\boldsymbol{\epsilon}} := \left\{ \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}; D) \;\middle|\; \nabla\cdot(\boldsymbol{\epsilon}\boldsymbol{v}) = 0,\ \omega\|\nabla_0\times\boldsymbol{v}\|_{\boldsymbol{\nu}} \le \beta_{\mathrm{st}} \right\},$$
$$\mathscr{B}_{\boldsymbol{\nu}} := \left\{ \boldsymbol{w} \in \boldsymbol{H}_0(\mathrm{div}=0; D) \;\middle|\; \left\|\nabla\times(\boldsymbol{\nu}\boldsymbol{w})\right\|_{\boldsymbol{\epsilon}^{-1}} \le 1+\beta_{\mathrm{st}} \right\}.$$

Owing to the definition (13) of the stability constant β_{st}, if $\boldsymbol{\theta} \in \boldsymbol{H}_0(\mathbf{curl}=\mathbf{0}; D)^{\perp}$ with $\|\boldsymbol{\theta}\|_{\boldsymbol{\epsilon}} = 1$ and $\boldsymbol{\xi}_{\boldsymbol{\theta}} \in \boldsymbol{V}_0^{\mathrm{c}}$ satisfy $b(\boldsymbol{w},\boldsymbol{\xi}_{\boldsymbol{\theta}}) = (\boldsymbol{w},\boldsymbol{\theta})_{\boldsymbol{\epsilon}}$ for all $\boldsymbol{w} \in \boldsymbol{V}_0^{\mathrm{c}}$, we have $\boldsymbol{\xi}_{\boldsymbol{\theta}} \in \mathscr{B}_{\boldsymbol{\epsilon}}$ and $\boldsymbol{\Phi}_{\boldsymbol{\theta}} := \nabla_0\times\boldsymbol{\xi}_{\boldsymbol{\theta}} \in \mathscr{B}_{\boldsymbol{\nu}}$.

Owing to the compact injections established in [41, Theorem 2.2], given any $\delta>0$, there exists a finite number N_δ of functions $\boldsymbol{v}_j \in \mathscr{B}_{\boldsymbol{\epsilon}}$ and $\boldsymbol{w}_\ell \in \mathscr{B}_{\boldsymbol{\nu}}$ such that, for all $\boldsymbol{v} \in \mathscr{B}_{\boldsymbol{\epsilon}}$ and all $\boldsymbol{w} \in \mathscr{B}_{\boldsymbol{\nu}}$, there exist indices $j,\ell \in \{1{:}N_\delta\}$ such that

$$\|\boldsymbol{v}-\boldsymbol{v}_j\|_{\boldsymbol{\epsilon}} \le \delta, \qquad \|\boldsymbol{w}-\boldsymbol{w}_\ell\|_{\boldsymbol{\nu}} \le \delta.$$

Furthermore, the density of $\boldsymbol{C}_{\mathrm{c}}^{\infty}(D)$ in $\boldsymbol{L}^2(D)$ implies that we can find $\widetilde{\boldsymbol{v}}_j, \widetilde{\boldsymbol{w}}_\ell \in \boldsymbol{C}_{\mathrm{c}}^{\infty}(D)$ such that

$$\|\boldsymbol{v}-\widetilde{\boldsymbol{v}}_j\|_{\boldsymbol{\epsilon}} \le 2\delta, \qquad \|\boldsymbol{w}-\widetilde{\boldsymbol{w}}_\ell\|_{\boldsymbol{\nu}} \le 2\delta.$$

We then write that

$$\min_{\boldsymbol{v}_h^{\mathrm{c}} \in \boldsymbol{V}_{h0}^{\mathrm{c}}} |||\boldsymbol{\xi}_{\boldsymbol{\theta}} - \boldsymbol{v}_h^{\mathrm{c}}|||^2 \le \left|\left|\left|\boldsymbol{\xi}_{\boldsymbol{\theta}} - \mathscr{J}_{h0}^{\mathrm{c}}(\boldsymbol{\xi}_{\boldsymbol{\theta}})\right|\right|\right|^2 = \omega^2\left\|\boldsymbol{\xi}_{\boldsymbol{\theta}} - \mathscr{J}_{h0}^{\mathrm{c}}(\boldsymbol{\xi}_{\boldsymbol{\theta}})\right\|_{\boldsymbol{\epsilon}}^2 + \left\|\boldsymbol{\Phi}_{\boldsymbol{\theta}} - \mathscr{J}_{h0}^{\mathrm{d}}(\boldsymbol{\Phi}_{\boldsymbol{\theta}})\right\|_{\boldsymbol{\nu}}^2.$$

Invoking the triangle inequality and the above bounds (with $\boldsymbol{v} := \boldsymbol{\xi}_{\boldsymbol{\theta}}$) gives

$$\begin{aligned}
\left\|\boldsymbol{\xi}_{\boldsymbol{\theta}} - \mathscr{J}_{h0}^{\mathrm{c}}(\boldsymbol{\xi}_{\boldsymbol{\theta}})\right\|_{\boldsymbol{\epsilon}} &\le \|\boldsymbol{\xi}_{\boldsymbol{\theta}} - \widetilde{\boldsymbol{v}}_j\|_{\boldsymbol{\epsilon}} + \left\|\widetilde{\boldsymbol{v}}_j - \mathscr{J}_{h0}^{\mathrm{c}}(\widetilde{\boldsymbol{v}}_j)\right\|_{\boldsymbol{\epsilon}} + \left\|\mathscr{J}_{h0}^{\mathrm{c}}(\widetilde{\boldsymbol{v}}_j - \boldsymbol{\xi}_{\boldsymbol{\theta}})\right\|_{\boldsymbol{\epsilon}} \\
&\lesssim \|\boldsymbol{\xi}_{\boldsymbol{\theta}} - \widetilde{\boldsymbol{v}}_j\|_{\boldsymbol{\epsilon}} + \left\|\widetilde{\boldsymbol{v}}_j - \mathscr{J}_{h0}^{\mathrm{c}}(\widetilde{\boldsymbol{v}}_j)\right\|_{\boldsymbol{\epsilon}} \\
&\le 2\delta + \left\|\widetilde{\boldsymbol{v}}_j - \mathscr{J}_{h0}^{\mathrm{c}}(\widetilde{\boldsymbol{v}}_j)\right\|_{\boldsymbol{\epsilon}},
\end{aligned}$$

where we used the $\boldsymbol{L}^2$-stability of $\mathcal{J}^{\mathrm{c}}_{h0}$. Similarly, we obtain

$$\left\|\boldsymbol{\Phi_\theta} - \mathcal{J}^{\mathrm{d}}_{h0}(\boldsymbol{\xi_\theta})\right\|_{\boldsymbol{\nu}} \lesssim \|\boldsymbol{\Phi_\theta} - \widetilde{\boldsymbol{w}}_\ell\|_{\boldsymbol{\nu}} + \left\|\widetilde{\boldsymbol{w}}_\ell - \mathcal{J}^{\mathrm{d}}_{h0}(\widetilde{\boldsymbol{w}}_\ell)\right\|_{\boldsymbol{\nu}} \leq 2\delta + \left\|\widetilde{\boldsymbol{w}}_\ell - \mathcal{J}^{\mathrm{d}}_{h0}(\widetilde{\boldsymbol{w}}_\ell)\right\|_{\boldsymbol{\nu}}.$$

Since the functions $\widetilde{\boldsymbol{v}}_j$ and $\widetilde{\boldsymbol{w}}_\ell$ are in finite number and smooth, we can assume that, if the mesh is sufficiently refined,

$$\left\|\widetilde{\boldsymbol{v}}_j - \mathcal{J}^{\mathrm{c}}_{h0}(\widetilde{\boldsymbol{v}}_j)\right\|_{\boldsymbol{\epsilon}} \leq \delta, \qquad \left\|\widetilde{\boldsymbol{w}}_j - \mathcal{J}^{\mathrm{d}}_{h0}(\widetilde{\boldsymbol{w}}_j)\right\|_{\boldsymbol{\nu}} \leq \delta.$$

This completes the proof since $\delta > 0$ is arbitrary. □

Lemma 20 (Convergence of divergence conformity factor). *We have* $\gamma_{\mathrm{dv}} \to 0$ *as* $h \to 0$.

Proof. We use the bound $\gamma_{\mathrm{dv}} \leq \gamma_{\mathcal{J}}$ with $\gamma_{\mathcal{J}}$ defined in (48). Thus, it suffices to show that $\gamma_{\mathcal{J}} \to 0$ as $h \to 0$.

We consider the unit ball $\mathcal{B} := \left\{\boldsymbol{w} \in \boldsymbol{H}_0(\mathbf{curl}; D); \nabla\cdot(\boldsymbol{\epsilon w}) = 0,\ \|\nabla_0 \times \boldsymbol{w}\|_{\boldsymbol{\nu}} \leq 1\right\}$. It is established in [41, Theorem 2.2] that the embedding $\boldsymbol{H}_0(\mathbf{curl}; D) \cap \boldsymbol{H}(\mathrm{div}; \boldsymbol{\epsilon}; D) \hookrightarrow \boldsymbol{L}^2(D)$ is compact. As a result, given any $\delta > 0$, there exists a finite number N_δ of elements $\boldsymbol{v}_j \in \mathcal{B}$ such that, for all $\boldsymbol{w} \in \mathcal{B}$, $\|\boldsymbol{w} - \boldsymbol{v}_j\|_{\boldsymbol{\epsilon}} \leq \delta$ for some index $j \in \{1{:}N_\delta\}$. Moreover, since $\boldsymbol{C}^\infty_{\mathrm{c}}(D)$ is dense in $\boldsymbol{L}^2(D)$, for each $j \in \{1{:}N_\delta\}$, there exists $\widetilde{\boldsymbol{v}}_j \in \boldsymbol{C}^\infty_{\mathrm{c}}(D)$ such that $\|\boldsymbol{v}_j - \widetilde{\boldsymbol{v}}_j\|_{\boldsymbol{\epsilon}} \leq \delta$. We have therefore shown that for all $\boldsymbol{w} \in \mathcal{B}$, there exists an index $j \in \{1{:}N_\delta\}$ such that

$$\|\boldsymbol{w} - \widetilde{\boldsymbol{v}}_j\|_{\boldsymbol{\epsilon}} \leq 2\delta.$$

We can now conclude by using the same arguments as in the above proof (notice that $\left\{\boldsymbol{w} \in \boldsymbol{X}^{\mathrm{c}}_0,\ \|\nabla_0 \times \boldsymbol{w}\|_{\boldsymbol{\nu}} \leq 1\right\} \subset \mathcal{B}$). □

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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