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Atypical values at infinity of real polynomial maps with 2-dimensional fibers

Valeurs atypiques à l'infini des applications polynomiales réelles à fibres bidimensionnelles

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Dedicated to Professor Ha Huy Vui on the occasion of his 75th birthday

Abstract. We characterize atypical values at infinity of a real polynomial function of three variables by a certain sum of indices of the gradient vector field of the function restricted to a sphere with a sufficiently large radius. This is a three-variable analogue of a result of Coste and de la Puente for real polynomial functions with two variables. We also give a characterization of atypical values at infinity of a real polynomial map whose regular fibers are 2-dimensional surfaces.

Résumé. Nous caractérisons les valeurs atypiques à l'infini d'une fonction polynomiale réelle de trois variables par une certaine somme d'indices du champ de vecteurs gradient de la fonction restreint à une sphère de rayon suffisamment grand. Il s'agit d'un analogue à trois variables d'un résultat de Coste et de la Puente pour les fonctions polynomiales réelles à deux variables. Nous donnons également une caractérisation des valeurs atypiques à l'infini d'une application polynomiale réelle dont les fibres régulières sont des surfaces bidimensionnelles.

Keywords. Polynomial map, singularity at infinity, atypical fiber, bifurcation set.

Mots-clés. Application polynomiale, singularité à l'infini, fibre atypique, ensemble de bifurcation.

2020 Mathematics Subject Classification. 58K15, 58K05, 58K30, 58K45.

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1. Introduction

Let $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a real polynomial map, $\text{Sing}(f)$ be the set of singular points of f in \mathbb{R}^n , and $K_0(f) = f(\text{Sing}(f))$. The *bifurcation set* $B_\infty(f)$ of f is the smallest set of values in \mathbb{R}^m outside which f is a locally trivial fibration. This is a semialgebraic set of codimension at least one [13,16,18]. A regular value $\lambda \in f(\mathbb{R}^n) \setminus K_0(f)$ is called a *typical value at ∞* of f if there is an open neighborhood over which f is a trivial fibration. Otherwise, λ is called an *atypical value at ∞* of f . For example, the polynomial map $f(x, y) = x(xy + 1)$ has no critical value but its bifurcation set is $\{0\}$ (cf. [2]). There are several studies about the bifurcation sets of real polynomial maps, see for instance [3,4,9,11,17].

Suppose $m = 1$, that is, $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is a real polynomial function. Let $B_{a,R}$ be the closed ball in \mathbb{R}^n centered at a point $a \in \mathbb{R}^n$ and of radius $R > 0$. Set

$$\Gamma = \left\{ x \in \mathbb{R}^n \mid \text{rank} \left(\frac{x-a}{\text{grad} f} \right) \leq 1 \right\}.$$

Note that $\text{Sing}(f) \subset \Gamma$. We choose a center $a \in \mathbb{R}^n$ and a sufficiently large $R > 0$ so that Γ is transverse to $\partial B_{a,r}$ for any $r > R$ and $\Gamma \setminus \text{Int} B_{a,R}$ is homeomorphic to $\Gamma \cap \partial B_{a,R} \times [0, 1)$. Each connected component of $\Gamma \setminus \text{Int} B_{a,R}$ is contained in either $\text{Sing}(f)$ or $\Gamma \setminus \text{Sing}(f)$. We can choose the center a generic so that each connected component of $\Gamma \setminus (\text{Sing}(f) \cup \text{Int} B_{a,R})$ is a curve, see [6]. We assume throughout the paper that the center satisfies this condition. These curves are called *tangency branches at ∞ of f* .

For each point $p \in (\Gamma \setminus \text{Sing}(f)) \cap \partial B_{a,R}$, let Γ_p denote the tangency branch at ∞ of f passing through p . Set $x_p(r) = \Gamma_p \cap \partial B_{a,r}$ for $r \geq R$ and define

$$\lambda_p = \lim_{r \rightarrow \infty} f(x_p(r)) \in \mathbb{R} \cup \{\pm\infty\}.$$

Let $T_\infty(f)$ denote the set of values $\lambda \in \mathbb{R}$ for which there exists a curve $x: [R, \infty) \rightarrow \Gamma$ with $x(r) \in \Gamma \cap \partial B_{a,r}$ and $\lim_{r \rightarrow \infty} f(x(r)) = \lambda$. Note that

$$T_\infty(f) \subset \{\lambda_p \in \mathbb{R} \mid p \in (\Gamma \setminus \text{Sing}(f)) \cap \partial B_{a,R}\} \cup K_0(f).$$

Note also that the bifurcation set $B_\infty(f)$ of f satisfies the inclusion $B_\infty(f) \subset T_\infty(f) \cup K_0(f)$, see for instance [5, Theorem 2.5].

The aim of this paper is to characterize atypical values of f by observing its behavior on the sphere $\partial B_{a,R}$ with a sufficiently large radius $R > 0$. Specifically, we focus on the vector field $X_{a,R}$ on $\partial B_{a,R}$ defined by the gradient vector field of the restriction of f to $\partial B_{a,R}$. For each isolated zero p of $X_{a,R}$, the index $\text{Ind}_p(X_{a,R})$ is defined, on a local coordinate chart $\psi: U \rightarrow \mathbb{R}^{n-1}$ from a small neighborhood U of p in ∂B_R satisfying $\psi(p) = 0$, by the degree of the map from the $(n-2)$ -dimensional sphere in $\psi(U) \subset \mathbb{R}^{n-1}$ centered at 0 with a sufficiently small radius to the $(n-2)$ -dimensional unit sphere given by

$$x \mapsto \frac{\psi_*(X_{a,R}(\psi^{-1}(x)))}{\|\psi_*(X_{a,R}(\psi^{-1}(x)))\|},$$

where ψ_* is the differential map induced by ψ . Note that each point of $(\Gamma \setminus \text{Sing}(f)) \cap \partial B_{a,R}$ is an isolated zero of $X_{a,R}$. For each $\lambda \in T_\infty(f)$, let $\Gamma^{(\lambda)}$ be the union of tangency branches Γ_p with $\lambda_p = \lambda$. For each connected component Ω of $\partial B_{a,R} \setminus f^{-1}(\lambda)$, set

$$\text{Ind}(\lambda, \Omega) = \sum_{p \in \Gamma^{(\lambda)} \cap \Omega} \text{Ind}_p(X_{a,R}).$$

We focus on the case $n = 3$. In this case, since regular fibers of f are of dimension 2, their topology can be determined by the indices of the vector field $X_{a,R}$. In consequence, we obtain the following theorem. For the definition of a vanishing component, see Section 2.1.

Theorem 1. *Let $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ be a polynomial function and $\lambda \in T_\infty(f) \setminus K_0(f)$. If $\text{Ind}(\lambda, \Omega) \neq 0$ for some connected component Ω of $\partial B_{a,R} \setminus f^{-1}(\lambda)$ then λ is an atypical value at ∞ of f . Conversely, if there does not exist a vanishing component at ∞ when t tends to λ and $\text{Ind}(\lambda, \Omega) = 0$ for any connected component Ω of $\partial B_R \setminus f^{-1}(\lambda)$ then λ is a typical value at ∞ of f .*

In [3], Coste and de la Puente studied atypical values of polynomial maps in two real variables. They focused on the tangency branches at ∞ of a polynomial function f with respect to the family of vertical lines $x = r$ and $x = -r$ for $r > R$, where $R > 0$ is sufficiently large. They obtained a necessary and sufficient condition for $\lambda \in \mathbb{R}$ to be an atypical value at ∞ of f , expressed in terms of the tangency between $f^{-1}(t)$ and $x = \pm r$ along these tangency branches, which they called “critical clusters”. Theorem 1 stated above is a three-variable analogue of their result, where we instead consider a family of round spheres to define tangency branches and analyze the indices of the vector field $X_{a,R}$, rather than considering only tangencies, since we are dealing with functions in three variables.

In the proof, it is shown that if $\text{Ind}(\lambda, \Omega) \neq 0$ for some Ω then, for t sufficiently close to λ , there exists a compact connected component of $f^{-1}(t) \setminus \text{Int } B_{a,R}$ diffeomorphic to a disk. This interpretation can be used when we generalize the assertion in Theorem 1 to polynomial maps $F: \mathbb{R}^n \rightarrow \mathbb{R}^{n-2}$ for $n \geq 3$. The statement is the following.

Theorem 2. *Let $F: \mathbb{R}^n \rightarrow \mathbb{R}^{n-2}$ be a polynomial map, where $n \geq 3$. Suppose that the radius $R > 0$ of $B_{a,R}$ is sufficiently large. Then, $\lambda \in F(\mathbb{R}^n) \setminus K_0(F)$ is a typical value at ∞ of F if and only if the following are satisfied:*

- (1) *there is no vanishing component at ∞ when t tends to λ ;*
- (2) *there exists a neighborhood D of λ in \mathbb{R}^{n-2} such that, for any $t \in D$:*
 - (2a) *$F^{-1}(t) \setminus \text{Int } B_R$ has no compact, connected component;*
 - (2b) *$\chi(F^{-1}(t)) = \chi(F^{-1}(\lambda))$ holds.*

The above theorem is stated again in Section 5 (Theorem 17), where a precise condition for the radius R is given. The condition (2a) is added instead of the condition about the indices in Theorem 1. Note that atypical values of an algebraic family of real curves, which can be seen as a restriction of a polynomial map from \mathbb{R}^n to \mathbb{R}^{n-1} , are characterized by the conditions (1) and (2b) [17]. See also [11]. Atypical values of a holomorphic map between connected complex manifolds $M \rightarrow B$ with $\dim_{\mathbb{C}} M = \dim_{\mathbb{C}} B + 1$ are also characterized by the conditions (1) and (2b) [12].

This paper is organized as follows. In Section 2, we prove a few lemmas concerning the choice of the center a and the radius R of the ball $B_{a,R}$. In Section 2.5, two examples of polynomial functions $f: \mathbb{R}^3 \rightarrow \mathbb{R}$, which are based on examples in [17] (also [3]), are given. In Section 3, we prove a theorem that characterizes a vanishing component at infinity of a real polynomial function. Using this theorem, we can obtain some argument for detecting a vanishing component at infinity, see Remark 12. Section 4 is devoted to the proof of Theorem 1, and Section 5 is devoted to the proof of Theorem 2.

2. Preliminaries

2.1. Vanishing component

In this section we give the definition of a vanishing component at ∞ for a polynomial map from \mathbb{R}^n to \mathbb{R}^m with $n > m \geq 1$.

Definition 3. Let $F: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a polynomial map. It is said that there is a vanishing component at ∞ when t tends to λ if there exists a sequence of points $\{t_k\}$ in \mathbb{R}^m such that

$$\lim_{k \rightarrow \infty} t_k = \lambda \quad \text{and} \quad \lim_{k \rightarrow \infty} \max_i \inf \{ \|x\| \in \mathbb{R} \mid x \in Y_{t_k, i} \} = \infty,$$

where $Y_{t,1}, \dots, Y_{t,n_t}$ are the connected components of $F^{-1}(t)$.

Remark 4. The existence of a vanishing component at ∞ does not change even if the distance function $\|x\|$ is replaced by $\|x - a\|$ for any point $a \in \mathbb{R}^n$.

2.2. The center of the ball $B_{a,R}$

Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be a polynomial function, $\text{Sing}(f)$ be the set of critical points of f in \mathbb{R}^n , and $K_0(f) = f(\text{Sing}(f))$.

Lemma 5. Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be a polynomial function, K be a finite set in $f(\mathbb{R}^n) \setminus K_0(f)$, and A_K be the set of points a in \mathbb{R}^n satisfying that, for each $\lambda \in K$, there exists an open interval I_λ in \mathbb{R} containing λ such that the function on $f^{-1}(t)$ defined by $x \mapsto \|x - a\|^2$ has only non-degenerate critical points for any $t \in I_\lambda \setminus \{\lambda\}$. Then the set A_K is dense in \mathbb{R}^n .

Proof. Set

$$S = \left\{ (x, v, t) \in \mathbb{R}^n \times \mathbb{R}^n \times (\mathbb{R} \setminus K_0(f)) \mid f(x) = t, \text{rank}(\text{grad}_f^v) \leq 1 \right\}.$$

It is easy to check that S is a semialgebraic set of dimensional $n + 1$ having no singular points.

Consider the “endpoint” map (see [15]):

$$E: S \rightarrow \mathbb{R}^n \times (\mathbb{R} \setminus K_0(f)), \quad (x, v, t) \mapsto (x + v, t).$$

By the Sard Theorem, the set $E(\text{Sing}(E))$ of critical values of E has measure 0. We can also check that $E(\text{Sing}(E))$ is a semialgebraic set in $\mathbb{R}^n \times (\mathbb{R} \setminus K_0(f))$ of dimension at most n . By [15, Lemma 6.5], $(a, t) \in E(\text{Sing}(E))$ if and only if the function on $f^{-1}(t)$ defined by $x \mapsto \|x - a\|^2$ has a degenerate critical point.

We will prove the following claim: for each point $a \in \mathbb{R}^n$, any neighborhood of a in \mathbb{R}^n contains at least one point $x \neq a$ such that the intersection $(\{x\} \times \mathbb{R}) \cap E(\text{Sing}(E))$ is an isolated set. This implies that A_K is dense in \mathbb{R}^n .

For a contradiction, we assume that there exist a point $a \in \mathbb{R}^n$ and a small neighborhood U of a in \mathbb{R}^n satisfying that, for each $x \in U \setminus \{a\}$, there is an open interval $I_x \subset \mathbb{R}$ such that $\{x\} \times I_x \subset E(\text{Sing}(E))$.

Since $E(\text{Sing}(E))$ is a semialgebraic set in $E(S)$ of codimension at least one, its Zariski closure V in $\mathbb{R}^n \times \mathbb{R}$ is an algebraic subset of dimension at most n . Let $\pi: V \rightarrow \mathbb{R}^n$ be the projection from $V \subset \mathbb{R}^n \times \mathbb{R}$ to \mathbb{R}^n defined by $(x, t) \mapsto x$. Since $\{x\} \times I_x \subset V$ for $x \in U \setminus \{a\}$, the inclusion $U \setminus \{a\} \subset \pi(V)$ holds.

On the other hand, it implies from [16, 18] that there exists an open ball $B \subset U \setminus \{a\}$ such that π is trivial on B , which means that $\pi^{-1}(B) \subset V$ is diffeomorphic to $B \times \pi^{-1}(x)$ for $x \in B$. From the inclusion $U \setminus \{a\} \subset \pi(V)$, we get $\{x\} \times I_x \subset \pi^{-1}(x)$. Therefore $\dim \pi^{-1}(B) = \dim U + 1 = n + 1$. This contradicts $\dim V \leq n$. \square

2.3. Topology of fibers and indices of vector fields on the sphere

Choose a center $a \in \mathbb{R}^n$ of $B_{a,R}$ generic and the radius $R > 0$ sufficiently large. The interior of $B_{a,R}$ is denoted by $\text{Int } B_{a,R}$ and its boundary is denoted by $\partial B_{a,R}$. For each point $p \in (\Gamma \setminus \text{Sing}(f)) \cap \partial B_{a,R}$, let Γ_p denote the tangency branch at ∞ of f passing through p . Set $x_p(r) = \Gamma_p \cap \partial B_{a,r}$, then $f(x_p(r))$ is monotone with respect to the parameter r . We use the following notations:

- $f \nearrow \lambda$ along Γ_p means that $f(x_p(r))$ is monotone increasing for $r \geq R$ and $\lim_{r \rightarrow \infty} f(x_p(r)) = \lambda$.
- $f \searrow \lambda$ along Γ_p means that $f(x_p(r))$ is monotone decreasing for $r \geq R$ and $\lim_{r \rightarrow \infty} f(x_p(r)) = \lambda$.

Remark 6. Let Γ_p be the tangency branch at ∞ of f passing through $p \in (\Gamma \setminus \text{Sing}(f)) \cap \partial B_{a,R}$. We have the following remarks.

- (1) The point p is a critical point of the following two functions:

$$\begin{aligned} f|_{\partial B_{a,r_a(p)}} : \partial B_{a,r_a(p)} &\rightarrow \mathbb{R}, \quad \text{where } r_a(p) = \|p - a\|, \\ r_a|_{f^{-1}(f(p))} : f^{-1}(f(p)) &\rightarrow \mathbb{R}, \quad \text{where } r_a(x) = \|x - a\|. \end{aligned}$$

- (2) Suppose that $f \nearrow \lambda_p$ along Γ_p . Then, p is a local maximum (resp. minimum) point of $f|_{\partial B_{a,r_a(p)}}$ if and only if it is a local minimum (resp. maximum) point of $r_a|_{f^{-1}(f(p))}$.
- (3) Suppose that $f \searrow \lambda_p$ along Γ_p . Then, p is a local maximum (resp. minimum) point of $f|_{\partial B_{a,r_a(p)}}$ if and only if it is a local maximum (resp. minimum) point of $r_a|_{f^{-1}(f(p))}$ (cf. Example 8).

For simplicity, we denote by \mathcal{P} one of the properties “local maximum”, “local minimum”, “neither local maximum nor local minimum”.

Lemma 7. *There exists a sufficiently large radius $R' > 0$ such that, for each $p \in (\Gamma \setminus \text{Sing}(f)) \cap \partial B_{a,R'}$, the property \mathcal{P} of $f|_{\partial B_{a,r}}$ is constant on Γ_p for $r \geq R'$.*

Proof. Let R be a radius of B_R such that Γ is transverse to $\partial B_{a,r}$ for any $r > R$ and $\Gamma \setminus \text{Int } B_{a,R}$ is homeomorphic to $\Gamma \cap \partial B_{a,R} \times [0, 1)$. For each property \mathcal{P} , define the subset $V_{\mathcal{P}}$ of \mathbb{R}^n by

$$V_{\mathcal{P}} = \left\{ x \in \mathbb{R}^n \mid x \text{ is a } \mathcal{P} \text{ point of } f|_{\partial B_{a,r_a(x)}} \text{ for } r_a(x) = \|x - a\| \geq R \right\}.$$

We will show that, for each \mathcal{P} , the set $V_{\mathcal{P}}$ is a semi-algebraic set. If \mathcal{P} is local maximum, the set $V_{\mathcal{P}}$ is represented in terms of the first-order formulas as follows (for the definitions of first-order formulas, see [1, 7]):

$$V_{\mathcal{P}} = \left\{ x \in \mathbb{R}^n \mid \exists \varepsilon \in \mathbb{R} \left((y \in \mathbb{R}^n, \|y\| = \|x\|, \|y - x\| < \varepsilon) \Rightarrow f(y) \leq f(x) \right) \right\}.$$

Hence, it implies from the Tarski–Seidenberg Theorem (see [1, Proposition 2.2.4] or [7, Theorem 1.6]) that $V_{\mathcal{P}}$ is a semialgebraic set. The set $V_{\mathcal{P}}$ for \mathcal{P} being local minimum is also semialgebraic by a similar argument. If \mathcal{P} is neither local maximum nor local minimum, then the set $V_{\mathcal{P}}$ is the complement of the above two semialgebraic sets. Therefore it is also semialgebraic.

Now, $\Gamma_p \cap V_{\mathcal{P}}$ is a semialgebraic subset of a curve for each \mathcal{P} . Hence we can choose $R' > 0$ sufficiently large so that each Γ_p is contained in some of $V_{\mathcal{P}}$. \square

2.4. Choice of the radius R

To discuss the choice of the radius R , we consider the set

$$K_{\infty}(f) = \left\{ t \in \mathbb{R} \mid \begin{array}{l} \text{there exists a sequence } \{x_k\} \text{ in } \mathbb{R}^n \text{ such that } \|x_k\| \rightarrow \infty, \\ f(x_k) \rightarrow t, \text{ and } \|x_k\| \|\text{grad } f(x_k)\| \rightarrow 0 \text{ as } k \rightarrow \infty \end{array} \right\}$$

called the set of *asymptotic critical values* of f , see [10]. Note that $K_{\infty}(f)$ is a finite set and satisfies $T_{\infty}(f) \subset K_0(f) \cup K_{\infty}(f)$ (cf. [5, Theorem 2.8]). We choose a generic point $a \in \mathbb{R}^n$ as in Lemma 5 with respect to the set $K = T_{\infty}(f) \setminus K_0(f)$. Choose an open interval I_{λ} for each $\lambda \in K$ so that $I_{\lambda} \cap I_{\lambda'} = \emptyset$ for $\lambda \neq \lambda' \in T_{\infty}(f)$. We choose the radius $R > 0$ sufficiently large so that the following properties hold:

- (i) $\Gamma \setminus \text{Int } B_{a,R}$ is homeomorphic to $(\Gamma \cap \partial B_{a,R}) \times [0, 1)$ and, for each $p \in (\Gamma \setminus \text{Sing}(f)) \cap \partial B_{a,R}$,

$$\Gamma_p \cap \bigcup_{\lambda \in T_\infty(f)} f^{-1}(\lambda) = \emptyset.$$

- (ii) $R > 0$ satisfies the condition in Lemma 7. Since the center a is chosen as in Lemma 5, the property “neither local maximum nor local minimum” for tangency branches is replaced by “saddle”.
- (iii) For each $\lambda \in K$, each connected component Y of $f^{-1}(\lambda) \setminus \text{Int } B_{a,r}$ intersects $\partial B_{a,r}$ transversely for any $r \geq R$. In particular, Y is diffeomorphic to $(Y \cap \partial B_{a,r}) \times [0, 1)$ for any $r \geq R$.
- (iv) $\{f(x) \mid x \in \Gamma_p\} \subset I_{\lambda_p}$ holds for any $p \in (\Gamma \setminus \text{Sing}(f)) \cap \partial B_{a,R}$ with $\lambda_p \neq \pm\infty$.

In the following sections, we always assume that the radius $R > 0$ is sufficiently large so that these properties hold.

2.5. Examples

We give two examples of polynomial functions $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ of the form $f(x, y, z) = g(x, y)$, where $g(x, y)$ is a polynomial function of two variables. In order to make use of the results in [3], in the following examples, instead of $B_{a,R}$, we use the ball \widehat{B}_R obtained from the cylinder $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + z^2 \leq R^2\}$ by truncating it at $y = \pm \widehat{R}$, where \widehat{R} is sufficiently large relative to R , and rounding the corners. Though all discussions in this paper are based on the round ball $B_{a,R}$, the same arguments can be applied to \widehat{B}_R . The vector field $X_{a,R}$ on $\partial B_{a,R}$ is replaced by the vector field \widehat{X}_R on $\partial \widehat{B}_R$ defined in the same manner.

Example 8. Let $g: \mathbb{R}^2 \rightarrow \mathbb{R}$ be the following polynomial function:

$$g(x, y) = 2y^5 + 4xy^4 + (2x^2 - 9)y^3 - 9xy^2 + 12y.$$

This example is given in [17, Example 3.4]. The shapes of fibers around the infinity are studied in [3] explicitly, which is given as in Figure 1. There are eight tangency branches, four of which are on the right-hand side and the other four are on the left-hand side. The arrow on each tangency branch represents the direction in which the value of f increases. For example, for the right-top tangency branch Γ_{p_1} , we have $g \searrow 0$ along Γ_{p_1} , p_1 is a local minimum of $g|_{x=x_1}$, and it is a local minimum of $x|_{g^{-1}(g(p_1))}$, where x_1 is the x -coordinate of p_1 . This function has no vanishing component at ∞ .

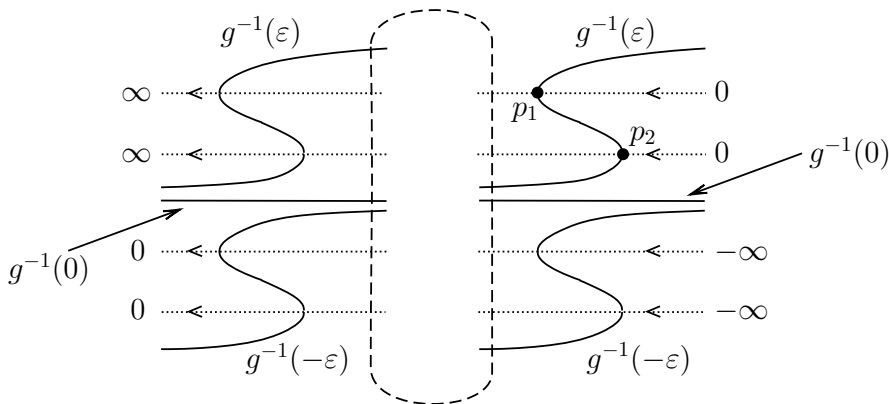


Figure 1. Fibers around the infinity in Example 8. The oriented dotted curves are tangency branches at ∞ .

Let $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ be a polynomial function given by $f(x, y, z) = g(x, y)$. In [17, Example 3.4], the function g is obtained from $h(x, y) = y(2x^2y^2 - 9xy + 12)$ as $g(x, y) = h(x + y, y)$. From this form, we can see that $g^{-1}(0) \cap \partial\widehat{B}_R$ is given by $\{y = 0\} \cap \partial\widehat{B}_R$, which is a connected, simple closed curve on the 2-sphere $\partial\widehat{B}_R$. The complement of this curve in $\partial\widehat{B}_R$ consists of two open disks. We denote the one where y is positive by Ω_1 and the other by Ω_2 . Because of the position of \widehat{B}_R , all tangency branches in Figure 1 are on the plane $z = 0$. Therefore, for example, the point p_1 is local minimum of $f|_{\widehat{B}_R(p_1)}$ and also local minimum of $\widehat{r}|_{f^{-1}(f(p_1))}$, where $\widehat{r}(x, y, z) = \sqrt{x^2 + z^2}$. This is in the case (3) of Remark 6. The index is $\text{Ind}_{p_1}(\widehat{X}_R) = 1$. On the other hand, the singularity of $\widehat{r}|_{f^{-1}(f(p_2))}$ on the tangency branch Γ_{p_2} passing through the point p_2 in the figure becomes a saddle, and therefore its index is $\text{Ind}_{p_2}(\widehat{X}_R) = -1$. The union $\Gamma^{(0)}$ of tangency branches at ∞ of f along which either $f \searrow 0$ or $f \nearrow 0$ has no other tangency branch passing through the region Ω_1 . Hence we have

$$\begin{aligned} \text{Ind}(\lambda, \Omega_1) &= \sum_{p \in \Gamma^{(0)} \cap \Omega_1} \text{Ind}_p(\widehat{X}_R) \\ &= \text{Ind}_{p_1}(\widehat{X}_R) + \text{Ind}_{p_2}(\widehat{X}_R) \\ &= 1 + (-1) \\ &= 0. \end{aligned}$$

By the same observation, we have $\text{Ind}(\lambda, \Omega_2) = 0$. Then, by Theorem 1, we can conclude that 0 is a typical value at ∞ of f .

Example 9. Let $g: \mathbb{R}^2 \rightarrow \mathbb{R}$ be the following polynomial function:

$$g(x, y) = x^2y^3(y^2 - 25)^2 + 2xy(y^2 - 25)(y + 25) - y^4 - y^3 + 50y^2 + 51y - 575.$$

This example is given in [17, Example 3.1]. The shapes of fibers around the infinity are studied in [3] explicitly after replacing x by $x + y$ to avoid vertical tangency at infinity. The fibers are given as in Figure 2. There are two components vanishing at ∞ when t tends to 0. The word “cleaving” means that the point on the tangency branch goes to ∞ when t tends to 0, so that the curve cleaves locally into two curves. There are two cleaving curves.

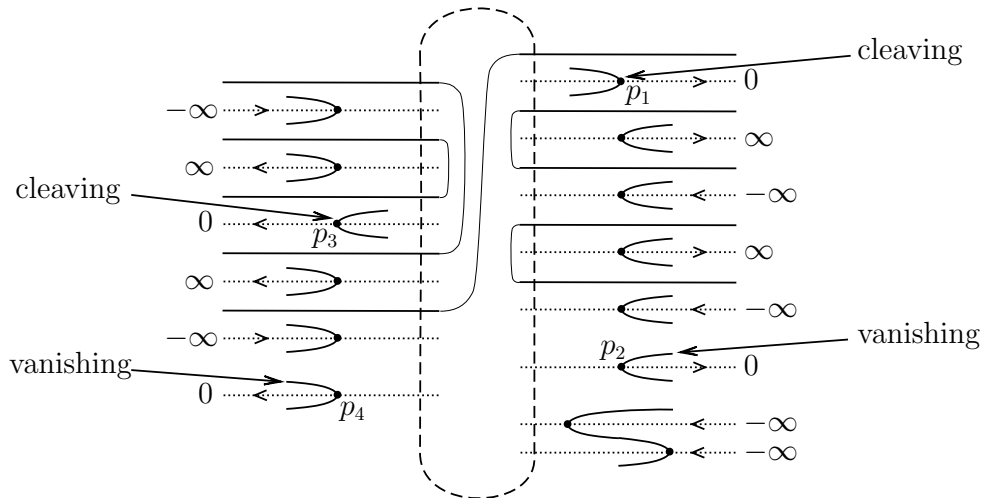


Figure 2. Fibers around the infinity in Example 9. The oriented dotted curves are tangency branches at ∞ . All horizontal solid lines are curves representing $g^{-1}(0)$.

Let $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ be a polynomial function given by $f(x, y, z) = g(x, y)$. Using Mathematica, we can see that the curve of $g^{-1}(0)$ inside the dotted circle is as shown in Figure 2. Note that it is

explained in [17, Example 3.1] that if $|t|$ is sufficiently small then $g^{-1}(t)$ is a disjoint union of five non-compact connected components. Thus, the curves $f^{-1}(0) \cap \partial \hat{B}_R$ on the sphere $\partial \hat{B}_R$ becomes as shown in Figure 3. There are five circles. Let \hat{X}_R be the gradient vector field of $f|_{\partial \hat{B}_R}$. We have $\text{Ind}_{p_1}(\hat{X}_R) = \text{Ind}_{p_3}(\hat{X}_R) = -1$ and $\text{Ind}_{p_2}(\hat{X}_R) = \text{Ind}_{p_4}(\hat{X}_R) = 1$. On the region Ω_1 depicted in the figure, we have

$$\begin{aligned} \text{Ind}(0, \Omega_1) &= \text{Ind}_{p_1}(\hat{X}_R) + \text{Ind}_{p_2}(\hat{X}_R) + \text{Ind}_{p_4}(\hat{X}_R) \\ &= (-1) + 1 + 1 = 1 \\ &\neq 0. \end{aligned}$$

Hence 0 is an atypical value at ∞ of f by Theorem 1. We can get the same conclusion from the region Ω_2 depicted in the figure since $\text{Ind}(0, \Omega_2) = \text{Ind}_{p_3}(\hat{X}_R) = -1 \neq 0$.

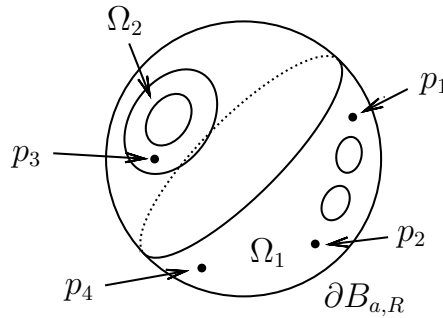


Figure 3. The curves $f^{-1}(0) \cap \partial \hat{B}_R$ on the sphere $\partial \hat{B}_R$.

3. A characterization of vanishing component at infinity

Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be a polynomial function. Hereafter we omit a in the suffix of $B_{a,r}$ for $r > 0$ and denote it by B_r for simplicity. Each critical point $p \in \partial B_R$ of $f|_{\partial B_R}$ not lying on $\text{Sing}(f)$ is a point in $(\Gamma \setminus \text{Sing}(f)) \cap \partial B_R$. Hence it has a tangency branch Γ_p .

Theorem 10. *Suppose $n \geq 3$ and $\lambda \in T_\infty(f) \setminus K_0(f)$. There is a vanishing component at ∞ when t tends to λ with $t > \lambda$ (resp. $t < \lambda$) if and only if there exists a local minimum (resp. maximum) point $p \in \partial B_R$ of $f|_{\partial B_R}$ with $f \searrow \lambda$ (resp. $f \nearrow \lambda$) along Γ_p such that the intersection of the connected component of $f^{-1}(f(p))$ containing p with ∂B_R consists of isolated points.*

Proof. We first prove the “only if” assertion. We only prove the assertion in the case where t tends to λ with $t > \lambda$. The proof for the other case is similar. Let $\{Y_t\}$ be a continuous family of connected components of $f^{-1}(t)$ that vanishes at ∞ when t tends to λ . Tangency branches intersecting $\{Y_t\}$ are contained in a connected component H of $\mathbb{R}^n \setminus (f^{-1}(\lambda) \cup \text{Int } B_R)$ by the property (i) about the choice of the radius R in Section 2.4. Remark that H is possibly $\mathbb{R}^n \setminus \text{Int } B_R$. Set $\Omega = H \cap \partial B_R$. Either $\Omega = \partial B_R$, or $\Omega \subset \partial B_R$ is bounded by a finite number of connected components of $f^{-1}(\lambda) \cap \partial B_R$. Since t tends to λ with $t > \lambda$, we have $\Omega \subset \{x \in \partial B_R \mid f(x) > \lambda\}$. Let S_Ω denote the set of local minimum points of $f|_{\partial B_R}$ in the intersection of Ω and the tangency branches intersecting all Y_t 's in the family $\{Y_t\}$. By the definition of a vanishing component at ∞ in Section 2.1 and Remark 6(1), the function $r_a(x) = \|x - a\|$ restricted to $f^{-1}(t)$ has a local minimum point y on $\Omega \cap \Gamma^{(\lambda)}$. Since $t > \lambda$, it satisfies that $f \searrow \lambda$ along Γ_y . Hence, by Remark 6(3), y is a local minimum point of $f|_{\partial B_R}$, that is, y is a point in S_Ω . In particular, S_Ω is non-empty.

Set $\delta = \min_{x \in S_\Omega} f(x)$ and let p be a point in S_Ω such that $f(p) = \delta$ and $f \searrow \lambda$ along Γ_p . Let $(\lambda, \delta]$ be the range of the parameter t of Y_t . We will show that $Y_\delta \cap \partial B_R$ consists of isolated points. Assume for a contradiction that $Y_\delta \cap \partial B_R$ is not isolated.

Claim 11. $Y_\delta \cap \Omega$ is not isolated.

Proof. Assume that $Y_\delta \cap \Omega$ is isolated. Since all points in $Y_\delta \cap \Omega$ are local minima of $r_a|_{Y_\delta} : Y_\delta \rightarrow \mathbb{R}$, where $r_a(x) = \|x - a\|$, the inequality $\lambda < \delta$ implies that $Y_\delta \cap f^{-1}(\lambda) = \emptyset$. Hence $Y_\delta \subset H$, see Figure 4. This inclusion implies $Y_\delta \cap \partial B_R = Y_\delta \cap \Omega$. However, the right-hand side is isolated while the left-hand side is not. This is a contradiction. \square

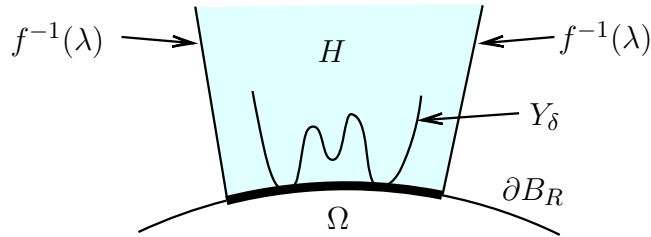


Figure 4. $Y_\delta \cap f^{-1}(\lambda) = \emptyset$ implies $Y_\delta \subset H$.

We continue the proof of Theorem 10. Let $Y_{[\lambda, \delta]}$ be the connected component of $f^{-1}([\lambda, \delta])$ containing p . There are two cases:

Case 1: $Y_{[\lambda, \delta]} \cap f^{-1}(\lambda) = \emptyset$. See Figure 5. Set $\Omega_{[\lambda, \delta]} = Y_{[\lambda, \delta]} \cap \bar{\Omega}$, where $\bar{\Omega}$ is the closure of Ω in ∂B_R . Since $Y_\delta \cap \Omega$ is not isolated by Claim 11, $Y_\delta \cap \Omega$ has a connected component C of dimension at least 1. A point in Ω at which Y_δ is tangent to Ω belongs to a tangency branch at ∞ of f and hence it is isolated in Ω . In particular, it cannot be in C . This means that Y_δ and Ω intersect along C transversely. Therefore, since f is continuous on $\bar{\Omega}$, $Y_{[\lambda, \delta]} \cap \Omega$ has a connected component C' of dimension $n - 1 \geq 2$. Since $Y_{[\lambda, \delta]}$ is closed, $\partial \bar{\Omega} \subset f^{-1}(\lambda)$, and $Y_{[\lambda, \delta]} \cap f^{-1}(\lambda) = \emptyset$, it follows that C' is a compact subset of Ω . Due to the generic choice of the center a in Lemma 5, the restriction of f to C' cannot be a constant function. Hence f is not a constant function on $\Omega_{[\lambda, \delta]}$. Since $\partial \bar{\Omega} \subset f^{-1}(\lambda)$ (possibly $\partial \bar{\Omega} = \emptyset$) and $Y_{[\lambda, \delta]} \cap f^{-1}(\lambda) = \emptyset$, we have $\partial \Omega_{[\lambda, \delta]} \subset f^{-1}(\delta)$ (possibly $\partial \Omega_{[\lambda, \delta]} = \emptyset$). Hence, there exists a local minimum point q of $f|_{\partial B_R}$ in the interior of $\Omega_{[\lambda, \delta]}$ with $\lambda < f(q) < \delta$.

Since $\lambda < f(q) < \delta = f(p)$, there exists a point q' on Γ_p such that $f(q) = f(q')$. If $f \searrow \lambda_q$ along Γ_q with $\lambda_q \neq \lambda$, then the two sets $\{f(x) \mid x \in \Gamma_p\}$ and $\{f(x) \mid x \in \Gamma_q\}$ should be disjoint by the property (iv). However $f(q) = f(q')$ is a common element of these two sets. If $f \searrow \lambda$ along Γ_q , then $q \in S_\Omega$. However, this and $f(q) < f(p)$ contradict $f(p) = \delta = \min_{x \in S_\Omega} f(x)$. Thus, in either case, a contradiction arises.

Case 2: $Y_{[\lambda, \delta]} \cap f^{-1}(\lambda) \neq \emptyset$. Take one point $q \in Y_{[\lambda, \delta]} \cap f^{-1}(\lambda)$, then q belongs to the connected component of $f^{-1}([\lambda, \varepsilon]) \setminus \text{Int } B_R$ contained in $Y_{[\lambda, \delta]}$ for any $\lambda < \varepsilon < \delta$. This contradicts the fact that $\{Y_t\}$ vanishes at ∞ when t tends to λ .

Next we prove the “if” assertion. Assume that there exists a local minimum point $p \in \Gamma_\lambda \cap \partial B_R$ of $f|_{\partial B_R}$ with $f \searrow \lambda$ along Γ_p such that the intersection of the connected component $Z_{f(p)}$ of $f^{-1}(f(p))$ containing p with the sphere ∂B_R consists of isolated points. Since p is a local minimum point of $f|_{\partial B_R}$, p is also a local minimum point of $r_a|_{Z_{f(p)}}$ by Remark 6(3). This and the isolatedness of $Z_{f(p)} \cap \partial B_R$ imply that $Z_{f(p)} \subset \mathbb{R}^n \setminus \text{Int } B_R$.

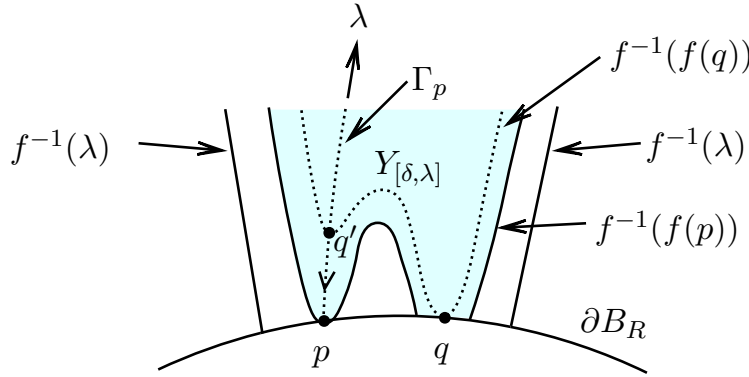


Figure 5. A schematic picture for the proof in Case 1.

Put $\delta = f(p)$. Let $Z_{(\lambda, \delta]}$ be the connected component of $f^{-1}((\lambda, \delta])$ containing p . We will show that $Z_{(\lambda, \delta]} \cap \partial B_R = Z_\delta \cap \partial B_R$. It is easy to see that the two connected components $\Gamma_p \setminus \{p\}$ and $\partial B_R \setminus Z_\delta$ are subsets of different connected components of $\mathbb{R}^n \setminus Z_\delta$. Assume that there exists a point $x \in Z_{(\lambda, \delta]} \cap (\partial B_R \setminus Z_\delta)$. Choose a point $y \in \Gamma_p \setminus \{p\}$. Note that the values $f(x)$ and $f(y)$ are in (λ, δ) . Since $Z_{(\lambda, \delta]}$ is connected, there exists a path in $Z_{(\lambda, \delta]}$ connecting x and y . Furthermore, since f is a trivial fibration on (λ, δ) , we can isotope this path so that it is in $Z_{(\lambda, \delta]} \setminus Z_\delta$. However, this is impossible since x and y belong to different connected components of $\mathbb{R}^n \setminus Z_\delta$. Therefore, $Z_{(\lambda, \delta]} \cap \partial B_R = Z_\delta \cap \partial B_R$.

Now it follows that for any $t \in (\lambda, \delta)$, the connected component Z_t of $f^{-1}(t)$ intersecting Γ_p does not intersect B_R . Hence the distance function $r_a|_{Z_t}$ on Z_t attains a minimum value at some point belonging to a tangency branch in $\mathbb{R}^n \setminus \text{Int } B_R$. Thus, we can find a sequence $\{t_k\}$ on (λ, δ) with $\lim_{k \rightarrow \infty} t_k = \lambda$ and a point $q \in \Gamma \cap \partial B_R$ such that

$$\min\{r_a(x) \mid x \in Z_{t_k}\} = r_a(q_k),$$

where $q_k = Z_{t_k} \cap \Gamma_q$. The distance $r_a(q_k)$ goes to ∞ as $k \rightarrow \infty$, otherwise Γ_q intersects $f^{-1}(\lambda)$ and this contradicts the property (i). Hence Z_{t_k} vanishes at ∞ as $k \rightarrow \infty$.

The proof for the case where p is a local maximum point is similar. \square

Remark 12. Using Theorem 10, a vanishing component at ∞ of $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is detected as follows.

Step 1. Choose a generic center a , calculate all tangency branches, and fix a sufficiently large radius $R > 0$ that satisfies the conditions written in Section 2.4.

Step 2. For each $p \in (\Gamma \setminus \text{Sing}(f)) \cap \partial B_R$, calculate $\lambda_p = \lim_{r \rightarrow \infty} f(x_p(r))$, where $x_p(r) = \Gamma_p \cap \partial B_r$. Then, make the following list of finite sets:

$$\begin{aligned} P_{\min}(\lambda) &= \{p \in (\Gamma \setminus \text{Sing}(f)) \cap \partial B_R \mid p \text{ is local minimum of } f|_{\partial B_R} \text{ with } f \searrow \lambda\}, \\ P_{\max}(\lambda) &= \{p \in (\Gamma \setminus \text{Sing}(f)) \cap \partial B_R \mid p \text{ is local maximum of } f|_{\partial B_R} \text{ with } f \nearrow \lambda\}, \\ \Lambda_{\min} &= \{\lambda \in \mathbb{R} \mid P_{\min}(\lambda) \neq \emptyset\}, \\ \Lambda_{\max} &= \{\lambda \in \mathbb{R} \mid P_{\max}(\lambda) \neq \emptyset\}. \end{aligned}$$

Step 3. For each element $\lambda \in \Lambda_{\min}$ (resp. $\lambda \in \Lambda_{\max}$), check if there exists $p \in P_{\min}(\lambda)$ (resp. $p \in P_{\max}(\lambda)$) such that the intersection $f^{-1}(f(p)) \cap \partial B_R$ consists of isolated points.

- (a) If it exists, then there exists a vanishing component at ∞ when t tends to λ by Theorem 10.

- (b) If it does not exist, then $\dim f^{-1}(f(p)) \cap \partial B_R \geq 1$ for any $p \in P_{\min}(\lambda) \cup P_{\max}(\lambda)$. For each $p \in P_{\min}(\lambda) \cup P_{\max}(\lambda)$, calculate all critical values c_1, \dots, c_k of $r_a(x) = \|x - a\|$ on $f^{-1}(f(p))$ and then choose a real number R' greater than $\max\{R, c_1, \dots, c_k\}$, see Figure 6. Make a list L' of the connected components of $\partial B_{R'} \setminus f^{-1}(f(p))$ and find a component $\Omega'_p \in L'$ intersecting Γ_p . If $f^{-1}(\lambda) \cap \Omega'_p = \emptyset$ then there exists a vanishing component at ∞ when t tends to λ as shown in the next lemma (Lemma 13).

All vanishing components at ∞ are detected by the above steps, which is proved in Lemma 14 below.

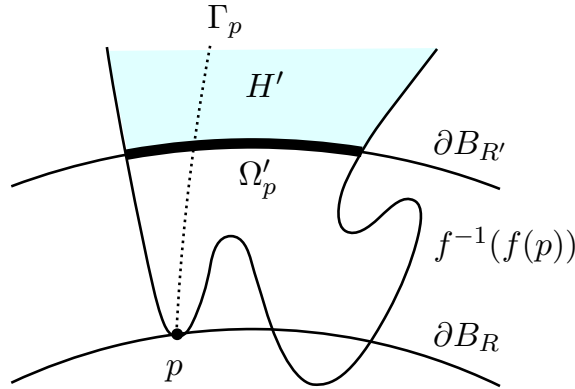


Figure 6. A schematic picture for Step 3(b).

Lemma 13. *If $f^{-1}(\lambda) \cap \Omega'_p = \emptyset$ then there exists a vanishing component at ∞ when t tends to λ .*

Proof. Consider the case where $p \in P_{\min}(\lambda)$. Let H' be the connected component of $\mathbb{R}^n \setminus (f^{-1}(f(p)) \cup \text{Int } B_{R'})$ intersecting Γ_p and $\overline{H'}$ be its closure. Let $\overline{\Omega'_p}$ be the closure of Ω'_p in $\partial B_{R'}$. Since $R' > \max\{R, c_1, \dots, c_k\}$, $\overline{H'}$ is diffeomorphic to $\overline{\Omega'_p} \times [0, 1)$. Let $Y_{(\lambda, f(p)]}$ be the connected component of $f^{-1}(([\lambda, f(p)]) \setminus \text{Int } B_{R'})$ intersecting Γ_p and $\overline{Y_{(\lambda, f(p)]}}$ be its closure. The inclusion $\overline{Y_{(\lambda, f(p)]}} \subset \overline{H'}$, the property (iii), and the assumption $f^{-1}(\lambda) \cap \Omega'_p = \emptyset$ imply that $\overline{Y_{(\lambda, f(p)]}} \cap f^{-1}(\lambda) = \emptyset$. Since $f \searrow \lambda$ along Γ_p and $f^{-1}(\lambda) \cap \Omega'_p = \emptyset$, we have $\lambda < f(x)$ for $x \in \overline{\Omega'_p}$, $f(x) = f(p)$ for $x \in \partial \overline{\Omega'_p}$, and there exists a point $x' \in \Omega'_p$ such that $f(x') < f(p)$. Set $\delta = \min\{f(x) \mid x \in \overline{\Omega'_p}\}$. Note that $\lambda < \delta < f(p)$. For $t \in (\lambda, \delta)$, the connected component Y_t of $f^{-1}(t)$ intersecting Γ_p does not intersect Ω'_p , and therefore it is contained in $Y_{(\lambda, f(p)]}$. Since $\overline{Y_{(\lambda, f(p)]}} \cap f^{-1}(\lambda) = \emptyset$, $\{Y_t\}$ is a vanishing component at ∞ when t tends to λ .

The assertion for the case $p \in P_{\max}(\lambda)$ is proved similarly. \square

Lemma 14. *If there exists a vanishing component at ∞ when t tends to λ , then there exists a point $p \in (\Gamma \setminus (\text{Sing}(f))) \cap \partial B_R$ with $f^{-1}(\lambda) \cap \Omega'_p = \emptyset$.*

Proof. Suppose that there exists a vanishing component at ∞ when t tends to λ . We prove only the case $t > \lambda$. By Theorem 10, there exists a local minimum point $p \in \Gamma_p \cap \partial B_R$ of $f|_{\partial B_R}$ such that the intersection of the connected component $Z_{f(p)}$ of $f^{-1}(f(p))$ containing p with the sphere ∂B_R consists of isolated points. Put $\delta = f(p)$ and let $Z_{(\lambda, \delta]}$ be the connected component of $f^{-1}((\lambda, \delta])$ containing p . Then, as shown in the proof of the “if” assertion of Theorem 10, we have $Z_{(\lambda, \delta]} \cap \partial B_R = Z_\delta \cap \partial B_R$. Let R' be the radius chosen as in Step 3(b) and Ω'_p be the connected component of $\partial B_{R'} \setminus Z_\delta$ intersecting Γ_p . Assume that there exists an intersection point $x \in f^{-1}(\lambda) \cap \Omega'_p$. By the property (iii), there exists an arc on $f^{-1}(\lambda)$ connecting x and a point on

$f^{-1}(\lambda) \cap \partial B_R$, but such an arc should intersect Z_δ . This contradicts the fact that the image of this arc is λ . \square

4. Proof of Theorem 1

Now, we restrict our setting to the case of polynomial functions with three variables. Let $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ be a polynomial function. For each $\lambda \in T_\infty(f) \setminus K_0(f)$, there exists a sufficiently small $\varepsilon > 0$ such that, for $J_\lambda^- = (\lambda - \varepsilon, \lambda)$ and $J_\lambda^+ = (\lambda, \lambda + \varepsilon)$, the restriction of f to $f^{-1}(J_\lambda^*)$ and the restriction of f to $f^{-1}(J_\lambda^*) \cap B_R$ are trivial fibrations unless $f^{-1}(J_\lambda^*) = \emptyset$, where $*$ $\in \{-, +\}$. Here ε is chosen so that $f^{-1}(t)$ intersects ∂B_R transversely for $t \in J_\lambda^*$. Then the restriction of f to $f^{-1}(J_\lambda^*) \cap (\mathbb{R}^n \setminus \text{Int } B_R)$ is also a trivial fibration.

The surface $f^{-1}(\lambda) \setminus \text{Int } B_R$ divides $\mathbb{R}^3 \setminus \text{Int } B_R$ into a finite number of connected components $H_{\lambda,1}^*, \dots, H_{\lambda,n_\lambda}^*$ by the property (iii), where $*$ = $-$ if $f(x) < \lambda$ on $H_{\lambda,i}^*$ and $*$ = $+$ if $f(x) > \lambda$ on $H_{\lambda,i}^*$. Each $H_{\lambda,i}^*$ is homeomorphic to $\Omega_{\lambda,i}^* \times [0, 1)$, where $\Omega_{\lambda,i}^* = H_{\lambda,i}^* \cap \partial B_R$.

Let $\text{Ind}(\lambda, \Omega_{\lambda,i}^*)$ be the sum of indices of the gradient vector field of f restricted to ∂B_R for all zeros belonging to $\Gamma^{(\lambda)}$ on $\Omega_{\lambda,i}^*$ as defined in the introduction.

Lemma 15. $\chi(f^{-1}(t) \cap H_{\lambda,i}^*) = \text{Ind}(\lambda, \Omega_{\lambda,i}^*)$ for any $t \in J_\lambda^*$.

Proof. Choose $R' > R$ sufficiently large so that $f^{-1}(t)$ intersects $\partial B_{R'}$ transversely and $f^{-1}(t) \setminus \text{Int } B_{R'}$ is diffeomorphic to $(f^{-1}(t) \cap \partial B_{R'}) \times [0, 1)$ for $t \in J_\lambda^*$. Then $f^{-1}(t) \cap H_{\lambda,i}^*$ has the same homotopy type as $f^{-1}(t) \cap H_{\lambda,i}^* \cap B_{R'}^{R'}$, where $B_{R'}^{R'} = \{x \in \mathbb{R}^3 \mid R \leq \|x - a\| \leq R'\}$. Hence we have

$$\chi(f^{-1}(t) \cap H_{\lambda,i}^*) = \chi(f^{-1}(t) \cap H_{\lambda,i}^* \cap B_{R'}^{R'}).$$

Consider the distance function $r_a(x) = \|x - a\|$ on $f^{-1}(t) \cap H_{\lambda,i}^* \cap B_{R'}^{R'}$. Due to the generic choice of the center of B_R in Section 2.2, this function has only non-degenerate critical points and has no critical point on the boundary. Hence, there is a one-to-one correspondence between critical points of r_a on $f^{-1}(t) \cap H_{\lambda,i}^*$ and the tangency branches Γ_p passing through $p \in \Gamma^{(\lambda)} \cap \Omega_{\lambda,i}^*$ as mentioned in Remark 6(1). Note that the non-degeneracy of the critical points of r_a implies the non-degeneracy of the critical points of $f|_{\partial B_R}$, which can be verified by comparing the two functions on a coordinate neighborhood around each critical point. If $p \in \Gamma^{(\lambda)} \cap \Omega_{\lambda,i}^*$ is local minimum or maximum of $f|_{\partial B_R}$ then $\text{Ind}_p(X_{a,R}) = 1$ and the Morse index $i(p_t)$ of the distance function r_a on $f^{-1}(t) \cap H_{\lambda,i}^*$ at the intersection point p_t of Γ_p with $f^{-1}(t)$ is 0 or 2. If $p \in \Gamma^{(\lambda)} \cap \Omega_{\lambda,i}^*$ is a saddle point of $f|_{\partial B_R}$ then $\text{Ind}_p(X_{a,R}) = -1$ and the Morse index $i(p_t)$ at the intersection point p_t of Γ_p with $f^{-1}(t)$ is 1. Hence we have $\text{Ind}_p(X_{a,R}) = (-1)^{i(p_t)}$. Since the Euler characteristic of $f^{-1}(t) \cap \Omega_{\lambda,i}^*$ is 0, by the Morse Theory, we have

$$\begin{aligned} \chi(f^{-1}(t) \cap H_{\lambda,i}^* \cap B_{R'}^{R'}) &= \sum_{p \in \Gamma^{(\lambda)} \cap \Omega_{\lambda,i}^*} (-1)^{i(p_t)} \\ &= \sum_{p \in \Gamma^{(\lambda)} \cap \Omega_{\lambda,i}^*} \text{Ind}_p(X_{a,R}) \\ &= \text{Ind}(\lambda, \Omega_{\lambda,i}^*). \end{aligned}$$

This completes the proof. \square

Example 16. Let $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ be the polynomial function defined as $f(x, y, z) = y(xy - 1)$. It satisfies $\text{Sing}(f) = \emptyset$ and 0 is an atypical value at ∞ of f since the number of connected components of $f^{-1}(0)$ is three although that of $f^{-1}(t)$ is two for $t \neq 0$. The thickened curves in Figure 7 are the intersection of $f^{-1}(0)$ with ∂B_R and the thin curves are the intersection of $f^{-1}(-\varepsilon)$ with ∂B_R , where $\varepsilon > 0$ is chosen such that $f^{-1}(-\varepsilon)$ is tangent to ∂B_R at a point near the positive x -axis. The tangent point p can be seen as a double point of the thin curve. The interval J_λ^- is chosen

such that f restricted to $f^{-1}(J_\lambda^-) \cap B_R$ is a trivial fibration. In this example, it is chosen such that $J_\lambda^- \subset (-\varepsilon, 0)$.

The equality in Lemma 15 for this example is confirmed as follows. Let $\Omega_{0,1}^-$ be the regions on ∂B_R bounded by $f^{-1}(0)$ and containing p , and $H_{0,1}^-$ be the corresponding connected component of $\mathbb{R}^3 \setminus (f^{-1}(0) \cup \text{Int } B_R)$. For each $t \in J_\lambda^-$, $f^{-1}(t) \cap H_{0,1}^-$ is homeomorphic to \mathbb{R}^2 removing two open disks. In particular, $\chi(f^{-1}(t) \cap H_{0,1}^-) = -1$. On the other hand, since the index of the vector field $X_{a,R}$ at p is -1 , we have $\text{Ind}(0, \Omega_{0,1}^-) = -1$. Thus the equality holds.

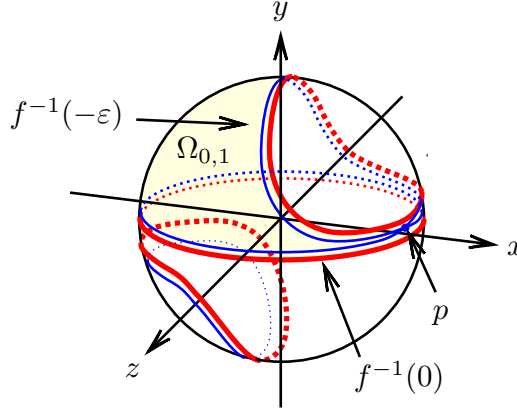


Figure 7. The curves $f^{-1}(0) \cap \partial B_R$ and $f^{-1}(-\varepsilon) \cap \partial B_R$.

Proof of Theorem 1. We prove the first assertion by contraposition. Assume that λ is a typical value of f . There exists a sufficiently small $\varepsilon > 0$ such that f is a trivial fibration on $J_\lambda = (\lambda - \varepsilon, \lambda + \varepsilon)$. Let $\Omega_{\lambda,i}^*$ be a connected component of $\partial B_R \setminus f^{-1}(\lambda)$ and $\partial \overline{\Omega_{\lambda,i}^*}$ be the boundary of the closure of $\Omega_{\lambda,i}^*$ in ∂B_R , which is a union of circles. By the property (iii), the connected component Y of $f^{-1}(\lambda) \setminus \text{Int } B_R$ intersecting $\partial \overline{\Omega_{\lambda,i}^*}$ is diffeomorphic to $\partial \overline{\Omega_{\lambda,i}^*} \times [0, 1)$, and hence $\chi(Y) = 0$. This and the triviality of f on J_λ imply that $\chi(f^{-1}(t) \cap H_{\lambda,i}^*) = 0$ for $t \in J_\lambda^*$, where $H_{\lambda,i}^*$ is the connected component of $\mathbb{R}^3 \setminus (f^{-1}(\lambda) \cup \text{Int } B_R)$ intersecting $\Omega_{\lambda,i}^*$. Combining this with Lemma 15 we obtain $\text{Ind}(\lambda, \Omega_{\lambda,i}^*) = 0$. This completes the proof of the first assertion.

Next we prove the second assertion. Because there does not exist a component of $f^{-1}(t)$ vanishing at ∞ when t tends to λ , there exists a sufficiently small $\varepsilon > 0$ such that each connected component of $f^{-1}(t)$ intersects ∂B_R for all $t \in J_\lambda^- \cup J_\lambda^+$. Let $H_{\lambda,i}^*$ be a connected component of $\mathbb{R}^3 \setminus (f^{-1}(\lambda) \cup \text{Int } B_R)$ and $\{Y_t^1, \dots, Y_t^s\}$ be the connected components of $f^{-1}(t) \cap H_{\lambda,i}^*$. Set $\Omega_{\lambda,i}^* = H_{\lambda,i}^* \cap \partial B_R$. Since $\text{Ind}(\lambda, \Omega_{\lambda,i}^*) = 0$, we have $\chi(f^{-1}(t) \cap H_{\lambda,i}^*) = 0$ by Lemma 15. Hence

$$\sum_{j=1}^s \chi(Y_t^j) = 0.$$

Choose $R' > R$ sufficiently large so that $f^{-1}(t)$ intersects $\partial B_{R'}$ transversely and $f^{-1}(t) \setminus \text{Int } B_{R'}$ is diffeomorphic to $(f^{-1}(t) \cap \partial B_{R'}) \times [0, 1)$ for $t \in J_\lambda^*$, and set $B_R^{R'} = \{x \in \mathbb{R}^3 \mid R \leq \|x - a\| \leq R'\}$. Then, as mentioned at the beginning of the proof of Lemma 15, $\chi(Y_t^j \cap B_R^{R'}) = \chi(Y_t^j)$ holds. Hence

$$\sum_{j=1}^s \chi(Y_t^j \cap B_R^{R'}) = 0. \quad (1)$$

Here each $Y_t^j \cap B_R^{R'}$ is a compact, connected, orientable surface embedded in \mathbb{R}^3 .

We claim that $\chi(Y_t^j \cap B_R^{R'}) = 0$ for any $j = 1, \dots, s$. If $s = 1$ then it follows from equation (1). Suppose that $s \geq 2$. Assume that $\chi(Y_t^{j_0} \cap B_R^{R'}) \neq 0$ for some $j_0 \in \{1, \dots, s\}$. Then there exists a connected component $Y_t^{j_1}$ with $\chi(Y_t^{j_1} \cap B_R^{R'}) > 0$ by (1). Since $D' = Y_t^{j_1} \cap B_R^{R'}$ is a compact, connected, orientable surface with boundary, it is diffeomorphic to a disk. The boundary $\partial D'$ of D' lies on ∂B_R since $\varepsilon > 0$ is chosen so that $Y_t^{j_1} \cap \partial B_R \neq \emptyset$. Moreover, the union of $\partial D'$ and a connected component C of the boundary of the closure $\overline{\Omega_{\lambda,i}^*}$ of $\Omega_{\lambda,i}^*$ bounds an annulus in $\overline{\Omega_{\lambda,i}^*}$ due to the property (iii). If $\overline{\Omega_{\lambda,i}^*}$ is not diffeomorphic to a disk, then the disk D' cannot exist since $H_{\lambda,i}^*$ is homeomorphic to $\Omega_{\lambda,i}^* \times [0, 1)$ and the union of $\partial D'$ and C bounds an annulus. If $\overline{\Omega_{\lambda,i}^*}$ is diffeomorphic to a disk, then the connectivity of the boundary of $\Omega_{\lambda,i}^*$ implies that $f^{-1}(t) \cap H_{\lambda,i}^*$ is also connected, which contradicts $s \geq 2$.

Now we have $\chi(Y_t^j \cap B_R^{R'}) = 0$ for any $t \in J_\lambda^*$ and $j = 1, \dots, s$. This means that all of these connected components are diffeomorphic to $S^1 \times [0, 1]$. Therefore, the relative homotopy groups $\pi_i(f^{-1}(J_\lambda^* \cup \{\lambda\}), f^{-1}(\lambda), x)$ are trivial for all $i \in \mathbb{N}$ and any base point $x \in f^{-1}(\lambda)$. Note that this conclusion holds for both of the cases $* = -$ and $* = +$. Hence, by [8, Proposition 3.3 and Theorem 1.2], for $J_\lambda = (\lambda - \varepsilon, \lambda + \varepsilon)$, the map

$$f|_{f^{-1}(J_\lambda)} : f^{-1}(J_\lambda) \rightarrow J_\lambda$$

is a Serre fibration. Then, this implies that $f|_{f^{-1}(J_\lambda)}$ is a trivial fibration by [14, Corollary 32]. Hence λ is a typical value at ∞ of f . \square

5. Typical values of polynomial maps with 2-dimensional fibers

In this section, we study polynomial maps from \mathbb{R}^n to \mathbb{R}^{n-2} . The case $n = 3$ is studied in the previous section.

Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^{n-2}$ be a polynomial map, where $n \geq 3$, and λ be a point in $F(\mathbb{R}^n) \setminus \overline{K_0}(F)$, where $\overline{K_0}(F)$ is the closure of $K_0(F)$ in \mathbb{R}^{n-2} . Let B_R be the n -dimensional ball in \mathbb{R}^n centered at $a \in \mathbb{R}^n$ and with radius $R > 0$. As shown in [8, Lemma 3.2], we can choose a sufficiently large radius $R > 0$ satisfying the following property:

- (v) each connected component Y of $F^{-1}(\lambda) \setminus \text{Int } B_R$ intersects ∂B_R transversely for any $r \geq R$; in particular, $Y \setminus \text{Int } B_r$ is diffeomorphic to $(Y \cap \partial B_r) \times [0, 1)$ for any $r \geq R$.

In particular, there is a deformation-retract from $F^{-1}(\lambda)$ to $F^{-1}(\lambda) \cap B_R$.

Theorem 17. *Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^{n-2}$ be a polynomial map, where $n \geq 3$. For $\lambda \in F(\mathbb{R}^n) \setminus \overline{K_0}(F)$, choose a radius R so that the property (v) holds. Then, λ is a typical value at ∞ of F if and only if the following are satisfied:*

- (1) *there is no vanishing component at ∞ when t tends to λ ;*
- (2) *there exists a neighborhood D of λ in \mathbb{R}^{n-2} such that, for all $t \in D$:*
 - (2a) *$F^{-1}(t) \setminus \text{Int } B_R$ has no compact, connected component;*
 - (2b) *$\chi(F^{-1}(t)) = \chi(F^{-1}(\lambda))$ holds.*

Proof. It is enough to show that if the conditions (1) and (2) are satisfied then F is a trivial fibration over some neighborhood of λ . Assume that the two conditions are satisfied. Let D be a small neighborhood of λ as in the condition (2). We can choose D small enough so that the fibers $F^{-1}(t)$ are regular and intersect ∂B_R transversely for all $t \in D$. The map $F|_{F^{-1}(D) \cap B_R} : F^{-1}(D) \cap B_R \rightarrow D$ is a trivial fibration.

By the conditions (1) and (2a), $F^{-1}(t) \setminus \text{Int } B_R$ does not have a connected component which is contractible for any $t \in D$. Hence we have $\chi(F^{-1}(t) \setminus \text{Int } B_R) \leq 0$. Then, by the condition (2b) and the property (v), we have

$$\begin{aligned}\chi(F^{-1}(\lambda)) &= \chi(F^{-1}(t)) = \chi(F^{-1}(t) \cap B_R) + \chi(F^{-1}(t) \setminus \text{Int } B_R) \\ &\leq \chi(F^{-1}(t) \cap B_R) = \chi(F^{-1}(\lambda) \cap B_R) = \chi(F^{-1}(\lambda)),\end{aligned}$$

which implies that $\chi(F^{-1}(t) \setminus \text{Int } B_R) = 0$. Here we used the fact that $F^{-1}(t) \cap \partial B_R$ is a disjoint union of circles and its Euler characteristic is 0. By the condition (2a), $F^{-1}(t) \setminus \text{Int } B_R$ is diffeomorphic to a disjoint union of a finite number of copies of $S^1 \times [0, 1)$.

The rest of the proof is the same as the last argument in the proof of Theorem 1. Since there exists a deformation-retract from $F^{-1}(t)$ to $F^{-1}(t) \cap B_R$ for each $t \in D$ and the map $F|_{F^{-1}(D) \cap B_R}$ is a trivial fibration, the relative homotopy groups $\pi_i(F^{-1}(D), F^{-1}(\lambda), x)$ are trivial for all $i \in \mathbb{N}$ and any base point $x \in F^{-1}(\lambda)$. Then, by [8, Proposition 3.3 and Theorem 1.2], the map $F|_{F^{-1}(D)}: F^{-1}(D) \rightarrow D$ is a Serre fibration and hence it is a trivial fibration by [14, Corollary 32]. Hence λ is a typical value at ∞ of F . \square

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Declaration of interests

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