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Fubini–Study forms on punctured Riemann surfaces

Formes de Fubini–Study sur des surfaces de Riemann épointées

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Abstract. In this paper we consider a punctured Riemann surface endowed with a Hermitian metric that equals the Poincaré metric near the punctures, and a holomorphic line bundle that polarizes the metric. We show that the quotient of the induced Fubini–Study forms by Kodaira maps of high tensor powers of the line bundle and the Poincaré form near the singularity grows polynomially uniformly on a neighborhood of the singularity as the tensor power tends to infinity, as an application of the method in [5].

Résumé. Dans cet article, nous considérons une surface de Riemann épointée munie d'une métrique hermitienne qui coïncide avec la métrique de Poincaré près des points de ponction, ainsi qu'un fibré en droites holomorphe qui polarise la métrique. Nous montrons que le quotient des formes induites de Fubini–Study par les applications de Kodaira des puissances tensorielles élevées du fibré en droites et de la forme de Poincaré près de la singularité croît de manière polynomiale et uniforme dans un voisinage de la singularité lorsque la puissance tensorielle tend vers l'infini, en application de la méthode décrite dans [5].

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1. Introduction

In this paper we study the asymptotics of the induced Fubini–Study metrics by Kodaira maps of high tensor powers of a singular Hermitian line bundle over a Riemann surface under the assumption that the curvature has singularities of Poincaré type at a finite set. We show namely that the *quotient* of the induced Fubini–Study metrics by Kodaira maps of high tensor powers of the line bundle and the Poincaré model near the singularity grows polynomially uniformly on a neighborhood of the singularity as the tensor power tends to infinity. In [3,4], Auvray, Ma and Marinescu obtained a weighted estimate in the C^m-norm near the punctures for the difference of the global Bergman kernel and of the Bergman kernel of the Poincaré model near the singularity, uniformly in the tensor power p of the given line bundle; and an application in arithmetic geometry is the sharp uniform asymptotics of the sup-norm of automorphic (cusp) forms associated with a Fuchsian group of the first kind which was studied by Ullmo, Kramer... [10,11,15]. Moreover, in [5] they also show that the quotient of the global Bergman kernel and the Bergman kernel of the Poincaré model near the singularity tends to one up to $\mathscr{O}(p^{-\infty})$. This paper is an application of the results and the method of [4,5].

Following [17], an important application of the expansion of the Bergman kernel is the convergence of the induced Fubini–Study forms by Kodaira maps. For more works on Bergman kernels, cf. [6,7,9,14,18], and we refer the readers to the book [13] for a comprehensive study of Bergman kernels. Donaldson [8] uses the expansion of the Bergman kernel to study the relation between constant scalar curvature Kähler metrics and Chow stability. Coming to our context, an interesting problem is the relation between the existence of special complete/singular metrics and the stability of the pair (X, D) where D is a smooth divisor of a compact Kähler manifold X; see e.g. the suggestions of [16, Section 3.1.2] and [12], for the case of "asymptotically hyperbolic Kähler metrics", which naturally generalize to higher dimensions the complete metrics ω studied here.

We place ourselves in the setting of [4] which we describe now. Let $\overline{\Sigma}$ be a compact Riemann surface and let $D = \{a_1, \dots, a_N\} \subset \overline{\Sigma}$ be a finite set. We consider the punctured Riemann surface $\Sigma = \overline{\Sigma} \smallsetminus D$ and a Hermitian form ω_Σ on Σ. Let L be a holomorphic line bundle on $\overline{\Sigma}$, and let h be a singular Hermitian metric on L such that:

(α) h is smooth over Σ, and for all $j = 1, \dots, N$, there is a trivialization of L in the complex neighborhood $\overline{V_j}$ of a_j in $\overline{\Sigma}$, with associated coordinate z_j such that $|1|^2_h(z_j) = \left|\log(|z_j|^2)\right|$;

(β) there exists $\varepsilon > 0$ such that the (smooth) curvature R^L of h satisfies $iR^L \geq \varepsilon\omega_\Sigma$ over Σ and moreover, $iR^L = \omega_\Sigma$ on $V_j := \overline{V_j} \smallsetminus \{a_j\}$; in particular, $\omega_\Sigma = \omega_{\mathbb{D}^*}$ in the local coordinate z_j on V_j and (Σ, ω_Σ) is complete.

Here $\omega_{\mathbb{D}^*}$ denotes the Poincaré metric on the punctured unit disc $\mathbb{D}^* = \mathbb{D} \smallsetminus \{0\}$, where $\mathbb{D}$ is the unit disc, normalized as follows:

$$\omega_{\mathbb{D}^*} := \frac{i\,dz \wedge d\bar{z}}{|z|^2 \log^2\left(|z|^2\right)}. \tag{1}$$

For $p \geq 1$, let $h^p := h^{\otimes p}$ be the metric induced by h on $L^p\big|_\Sigma$, where $L^p := L^{\otimes p}$. We denote by $H^0_{(2)}(\Sigma, L^p)$ the space of $\boldsymbol{L}^2$-holomorphic sections of L^p relative to the metrics h^p and ω_Σ endowed with the obvious inner product. By [13, (6.2.17)], the sections from $H^0_{(2)}(\Sigma, L^p)$ extend to holomorphic sections of L^p over $\overline{\Sigma}$ which vanish on D. In particular, the dimension d_p of $H^0_{(2)}(\Sigma, L^p)$ is finite.

We denote by $B_p(\cdot)$ the Bergman kernel function of the orthogonal projection B_p from the space of $\boldsymbol{L}^2$-sections of L^p over Σ onto $H^0_{(2)}(\Sigma, L^p)$. If $\{S^p_\ell\}_{\ell=1}^{d_p}$ is an orthonormal basis of $H^0_{(2)}(\Sigma, L^p)$, then

$$B_p(x) := \sum_{\ell=1}^{d_p} \left|S^p_\ell(x)\right|^2_{h^p}. \tag{2}$$

Note that these are independent of the choice of basis (see [13, (6.1.10)]). Similarly, let $B^{\mathbb{D}^*}_p(x)$ be the Bergman kernel function of $\left(\mathbb{D}^*, \omega_{\mathbb{D}^*}, \mathbb{C}, \left|\log(|z|^2)\right|^p h_0\right)$ with h_0 the flat Hermitian metric on the trivial line bundle $\mathbb{C}$.

We fix a point $\mathbf{a} \in D$ and work in coordinates centered at $\mathbf{a}$. Let $\mathfrak{e}_L$ be the holomorphic frame of L near $\mathbf{a}$ corresponding to the trivialization in the condition (α). By the assumptions (α) and (β)

we have the following identification of the geometric data in the coordinate z on the punctured disc $\mathbb{D}^*_{4r}$ of radius $4r$ centered at $\mathbf{a}$, via the trivialization $\mathfrak{e}_L$ of L,

$$\left(\Sigma,\omega_\Sigma,L,h\right)\big|_{\mathbb{D}^*_{4r}}=\left(\mathbb{D}^*,\omega_{\mathbb{D}^*},\mathbb{C},h_{\mathbb{D}^*}=\left|\log\left(|z|^2\right)\right|\cdot h_0\right)\Big|_{\mathbb{D}^*_{4r}},\quad \text{with } 0<r<(4e)^{-1}. \tag{3}$$

In [5], Auvray, Ma and Marinescu proved the following estimates.

Theorem 1 ([5, Theorems 1.2 and 1.3]). *If $(\Sigma,\omega_\Sigma,L,h)$ fulfill conditions (α) and (β), then for all $k\in\mathbb{N}$, $l\in\mathbb{N}^*$ and $D_1,\dots,D_k\in\left\{\frac{\partial}{\partial z},\frac{\partial}{\partial\bar z}\right\}$, there exists $C>0$ such that for any $p\in\mathbb{N}^*$ we have*

$$\sup_{z\in V_1\cup\dots\cup V_N}\left|(D_1\cdots D_k)\left(\frac{B_p}{B_p^{\mathbb{D}^*}}(z)-1\right)\right|\le Cp^{-\ell}. \tag{4}$$

Let us consider the Kodaira map at level $p\ge 2$ induced by $H^0_{(2)}(\Sigma,L^p)$, which is a meromorphic map defined by

$$J_{p,(2)}\colon\Sigma\dashrightarrow\mathbb{P}\left(H^0_{(2)}(\Sigma,L^p)^*\right)\cong\mathbb{CP}^{d_p-1},\quad x\longmapsto\left\{\sigma\in H^0_{(2)}(\Sigma,L^p):\sigma(x)=0\right\}. \tag{5}$$

By [5, p. 2364], $J_{p,(2)}$ is an embedding for $p\gg 1$.

The $\boldsymbol{L}^2$-metric on $H^0_{(2)}(\Sigma,L^p)$ induces a Fubini–Study form $\omega_{\mathrm{FS},p}$ on the projective space $\mathbb{P}\left(H^0_{(2)}(\Sigma,L^p)^*\right)$. We have by [13, Theorem 5.1.3, (5.1.21)]

$$\frac{1}{p}J^*_{p,(2)}\omega_{\mathrm{FS},p}=\frac{i}{2\pi}R^L+\frac{i}{2\pi p}\partial\bar\partial\log(B_p). \tag{6}$$

From (4) and (6) (cf. [5, Theorem 4.1]), we have, as $p\to+\infty$,

$$\frac{1}{p}J^*_{p,(2)}\omega_{\mathrm{FS},p}=\frac{1}{2\pi}\omega_\Sigma+\frac{i}{2\pi p}\partial\bar\partial\log(B_p^{\mathbb{D}^*})+\mathscr{O}(p^{-\infty}), \tag{7}$$

uniformly on $V_1\cup V_2\cup\dots\cup V_N$. Note that, by condition (β), we have

$$\omega_{\mathbb{D}^*}=\omega_\Sigma=iR^L=-i\,\partial\bar\partial\log\left|\log\left(|z|^2\right)\right|\quad\text{on } V_j. \tag{8}$$

The main result of the present paper is the following estimate of the quotient of the induced Fubini–Study forms $\frac{1}{p}J^*_{p,(2)}\omega_{\mathrm{FS},p}$ by Kodaira maps of L^p and the Poincaré form $\omega_{\mathbb{D}^*}(z)$.

Theorem 2. *If $(\Sigma,\omega_\Sigma,L,h)$ fulfill conditions (α) and (β), then*

$$\sup_{z\in V_1\cup\dots\cup V_N}\left|\frac{J^*_{p,(2)}\omega_{\mathrm{FS},p}(z)}{p\,\omega_{\mathbb{D}^*}(z)}\right|=\mathscr{O}(p^3)\quad\text{as } p\to+\infty. \tag{9}$$

Let us mention that the difficulty of the estimate (9) consists in the fact that $B_p^{\mathbb{D}^*}(\cdot)$ vanishes at 0. Moreover, by [13, Theorem 6.1.1] and (6), for any $K\subset\Sigma$ compact,

$$\frac{1}{p}J^*_{p,(2)}\omega_{\mathrm{FS},p}-\frac{i}{2\pi}R^L=\mathscr{O}(p^{-1})\quad\text{on } K. \tag{10}$$

We give an important example where Theorem 2 applies. Let $\Gamma\subset\mathrm{PSL}(2,\mathbb{R})$ be a geometrically finite Fuchsian group of the first kind without elliptic elements. Then $\Sigma:=\Gamma\backslash\mathbb{H}$ can be compactified by finitely many points $D=\{a_1,\dots,a_N\}$ into a compact Riemann surface $\overline{\Sigma}$ [4, p. 956]. Let ω_Σ be the Kähler–Einstein metric on Σ induced by the Poincaré metric $\omega_{\mathbb{H}}=\frac{i\,dz\wedge d\bar z}{4|\mathrm{Im}\,z|^2}$ on the upper-half plane $\mathbb{H}$.

Let $\mathscr{S}^\Gamma_{2p}$ be the space of cusp forms (Spitzenformen) of weight $2p$ of Γ endowed with the Petersson inner product (cf. [4, p. 995]). We can form the Bergman kernel function of $\mathscr{S}^\Gamma_{2p}$ as in (2), denoted by S^Γ_p [4, p. 995], then

$$S^\Gamma_p(z)=\sum_j\left|f_j(z)\right|^2(2\,\mathrm{Im}\,z)^{2p} \tag{11}$$

with $\{f_j\}$ any orthonormal basis of $\mathscr{S}^{\Gamma}_{2p}$. For any $p > 0$ and $z \in \Sigma$, define

$$\omega_{\Sigma}^{\mathrm{Ber},p} := \frac{i}{2\pi}\partial\bar{\partial}\log\big(S_p^{\Gamma}(z)\big). \tag{12}$$

As a corollary of Theorem 2, we get the following:

Corollary 3. *Let $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ be a geometrically finite Fuchsian group of the first kind without elliptic elements and $\Sigma := \Gamma\backslash\mathbb{H}$. Then*

$$\sup_{z\in\Sigma}\left|\frac{\omega_{\Sigma}^{\mathrm{Ber},p}(z)}{p\,\omega_{\Sigma}(z)}\right| = \mathscr{O}(p^3) \quad \text{as } p \to +\infty. \tag{13}$$

Remark 4. Corollary 3 still holds if Γ has elliptic elements. In this case, the quotient $\Sigma = \Gamma\backslash\mathbb{H}$ is an orbifold, by the result of Dai–Liu–Ma [7, (5.25)], Ma–Marinescu [13, Section 5.4.3] on the Bergman kernel on orbifolds and the localization of the asymptotics of the Bergman kernel on compact part of Σ, from [4, (1.15), p. 998], we know near the orbifold points, $\mathscr{O}(p^3)$ in (13) can be estimated as $\mathscr{O}(1)$.

Remark 5. In [2, Main Theorem 1], the authors claim a stronger estimate than (13) and they prove their estimate by using a computation on classical modular forms, but unfortunately their proof is not complete. As a comparison, the difficulty in proving (13) arises from the fact that the Bergman kernel function vanishes on D, but [2, Proposition 2.3] claims that it has a uniform strictly positive lower bound, cf. [4, Corollary 3.6] and the explanation after (15).

In [1, Main Theorem 2], the authors claim a high dimension version of (13) for ball quotients, but unfortunately their proof is also not complete. The similar incomplete argument occurs in the proof of [1, Proposition 3.6].

2. Proof of Theorem 2 and Corollary 3

In this section, we establish Theorem 2 and Corollary 3 by using the techniques in [5]. Our starting point is (7), and thus we only need to work on the punctured unit disc $\mathbb{D}^*$. The key part in our proof of Theorem 2 in Section 2.1 is Lemma 6 which is established in Section 2.2: for the four terms in (26) and (27a), we get $\mathscr{O}(p^{-\infty})$. The main estimate $\mathscr{O}(p^3)$ is from (27b) as $B_p^{\mathbb{D}^*}(z)$ oscillates too violently near $|z| = e^{-p}$ (cf. [4, Section 3.2]). In Section 2.3, we explain the proof of Corollary 3.

2.1. *Proof of Theorem 2*

From [5, (2.6) and (2.7)], we have

$$B_p^{\mathbb{D}^*}(z) = \big|\log(|z|^2)\big|^p \beta_p^{\mathbb{D}^*}(z), \quad \text{for } z \in \mathbb{D}^*, \tag{14}$$

where

$$\beta_p^{\mathbb{D}^*}(z) = \sum_{l=1}^{\infty}(c_l^{(p)})^2|z|^{2l}, \quad \text{with } c_l^{(p)} = \left(\frac{l^{p-1}}{2\pi(p-2)\,!}\right)^{1/2}. \tag{15}$$

From (14) and (15), we know $\lim_{z\to 0} B_p^{\mathbb{D}^*}(z) = 0$ for any p. Thus, $B_p^{\mathbb{D}^*}$ has no strictly positive lower bound. This is the main difficulty to establish Theorem 1.

By [4, Proposition 3.3], for any $m \in \mathbb{N}$, $0 < b < 1$ and $0 < \gamma < \frac{1}{2}$, there exists $\varepsilon = \varepsilon(b,\gamma) > 0$ such that

$$\left\|B_p^{\mathbb{D}^*}(z) - \frac{p-1}{2\pi}\right\|_{C^m(\{be^{-p^{\gamma}} \le |z| < 1\},\omega_{\mathbb{D}^*})} = \mathscr{O}(e^{-\varepsilon p^{1-2\gamma}}) \quad \text{as } p \to +\infty. \tag{16}$$

After reducing to some V_j and identifying the geometric data on $\mathbb{D}^*_{4r}$ and Σ via (3), by (1) and (7), we have

$$\begin{aligned}\frac{J^*_{p,(2)}\omega_{\mathrm{FS},p}(z)}{p\,\omega_{\mathbb{D}^*}(z)}-\frac{1}{2\pi}&=\frac{i\partial\bar\partial\log\big(B_p^{\mathbb{D}^*}(z)\big)}{2\pi p\,\omega_{\mathbb{D}^*}(z)}+\mathcal{O}(p^{-\infty})\\&=\frac{1}{2\pi p}|z|^2\big(\log(|z|^2)\big)^2\frac{B_p^{\mathbb{D}^*}(z)\cdot\frac{\partial^2}{\partial z\partial\bar z}B_p^{\mathbb{D}^*}(z)-\frac{\partial}{\partial z}B_p^{\mathbb{D}^*}(z)\cdot\frac{\partial}{\partial\bar z}B_p^{\mathbb{D}^*}(z)}{\big(B_p^{\mathbb{D}^*}(z)\big)^2}+\mathcal{O}(p^{-\infty}).\end{aligned}\tag{17}$$

As $|z|\log|z|^2\frac{\partial}{\partial z}$ is an orthonormal frame of $(T^{(1,0)}\mathbb{D}^*,\omega_{\mathbb{D}^*})$, from (16) with $m=1,2$, we have

$$\begin{aligned}&\sup_{be^{-p^\gamma}\le|z|<1}|z|^2\big(\log(|z|^2)\big)^2\left|\frac{\partial^2}{\partial z\partial\bar z}B_p^{\mathbb{D}^*}(z)\right|=\mathcal{O}(e^{-\varepsilon p^{1-2\gamma}})\quad\text{as } p\to+\infty,\\&\sup_{be^{-p^\gamma}\le|z|<1}|z|\big|\log(|z|^2)\big|\left|\frac{\partial}{\partial z}B_p^{\mathbb{D}^*}(z)\right|=\mathcal{O}(e^{-\varepsilon p^{1-2\gamma}})\quad\text{as } p\to+\infty.\end{aligned}\tag{18}$$

From (16) with $m=0$, (17) and (18), we get

$$\sup_{be^{-p^\gamma}\le|z|<1}\left|\frac{J^*_{p,(2)}\omega_{\mathrm{FS},p}(z)}{p\,\omega_{\mathbb{D}^*}(z)}-\frac{1}{2\pi}\right|=\mathcal{O}(p^{-\infty}).\tag{19}$$

Thus, in order to prove Theorem 2 it suffices to show that

$$\sup_{0<|z|<be^{-p^\gamma}}\left|\frac{J^*_{p,(2)}\omega_{\mathrm{FS},p}(z)}{p\,\omega_{\mathbb{D}^*}(z)}\right|=\mathcal{O}(p^3).\tag{20}$$

From (1), (7), (8) and (14), we get

$$\frac{J^*_{p,(2)}\omega_{\mathrm{FS},p}(z)}{p\,\omega_{\mathbb{D}^*}(z)}=|z|^2\big(\log|z|^2\big)^2\cdot\frac{\frac{\partial^2}{\partial z\partial\bar z}\beta_p^{\mathbb{D}^*}(z)\cdot\beta_p^{\mathbb{D}^*}(z)-\frac{\partial}{\partial z}\beta_p^{\mathbb{D}^*}(z)\frac{\partial}{\partial\bar z}\beta_p^{\mathbb{D}^*}(z)}{2\pi p\big(\beta_p^{\mathbb{D}^*}(z)\big)^2}+\mathcal{O}(p^{-\infty}).\tag{21}$$

A key idea of [5] is that we use the precise formula (15), and only the lower degree (i.e., $l=\mathcal{O}(p)$) part of (15) plays the essential contribution in (21). We define δ_p as follows,

$$\delta_p=\left\lfloor\frac{(p-2)}{2|\log r|}\right\rfloor,\tag{22}$$

and $\lfloor x\rfloor$ is the integral part of $x\in\mathbb{R}$. Set

$$\begin{aligned}I_{1,p}&:=\sum_{l=1}^{\delta_p}\sum_{m=1}^{\delta_p}l(l-m)(c_l^{(p)})^2(c_m^{(p)})^2|z|^{2l+2m},\\I_{2,p}&:=\sum_{l=1}^{\delta_p}\sum_{m=\delta_p+1}^{\infty}l(l-m)(c_l^{(p)})^2(c_m^{(p)})^2|z|^{2l+2m},\\I_{3,p}&:=\sum_{l=\delta_p+1}^{\infty}\sum_{m=1}^{\delta_p}l(l-m)(c_l^{(p)})^2(c_m^{(p)})^2|z|^{2l+2m},\\I_{4,p}&:=\sum_{l=\delta_p+1}^{\infty}\sum_{m=\delta_p+1}^{\infty}l(l-m)(c_l^{(p)})^2(c_m^{(p)})^2|z|^{2l+2m}.\end{aligned}\tag{23}$$

From (15) and (23), we have

$$\begin{aligned}
&|z|^2\left(\frac{\partial^2}{\partial z\partial\bar z}\beta_p^{\mathbb{D}^*}(z)\cdot\beta_p^{\mathbb{D}^*}(z)-\frac{\partial}{\partial z}\beta_p^{\mathbb{D}^*}(z)\frac{\partial}{\partial\bar z}\beta_p^{\mathbb{D}^*}(z)\right)\\
&\qquad=|z|^2\sum_{l,m=1}^{\infty}l^2(c_l^{(p)})^2(c_m^{(p)})^2|z|^{2l+2m-2}-|z|^2\sum_{l,m=1}^{\infty}lm(c_l^{(p)})^2(c_m^{(p)})^2|z|^{2l+2m-2}\\
&\qquad=\sum_{l,m=1}^{\infty}l(l-m)(c_l^{(p)})^2(c_m^{(p)})^2|z|^{2l+2m}\\
&\qquad=I_{1,p}+I_{2,p}+I_{3,p}+I_{4,p}.
\end{aligned}\tag{24}$$

Thus, by (21) and (24) we get

$$\frac{J^*_{p,(2)}\omega_{\mathrm{FS},p}(z)}{p\,\omega_{\mathbb{D}^*}(z)}=\left(\log|z|^2\right)^2\cdot\frac{I_{1,p}+I_{2,p}+I_{3,p}+I_{4,p}}{2\pi p\left(\beta_p^{\mathbb{D}^*}(z)\right)^2}+\mathscr{O}(p^{-\infty}).\tag{25}$$

We will show the following result.

Lemma 6.

(1) *For $j=2,3,4$, we have*

$$\sup_{0<|z|<be^{-p^\gamma}}\left|\left(\log|z|^2\right)^2\cdot\frac{I_{j,p}}{\left(\beta_p^{\mathbb{D}^*}(z)\right)^2}\right|=\mathscr{O}(p^{-\infty}).\tag{26}$$

(2) *We have*

$$\sup_{0<|z|<2e^{-p}}\left|\left(\log|z|^2\right)^2\cdot\frac{I_{1,p}}{\left(\beta_p^{\mathbb{D}^*}(z)\right)^2}\right|=\mathscr{O}(p^{-\infty}),\tag{27a}$$

$$\sup_{e^{-p}<|z|<be^{-p^\gamma}}\left|\left(\log|z|^2\right)^2\cdot\frac{I_{1,p}}{\left(\beta_p^{\mathbb{D}^*}(z)\right)^2}\right|=\mathscr{O}(p^{4}).\tag{27b}$$

How to get Theorem 2 from Lemma 6. Since $\left\{z\in\mathbb{C}:0<|z|<be^{-p^\gamma}\right\}=\left\{z\in\mathbb{C}:0<|z|<2e^{-p}\right\}\cup\left\{z\in\mathbb{C}:e^{-p}<|z|<be^{-p^\gamma}\right\}$, from (27a) and (27b), we get

$$\sup_{0<|z|<be^{-p^\gamma}}\left|\left(\log|z|^2\right)^2\cdot\frac{I_{1,p}}{\left(\beta_p^{\mathbb{D}^*}(z)\right)^2}\right|=\mathscr{O}(p^{4}).\tag{28}$$

From (25), (26) and (28), we get (20). The proof of Theorem 2 is completed.

2.2. *Proof of Lemma 6*

From (23), we have

$$|I_{1,p}|\le\sum_{m=2}^{\delta_p}|1-m|(c_1^{(p)})^2(c_m^{(p)})^2|z|^{2+2m}+\sum_{l=2}^{\delta_p}\sum_{m=2}^{\delta_p}l|l-m|(c_l^{(p)})^2(c_m^{(p)})^2|z|^{2l+2m}.\tag{29}$$

We first estimate the first term on the right hand side of (29). By (15), we have

$$(c_1^{(p)})^2|z|^2<\beta_p^{\mathbb{D}^*}(z),\tag{30}$$

and

$$(c_m^{(p)})^2=\frac{m^{p-1}}{2\pi(p-2)\,!}=\left(\frac{m}{m-1}\right)^{p-1}\cdot\frac{(m-1)^{p-1}}{2\pi(p-2)\,!}=\left(\frac{m}{m-1}\right)^{p-1}(c_{m-1}^{(p)})^2.\tag{31}$$

From (15) and (31), as $\frac{l}{l-1} \le 2$ for $l \ge 2$, we have

$$\begin{aligned}\sum_{l=2}^{\delta_p}(c_l^{(p)})^2|z|^{2(l-1)} &= \sum_{l=2}^{\delta_p}\left(\frac{l}{l-1}\right)^{p-1}(c_{l-1}^{(p)})^2|z|^{2(l-1)} \\ &\le 2^{p-1}\sum_{l=2}^{\delta_p}(c_{l-1}^{(p)})^2|z|^{2(l-1)} \le 2^{p-1}\beta_p^{\mathbb{D}^*}(z).\end{aligned} \tag{32}$$

From (22), (30) and (32), we have

$$\begin{aligned}\sum_{m=2}^{\delta_p}|1-m|(c_1^{(p)})^2(c_m^{(p)})^2|z|^{2+2m} &\le (\delta_p-1)(c_1^{(p)})^2|z|^2\sum_{m=2}^{\delta_p}(c_m^{(p)})^2|z|^{2(m-1)}\cdot|z|^2 \\ &\le (\delta_p-1)2^{p-1}|z|^2\,\beta_p^{\mathbb{D}^*}(z)^2 \\ &\le p\,2^{p-1}|z|^2\beta_p^{\mathbb{D}^*}(z)^2.\end{aligned} \tag{33}$$

Proof of (27a). For $0<|z|<2e^{-p}$, we have

$$2^{p-1}|z| < 2^p e^{-p} = \left(\frac{2}{e}\right)^p. \tag{34}$$

Thus, by (33) and (34), for $|z|<2e^{-p}$, we get

$$\sum_{m=2}^{\delta_p}|1-m|(c_1^{(p)})^2(c_m^{(p)})^2|z|^{2+2m} \le p\left(\frac{2}{e}\right)^p|z|\,\beta_p^{\mathbb{D}^*}(z)^2. \tag{35}$$

Next we estimate the second term on the right hand side of (29):

$$\begin{aligned}\sum_{l=2}^{\delta_p}\sum_{m=2}^{\delta_p}l|l-m|(c_l^{(p)})^2(c_m^{(p)})^2|z|^{2l+2m} &\le \delta_p\cdot\delta_p\sum_{l=2}^{\delta_p}\sum_{m=2}^{\delta_p}(c_l^{(p)})^2(c_m^{(p)})^2|z|^{2l+2m} \quad (\text{as } |l-m|<\delta_p) \\ &= \delta_p^2|z|^4\sum_{l=2}^{\delta_p}(c_l^{(p)})^2|z|^{2(l-1)}\cdot\sum_{m=2}^{\delta_p}(c_m^{(p)})^2|z|^{2(m-1)}.\end{aligned} \tag{36}$$

By (22), (32), (34) and (36), for $|z|<2e^{-p}$, as in (35), we get

$$\begin{aligned}\sum_{l=2}^{\delta_p}\sum_{m=2}^{\delta_p}l|l-m|(c_l^{(p)})^2(c_m^{(p)})^2|z|^{2l+2m} &\le \delta_p^2|z|^4\cdot 2^{p-1}\beta_p^{\mathbb{D}^*}(z)\cdot 2^{p-1}\beta_p^{\mathbb{D}^*}(z) \\ &\le p^2\cdot\left(\frac{2}{e}\right)^{2p}|z|^2\beta_p^{\mathbb{D}^*}(z)^2.\end{aligned} \tag{37}$$

Combining (29), (35) and (37), we obtain

$$\left|\left(\log|z|^2\right)^2\cdot\frac{I_{1,p}}{\left(\beta_p^{\mathbb{D}^*}(z)\right)^2}\right| \le p\left(\frac{2}{e}\right)^p|z|\left(\log|z|^2\right)^2 + p^2\left(\frac{2}{e}\right)^{2p}|z|^2\left(\log|z|^2\right)^2. \tag{38}$$

Since the functions $f(x)=x(\log x^2)^2$ and $g(x)=x^2(\log x^2)^2$ are bounded on the interval $]0,2e^{-p}[\,\subset\,]0,e^{-2}[$, by (38), we get that there exists $C_1>0$ such that

$$\sup_{0<|z|<2e^{-p}}\left|\left(\log|z|^2\right)^2\cdot\frac{I_{1,p}}{\left(\beta_p^{\mathbb{D}^*}(z)\right)^2}\right| \le C_1\left[p\left(\frac{2}{e}\right)^p + p^2\left(\frac{2}{e}\right)^{2p}\right]. \tag{39}$$

By (39), we get (27a).

Proof of (27b). From (15), (22) and (23), we have

$$\begin{aligned} |I_{1,p}| &\le \sum_{l=1}^{\delta_p}\sum_{m=1}^{\delta_p} l|l-m|(c_l^{(p)})^2(c_m^{(p)})^2|z|^{2l+2m} \\ &\le \delta_p\cdot\delta_p \sum_{l=1}^{\delta_p}\sum_{m=1}^{\delta_p}(c_l^{(p)})^2(c_m^{(p)})^2|z|^{2l+2m} \\ &\le p^2\beta_p^{\mathbb{D}^*}(z)^2. \end{aligned} \tag{40}$$

Therefore,

$$\left|\left(\log|z|^2\right)^2\cdot\frac{I_{1,p}}{\left(\beta_p^{\mathbb{D}^*}(z)\right)^2}\right| \le p^2\left(\log|z|^2\right)^2. \tag{41}$$

For $z\in\left\{u\in\mathbb{C}: e^{-p}<|u|<be^{-p^\gamma}\right\}$, we have $e^{-2p}<|z|^2<1$. Thus, $-2p<\log|z|^2<0$. Therefore, $\left(\log|z|^2\right)^2<4p^2$. Hence, by (41), we have

$$\sup_{e^{-p}<|z|<be^{-p^\gamma}}\left|\left(\log|z|^2\right)^2\cdot\frac{I_{1,p}}{\left(\beta_p^{\mathbb{D}^*}(z)\right)^2}\right| \le p^2\cdot 4p^2 = 4p^4. \tag{42}$$

From (42), we get (27b).

Proof of Lemma 6(1) for $j=2$. From (23), we have

$$\begin{aligned} |I_{2,p}| &\le \sum_{l=1}^{\delta_p}\sum_{m=\delta_p+1}^{\infty} l|l-m|(c_l^{(p)})^2(c_m^{(p)})^2|z|^{2l+2m} \\ &< \sum_{l=1}^{\delta_p} l(c_l^{(p)})^2|z|^{2l}\cdot\sum_{m=\delta_p+1}^{\infty} m(c_m^{(p)})^2|z|^{2m}, \end{aligned} \tag{43}$$

as $l|l-m| = l(m-l) < lm$.

From [5, (3.70)], there exists $C>0$ such that for any $s\in\mathbb{N}^*$, $p\ge 2$, we have

$$|c_s^{(p)}| \le Cp^{\frac{1}{2}}\frac{1}{(2r)^s}\frac{1}{\left(\log|2r|^2\right)^{p/2}}. \tag{44}$$

Next, we need the following.

Lemma 7 ([5, (3.60)]). *For any $\tau\in\mathbb{N}$ fixed, we have*

$$\frac{1}{\left|\log\left(|2r|^2\right)\right|^{p/2}}\left(\frac{|z|}{2r}\right)^{\delta_p-\tau+1} \le Cp^{-\frac{1}{2}}\frac{1}{2^{\alpha' p}}\beta_p^{\mathbb{D}^*}(z)^{\frac{1}{2}} \tag{45}$$

for all $p\gg 1$ and $|z|\le c'p^{-A'}$, where c' and A' are constants defined as follows:

$$A'=\frac{1}{2\alpha'},\quad \alpha'=\frac{1}{4|\log r|}\quad \textit{and}\quad c' = re^{1/2\alpha'}\left|\log\left(|2r|^2\right)\right|^{1/2\alpha'}. \tag{46}$$

Proof. For the sake of completeness, we include a proof here. By (22), for any $\tau\in\mathbb{N}$ fixed, we obtain

$$\alpha' p \le \delta_p-\tau \quad \text{for } p\gg 1. \tag{47}$$

Thus, by (46) and (47) for $\tau\in\mathbb{N}$ fixed, we have

$$\left(\frac{|z|}{2r}\right)^{2(\delta_p-\tau)/p}\frac{1}{\left|\log\left(|2r|^2\right)\right|} \le \left(\frac{|z|}{2r}\right)^{2\alpha'}\frac{1}{\left|\log\left(|2r|^2\right)\right|} \le 2^{-2\alpha'}\frac{e}{p} \tag{48}$$

for $p\gg 1$, $|z|\le c'p^{-A'}$. Recall that the Stirling formula states

$$\frac{p^p}{p!} = (2\pi p)^{-1/2}e^p\left(1+\mathscr{O}(p^{-1})\right) \quad \text{as } p\to+\infty. \tag{49}$$

By (15), (48) and (49), for any $\tau \in \mathbb{N}$ fixed, we have

$$\begin{aligned} \frac{1}{\left|\log(|2r|^2)\right|^{p/2}}\left(\frac{|z|}{2r}\right)^{\delta_p-\tau+1} &= \frac{1}{2r}\left(\left(\frac{|z|}{2r}\right)^{2(\delta_p-\tau)/p}\frac{1}{\left|\log(|2r|^2)\right|}\right)^{p/2}\left(2\pi(p-2)\,!\right)^{1/2}c_1^{(p)}|z| \\ &\leq Cp^{-\frac{1}{2}}\frac{1}{2^{\alpha' p}}\beta_p^{\mathbb{D}^*}(z)^{\frac{1}{2}} \end{aligned} \tag{50}$$

for $p \gg 1$, $|z| \leq c' p^{-A'}$. □

From now on, we assume $|z| \leq c' p^{-A'}$, $p \gg 1$. By (15) and (22) we get

$$\sum_{l=1}^{\delta_p} l(c_l^{(p)})^2|z|^{2l} \leq \delta_p \sum_{l=1}^{\delta_p}(c_l^{(p)})^2|z|^{2l} \leq \delta_p \beta_p^{\mathbb{D}^*}(z) \leq p\beta_p^{\mathbb{D}^*}(z). \tag{51}$$

On the other hand, by (44), we have

$$\begin{aligned} \sum_{m=\delta_p+1}^{\infty} m(c_m^{(p)})^2|z|^{2m} &\leq \sum_{m=\delta_p+1}^{\infty} mCp\frac{1}{(2r)^{2m}}\frac{1}{\left|\log(|2r|^2)\right|^p}|z|^{2m} \\ &= Cp\frac{1}{\left|\log(|2r|^2)\right|^p}\sum_{m=\delta_p+1}^{\infty} m\left(\frac{|z|}{2r}\right)^{2m}. \end{aligned} \tag{52}$$

Since for $0 \leq \xi < 1$,

$$\sum_{q=N+1}^{\infty} q\xi^{q-1} = \left(\sum_{q=N+1}^{\infty}\xi^q\right)' = \left(\frac{\xi^{N+1}}{1-\xi}\right)' = \frac{(N+1)\xi^N - N\xi^{N+1}}{(1-\xi)^2} \leq \frac{(N+1)\xi^N}{(1-\xi)^2},$$

we have

$$\begin{aligned} \sum_{m=\delta_p+1}^{\infty} m\left(\frac{|z|}{2r}\right)^{2m} &= \left(\frac{|z|}{2r}\right)^2 \sum_{m=\delta_p+1}^{\infty} m\left(\frac{|z|}{2r}\right)^{2(m-1)} \\ &\leq \left(\frac{|z|}{2r}\right)^2(\delta_p+1)\frac{\left(\frac{|z|}{2r}\right)^{2\delta_p}}{\left(1-\left(\frac{|z|}{2r}\right)^2\right)^2} \\ &\leq \left(\frac{|z|}{2r}\right)^2(\delta_p+1)\cdot\left(\frac{4}{3}\right)^2\left(\frac{|z|}{2r}\right)^{2\delta_p}. \end{aligned} \tag{53}$$

Therefore, by (22), (45) for $\tau = 1$, (52) and (53), we obtain

$$\begin{aligned} \sum_{m=\delta_p+1}^{\infty} m(c_m^{(p)})^2|z|^{2m} &\leq \left(\frac{4}{3}\right)^2\left(\frac{|z|}{2r}\right)^2(\delta_p+1)\cdot Cp\frac{1}{\left|\log(|2r|^2)\right|^p}\left(\frac{|z|}{2r}\right)^{2\delta_p} \\ &\leq \left(\frac{4}{3}\right)^2\left(\frac{|z|}{2r}\right)^2(p+1)C'\frac{1}{2^{2\alpha' p}}\beta_p^{\mathbb{D}^*}(z). \end{aligned} \tag{54}$$

From (43), (51) and (54), we get

$$\left|\left(\log|z|^2\right)^2\cdot\frac{I_{2,p}}{\left(\beta_p^{\mathbb{D}^*}(z)\right)^2}\right| \leq \left(\log|z|^2\right)^2\left(\frac{|z|}{2r}\right)^2\left(\frac{4}{3}\right)^2 C'p(p+1)\frac{1}{2^{2\alpha' p}}. \tag{55}$$

Since the function $f(x) = x^2(\log x^2)^2$ is bounded on the interval $]0,1[$, from (55), we get

$$\sup_{0<|z|\leq c'p^{-A'}}\left|\left(\log|z|^2\right)^2\cdot\frac{I_{2,p}}{\left(\beta_p^{\mathbb{D}^*}(z)\right)^2}\right| \leq C''p^2\frac{1}{2^{2\alpha' p}}. \tag{56}$$

Since $c'p^{-A'} > be^{-p^\gamma}$ for $p \gg 1$, we get (26) for $j = 2$ from (56).

Proof of Lemma 6(1) for $j=3$. We assume $|z| \le c' p^{-A'}$, $p \gg 1$ as in Lemma 7. From (23), we have

$$
\begin{aligned}
|I_{3,p}| &\le \sum_{l=\delta_p+1}^{\infty} \sum_{m=1}^{\delta_p} l|l-m|(c_l^{(p)})^2 (c_m^{(p)})^2 |z|^{2l+2m} \\
&\le \sum_{l=\delta_p+1}^{\infty} l^2 (c_l^{(p)})^2 |z|^{2l} \cdot \sum_{m=1}^{\delta_p} (c_m^{(p)})^2 |z|^{2m},
\end{aligned}
\tag{57}
$$

as $l|l-m| = l(l-m) < l^2$.

From (44), we have

$$
\begin{aligned}
\sum_{l=\delta_p+1}^{\infty} l^2 (c_l^{(p)})^2 |z|^{2l} &\le \sum_{l=\delta_p+1}^{\infty} l^2 C p \frac{1}{(2r)^{2l}} \frac{1}{\left|\log(|2r|^2)\right|^p} |z|^{2l} \\
&= Cp \frac{1}{\left|\log(|2r|^2)\right|^p} \sum_{l=\delta_p+1}^{\infty} l^2 \left(\frac{|z|}{2r}\right)^{2l}.
\end{aligned}
\tag{58}
$$

By a simple computation, we get for $0 \le \xi < 1$,

$$
\begin{aligned}
\sum_{q=N+1}^{\infty} q(q+1)\xi^{q-1} &= \left(\sum_{q=N+1}^{\infty} \xi^{q+1}\right)'' \\
&= \left(\frac{\xi^{N+2}}{1-\xi}\right)'' \\
&= \frac{(N+2)(N+1)\xi^N - 2N(N+2)\xi^{N+1} + N(N+1)\xi^{N+2}}{(1-\xi)^3} \\
&\le \frac{(N+2)(N+1)\xi^N}{(1-\xi)^3}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\sum_{l=\delta_p+1}^{\infty} l^2 \left(\frac{|z|}{2r}\right)^{2l} &\le \left(\frac{|z|}{2r}\right)^2 \sum_{l=\delta_p+1}^{\infty} l(l+1) \left(\frac{|z|}{2r}\right)^{2(l-1)} \\
&\le \left(\frac{|z|}{2r}\right)^2 (\delta_p+2)(\delta_p+1) \frac{\left(\frac{|z|}{2r}\right)^{2\delta_p}}{\left(1-\left(\frac{|z|}{2r}\right)^2\right)^3} \\
&\le \left(\frac{|z|}{2r}\right)^2 (\delta_p+2)(\delta_p+1) \cdot \left(\frac{4}{3}\right)^3 \left(\frac{|z|}{2r}\right)^{2\delta_p}.
\end{aligned}
\tag{59}
$$

Therefore, by (58) and (59), we get

$$
\sum_{l=\delta_p+1}^{\infty} l^2 (c_l^{(p)})^2 |z|^{2l} \le C \left(\frac{|z|}{2r}\right)^2 (\delta_p+2)(\delta_p+1) \left(\frac{4}{3}\right)^3 \cdot p \frac{1}{\left|\log(|2r|^2)\right|^p} \left(\frac{|z|}{2r}\right)^{2\delta_p}.
\tag{60}
$$

By (22), (45) for $\tau = 1$ and (60), we get

$$
\begin{aligned}
\sum_{l=\delta_p+1}^{\infty} l^2 (c_l^{(p)})^2 |z|^{2l} &\le C \left(\frac{|z|}{2r}\right)^2 (\delta_p+2)(\delta_p+1) \left(\frac{4}{3}\right)^3 \cdot C^2 \frac{1}{2^{2\alpha' p}} \beta_p^{\mathbb{D}^*}(z) \\
&\le C' \left(\frac{|z|}{2r}\right)^2 (p+2)(p+1) \frac{1}{2^{2\alpha' p}} \beta_p^{\mathbb{D}^*}(z).
\end{aligned}
\tag{61}
$$

From (15), (57) and (61), we get

$$
\left| (\log|z|^2)^2 \cdot \frac{I_{3,p}}{\left(\beta_p^{\mathbb{D}^*}(z)\right)^2} \right| \le \left(\log|z|^2\right)^2 \left(\frac{|z|}{2r}\right)^2 \left[C'(p+2)(p+1) \frac{1}{2^{2\alpha' p}} \right].
\tag{62}
$$

Since the function $f(x) = x^2(\log x^2)^2$ is bounded on the interval $]0,1[$, from (62), we get

$$\sup_{0<|z|\le c' p^{-A'}} \left| \left(\log|z|^2\right)^2 \cdot \frac{I_{3,p}}{\left(\beta_p^{\mathbb{D}^*}(z)\right)^2} \right| \le C_2\, p^2 \frac{1}{2^{2\alpha' p}}. \tag{63}$$

Since $c' p^{-A'} > b e^{-p^\gamma}$ for $p \gg 1$, we get (26) for $j = 3$ from (63).

Proof of Lemma 6(1) for $j = 4$. We assume $|z| \le c' p^{-A'}$, $p \gg 1$ as in Lemma 7. By (23), we have

$$\begin{aligned} |I_{4,p}| &\le \sum_{l=\delta_p+1}^{\infty} \sum_{m=\delta_p+1}^{\infty} l|l-m|(c_l^{(p)})^2 (c_m^{(p)})^2 |z|^{2l+2m} \\ &\le \sum_{l=\delta_p+1}^{\infty} l^2 (c_l^{(p)})^2 |z|^{2l} \cdot \sum_{m=\delta_p+1}^{\infty} (c_m^{(p)})^2 |z|^{2m} \\ &\quad + \sum_{l=\delta_p+1}^{\infty} l (c_l^{(p)})^2 |z|^{2l} \cdot \sum_{m=\delta_p+1}^{\infty} m (c_m^{(p)})^2 |z|^{2m}. \end{aligned} \tag{64}$$

From (44), as in (54), we get

$$\begin{aligned} \sum_{m=\delta_p+1}^{\infty} (c_m^{(p)})^2 |z|^{2m} &\le \sum_{m=\delta_p+1}^{\infty} C p \frac{1}{(2r)^{2m}} \frac{1}{\left|\log\left(|2r|^2\right)\right|^p} |z|^{2m} \\ &\le C p \frac{1}{\left|\log\left(|2r|^2\right)\right|^p} 4 \cdot \left(\frac{|z|}{2r}\right)^{2\delta_p+2}. \end{aligned} \tag{65}$$

From (45) for $\tau = 1$ and (65), we get

$$\sum_{m=\delta_p+1}^{\infty} (c_m^{(p)})^2 |z|^{2m} \le 4C \frac{1}{2^{2\alpha' p}} \beta_p^{\mathbb{D}^*}(z) \left(\frac{|z|}{2r}\right)^2. \tag{66}$$

From (54), (61), (64) and (66), we get

$$\begin{aligned} |I_{4,p}| &\le C' \left(\frac{|z|}{2r}\right)^2 (p+2)(p+1) \frac{1}{2^{2\alpha' p}} \beta_p^{\mathbb{D}^*}(z) \cdot 4C \frac{1}{2^{2\alpha' p}} \beta_p^{\mathbb{D}^*}(z) \left(\frac{|z|}{2r}\right)^2 \\ &\quad + \left[\left(\frac{4}{3}\right)^2 C' \left(\frac{|z|}{2r}\right)^2 (p+1) \frac{1}{2^{2\alpha' p}} \beta_p^{\mathbb{D}^*}(z) \right]^2 \\ &\le C'' p^2 \frac{1}{2^{4\alpha' p}} \left(\frac{|z|}{2r}\right)^4 \beta_p^{\mathbb{D}^*}(z)^2. \end{aligned} \tag{67}$$

Thus, by (67) we obtain

$$\left| \left(\log|z|^2\right)^2 \cdot \frac{I_{4,p}}{\left(\beta_p^{\mathbb{D}^*}(z)\right)^2} \right| \le C'' p^2 \frac{1}{2^{4\alpha' p}} \left(\frac{|z|}{2r}\right)^4 \left(\log|z|^2\right)^2. \tag{68}$$

Since the function $f(x) = x^4(\log x^2)^2$ is bounded on the interval $]0,1[$, from (68), we get

$$\sup_{0<|z|\le c' p^{-A'}} \left| \left(\log|z|^2\right)^2 \cdot \frac{I_{4,p}}{\left(\beta_p^{\mathbb{D}^*}(z)\right)^2} \right| \le C''' p^2 \frac{1}{2^{4\alpha' p}}. \tag{69}$$

Since $c' p^{-A'} > b e^{-p^\gamma}$ for $p \gg 1$, we get (26) for $j = 4$ from (69).

The proof of Lemma 6 is completed.

2.3. *Proof of Corollary 3*

Let Γ be a geometrically finite Fuchsian group of the first kind, without elliptic elements, then as explained in [4, p. 956], $\Sigma := \Gamma\backslash\mathbb{H}$ can be compactified by finitely many points $D = \{a_1, \dots, a_N\}$ into a compact Riemann surface $\overline{\Sigma}$ with genus g such that the following equivalent conditions (i)–(iv) are fulfilled:

(i) $\Sigma = \overline{\Sigma} \smallsetminus D$ admits a complete Kähler–Einstein metric ω_Σ with $\mathrm{Ric}_{\omega_\Sigma} = -\omega_\Sigma$;
(ii) $2g - 2 + N > 0$;
(iii) the universal cover of Σ is the upper-half plane $\mathbb{H}$;
(iv) $L = K_{\overline{\Sigma}} \otimes \mathscr{O}_{\overline{\Sigma}}(D)$ is ample, where $K_{\overline{\Sigma}} = T^{*(1,0)}\overline{\Sigma}$ is the canonical line bundle on $\overline{\Sigma}$.

From [4, Lemma 6.2], there exists a singular Hermitian metric on L such that (Σ, ω_Σ) and the formal square root of (L, h) satisfy the conditions (α) and (β).

From [4, the proof of Lemma 6.2], ω_Σ is the Kähler–Einstein form of constant negative curvature -4, induced by the Poincaré form $\omega_{\mathbb{H}} = \frac{i\,dz\wedge d\bar{z}}{4|\mathrm{Im}\,z|^2}$ on $\mathbb{H}$, and every $a \in D$ has a coordinate neighborhood $(\overline{U}_a, z)$ in $\overline{\Sigma}$ such that in this coordinate ω_Σ is exactly given by $\omega_{\mathbb{D}^*}(z)$ on $U_a = \overline{U}_a \smallsetminus \{a\}$. Moreover, the curvature of the line bundle $\big(L|_\Sigma, h\big)$ is given by $-2i\omega_\Sigma$ and thus the curvature of the (formal) square root of (L, h) is $-i\omega_\Sigma$. Therefore, from [4, Corollary 2.4], we get that for any $k, m \in \mathbb{N}$ and any compact set $K \subset \Sigma$,

$$B_p(x) = \frac{1}{\pi}p - \frac{1}{2\pi} + \mathscr{O}(p^{-k}), \quad \text{in } C^m(K) \text{ as } p \to +\infty. \tag{70}$$

Hence, from (6), we get

$$\sup_K \left| \frac{J^*_{p,(2)}\omega_{\mathrm{FS},p}(z)}{2p\,\omega_\Sigma(z)} - \frac{1}{2\pi} \right| = \mathscr{O}(p^{-\infty}). \tag{71}$$

On the coordinate neighborhood U_a of $a \in D$, we have $\omega_\Sigma = \omega_{\mathbb{D}^*}$. Combine Theorem 2 and (71), we get

$$\sup_\Sigma \left| \frac{J^*_{p,(2)}\omega_{\mathrm{FS},p}(z)}{p\,\omega_\Sigma(z)} \right| = \mathscr{O}(p^3). \tag{72}$$

As explained in [4, (6.36)], the space $\mathscr{S}^\Gamma_{2p}$ of cusp forms (Spitzenformen) of weight $2p$ of Γ endowed with the Petersson scalar product is isometric with the space of $\boldsymbol{L}^2$ holomorphic sections of L^p over Σ (cf. (2)) with respect to the volume form ω_Σ and the metric h^p on L^p. The Bergman kernel B_p of $H^0_{(2)}(\Sigma, L^p)$ can be identified to the Bergman kernel S^Γ_p of $\mathscr{S}^\Gamma_{2p}$ (cf. [4, (6.37)]). Thus, by (6) and (12), we get

$$\omega^{\mathrm{Ber},p}_\Sigma = \frac{i}{2\pi}\partial\bar{\partial}\log\big(B_p(z)\big), \qquad \frac{1}{p}J^*_{p,(2)}\omega_{\mathrm{FS},p} = \frac{1}{\pi}\omega_\Sigma + \frac{1}{p}\omega^{\mathrm{Ber},p}_\Sigma. \tag{73}$$

From (72) and (73), we get (13). The proof of Corollary 3 is completed.

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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