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Cyclicity of composition operators on the Paley–Wiener spaces

Cyclicité des opérateurs de composition sur les espaces de Paley–Wiener

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Abstract. In this article we characterize the cyclicity of bounded composition operators $C_\phi f = f \circ \phi$ on the Paley–Wiener spaces of entire functions B^2_σ for $\sigma > 0$. We show that C_ϕ is cyclic precisely when $\phi(z) = z + b$ where either $b \in \mathbb{C} \setminus \mathbb{R}$ or $b \in \mathbb{R}$ with $0 < |b| \leq \pi/\sigma$. We also describe when the reproducing kernels of B^2_σ are cyclic vectors for C_ϕ and see that this is related to a question of completeness of exponential sequences in $L^2[-\sigma,\sigma]$. The interplay between cyclicity and complex symmetry plays a key role in this work.

Résumé. Dans cet article, nous caractérisons la cyclicité des opérateurs de composition bornée $C_\phi f = f \circ \phi$ sur les espaces de Paley–Wiener des fonctions entières B^2_σ pour $\sigma > 0$. Nous montrons que C_ϕ est cyclique précisément lorsque $\phi(z) = z + b$ où $b \in \mathbb{C} \setminus \mathbb{R}$ ou $b \in \mathbb{R}$ avec $0 < |b| \leq \pi/\sigma$. Nous décrivons également lorsque les noyaux reproducteurs de B^2_σ sont des vecteurs cycliques pour C_ϕ et voyons que cela est lié à une question de complétude des suites exponentielles dans $L^2[-\sigma,\sigma]$. L'interaction entre cyclicité et symétrie complexe joue un rôle clé dans ce travail.

Keywords. Cyclic operator, composition operator, Paley–Wiener space.

Mots-clés. Opérateur cyclique, opérateur de composition, espace Paley–Wiener.

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1. Introduction

A bounded linear operator T on a separable Hilbert space $\mathscr{H}$ is *cyclic* with *cyclic vector* $f \in \mathscr{H}$ if the linear span of the *orbit* $\mathrm{Orb}(f,T) = \{T^n f : n = 0,1,2,\ldots\}$ is dense in $\mathscr{H}$. Similarly T is *supercyclic* if all scalar multiples of elements in $\mathrm{Orb}(f,T)$ are dense in $\mathscr{H}$, and *hypercyclic* if the orbit itself is dense. Cyclicity is a central theme in linear dynamics and has been studied widely due to its connection with the Invariant Subspace Problem (ISP). See the recent monographs [1] and [10] to learn more about linear dynamics.

On the other hand a bounded linear operator T on $\mathscr{H}$ is *complex symmetric* if there exists an orthonormal basis for $\mathscr{H}$ with respect to which T has a self-transpose matrix representation. An equivalent definition also exists. A *conjugation* is a conjugate-linear operator $J\colon \mathscr{H} \to \mathscr{H}$ that satisfies the conditions:

(a) J is *isometric*: $\langle Jf, Jg\rangle = \langle g, f\rangle \ \forall\ f, g \in \mathscr{H}$;
(b) J is *involutive*: $J^2 = I$.

We say that T is *J-symmetric* if $JT = T^*J$, and complex symmetric if there exists a conjugation J with respect to which T is J-symmetric. Complex symmetric operators are generalizations of complex symmetric matrices and of normal operators, and their study was initiated by Garcia, Putinar and Wogen [6–9]. If $JT = T^*J$ for some operator T and conjugation J, then T is cyclic (supercyclic) if and only if T^* is cyclic (supercyclic). The conjugation J serves as a bijection between the cyclic vectors of T and T^*.

Let $\mathscr{S}$ be a space of functions defined on a set Ω. A *composition operator* C_ϕ on $\mathscr{S}$ with *symbol* $\phi\colon \Omega \to \Omega$ is defined as

$$C_\phi f = f \circ \phi, \quad f \in \mathscr{S}.$$

The cyclicity of composition operators has been studied on several holomorphic function spaces (see [2,5,15]) and has become an active research topic within linear dynamics. In [4], Chácon, Chácon and Giménez initiated the study of composition operators on the classical Paley–Wiener space B^2_π. They prove that C_ϕ is bounded on B^2_π precisely when

$$\phi(z) = az + b, \quad \text{where } a \in \mathbb{R} \text{ with } 0 < |a| \le 1 \text{ and } b \in \mathbb{C}. \tag{1}$$

More recently Ikeda, Ishikawa and Yoshihiro [12] show that this is true even in the general context of reproduction kernel Hilbert spaces of entire functions on $\mathbb{C}^n$.

The cyclicity and complex symmetry of C_ϕ on the Hardy–Hilbert space $H^2(\mathbb{C}_+)$ of the half-plane were characterized by Noor and Severiano [15] for affine symbols ϕ. In this article we study the cyclicity and complex symmetry of composition operators on the *Paley–Wiener spaces* B^2_σ for all $\sigma > 0$. From results in [12] it follows that the only bounded composition operators C_ϕ on B^2_σ for $\sigma > 0$ are those induced by symbols of the form (1). The Paley–Wiener spaces B^2_σ are isometrically embedded into $H^2(\mathbb{C}_+)$ as so-called *model subspaces* K_Θ (see [13, p. 305]) defined as

$$K_\Theta := H^2(\mathbb{C}_+) \ominus \Theta H^2(\mathbb{C}_+) \quad \text{where } \Theta(z) = e^{i\sigma z}.$$

Therefore one can view this work as the study of C_ϕ on some model subspaces of $H^2(\mathbb{C}_+)$. In contrast with $H^2(\mathbb{C}_+)$, our results show the existence of non-normal complex symmetric C_ϕ (see Theorem 3), and that the cyclicity of C_ϕ and its adjoint C^*_ϕ in B^2_σ depends on $\sigma > 0$ (see Theorem 4). In particular, we show that no C_ϕ is supercyclic on any B^2_σ. Finally we characterize the reproducing kernels $(k_w)_{w\in\mathbb{C}}$ in B^2_σ that are cyclic vectors for C_ϕ and show that this is closely related to the completeness of exponential sequences $(e^{i\lambda_n t})_{n\in\mathbb{N}}$ in $L^2[-\sigma,\sigma]$ (see Theorem 5). The latter is a central question in non-harmonic Fourier analysis (see [16]). Some of the main results are summarized in the following table.

Table 1. Main results for C_ϕ on B^2_σ where $\phi(z) = az + b$.

C_ϕ cyclic	$a = 1$ with $b \in \mathbb{C}\setminus\mathbb{R}$ or $0 < \lvert b\rvert \le \pi/\sigma$, $b \in \mathbb{R}$
C^*_ϕ cyclic	$0 < \lvert a\rvert < 1$, $b \in \mathbb{C}$ or C_ϕ cyclic
C_ϕ supercyclic	never
C_ϕ complex symmetric	$a = 1$, $b \in \mathbb{C}$ or $a = -1$, $b \in \mathbb{C}$
C_ϕ normal	$a = 1$, $b \in \mathbb{C}$ or $a = -1$, $b \in \mathbb{R}$

Before we begin, it is necessary to mention an error in [4] during the computation of an adjoint formula for C_ϕ on B^2_π which then leads to an incomplete description of normal C_ϕ. This is discussed and corrected in Section 2.

2. Preliminaries

An entire function f is of *exponential type* if the inequality $\left|f(z)\right| \le Ae^{B|z|}$ holds for all $z \in \mathbb{C}$ and for some constants $A, B > 0$. The exponential type σ of f is defined as the infimum of all $B > 0$ for which this inequality holds and can be determined by

$$\sigma = \limsup_{r\to\infty} \frac{\log M_f(r)}{r} \quad \text{where } M_f(r) = \max_{|z|=r}\left|f(z)\right|.$$

2.1. *Paley–Wiener spaces* B^2_σ

For $\sigma > 0$, the *Paley–Wiener space* B^2_σ consists of all entire functions of exponential type $\le \sigma$ whose restrictions to $\mathbb{R}$ belong to $L^2(\mathbb{R})$. The space B^2_σ is a reproducing kernel Hilbert space when endowed with the norm

$$\|f\|^2 = \int_{-\infty}^{\infty} \left|f(t)\right|^2 \mathrm{d}t, \quad f \in B^2_\sigma.$$

For each $w \in \mathbb{C}$, let k_w denote the reproducing kernel for B^2_σ at w defined by

$$k_w(z) = \frac{\sin\sigma(z-\overline{w})}{\pi(z-\overline{w})}, \quad z \in \mathbb{C}$$

and which satisfies the basic relation $\langle f, k_w\rangle = f(w)$ for all $f \in B^2_\sigma$. The Paley–Wiener Theorem states that any $f \in B^2_\sigma$ can be represented as the *inverse* Fourier transform

$$f(z) = (\mathfrak{F}^{-1}F)(z) := \frac{1}{\sqrt{2\pi}} \int_{-\sigma}^{\sigma} F(t)e^{izt}\,\mathrm{d}t$$

of some $F \in L^2[-\sigma,\sigma]$ where the Fourier transform $\mathfrak{F}\colon B^2_\sigma \to L^2[-\sigma,\sigma]$ is an isometric isomorphism. We denote $\widehat{f} := \mathfrak{F}f$ for $f \in B^2_\sigma$ and in particular $(\widehat{k_w})(t) = e^{-i\overline{w}t}$ for $w \in \mathbb{C}$. The monographs [3] and [16] may be consulted for more information about these spaces.

2.2. *Composition operators*

The composition operator C_ϕ is bounded on B^2_σ if and only if its inducing symbol ϕ has the form

$$\phi(z) = az + b, \quad \text{where } a \in \mathbb{R} \text{ with } 0 < |a| \le 1 \text{ and } b \in \mathbb{C} \tag{2}$$

with the usual action of C^*_ϕ on reproducing kernels determined by $C^*_\phi k_w = k_{\phi(w)}$ for $w \in \mathbb{C}$. For each $n \in \mathbb{N}$, the n-iterate of the self-map $\phi\colon \mathbb{C} \to \mathbb{C}$ is denoted by $\phi^{[n]}$ where

$$\phi^{[n]}(z) = \begin{cases} z + nb, & \text{if } a = 1, \\ a^n z + \frac{(1-a^n)}{1-a} b, & \text{if } a \ne 1, \end{cases} \tag{3}$$

and $C^n_\phi = \mathbb{C}_{\phi^{[n]}}$. We see from (3) that if $0 < |a| < 1$ and $\alpha := \frac{b}{1-a}$, then α is an attractive fixed point of ϕ, that is, $\phi^{[n]}(z) \to \alpha$ as $n \to \infty$ for all $z \in \mathbb{C}$.

2.3. *Adjoints of composition operators*

In [4] the authors show that C_ϕ and C_ϕ^* on B_π^2 are unitarily equivalent, via the Fourier transform $\mathfrak{F}$, to a pair of weighted composition operators $\widehat{C}_\phi$ and $\widehat{C}_\phi^*$ on $L^2[-\pi,\pi]$ respectively. For simplicity they assume $a>0$ which is sufficient for their results. However, as we shall see in the next section, the cases $a>0$ and $a<0$ lead to very distinct outcomes for the complex symmetry and cyclicity of C_ϕ. Therefore for the sake of completeness we provide a proof of their result for all $0<|a|\le 1$.

Proposition 1. *If $\phi(z)=az+b$ where $a\in\mathbb{R}$ with $0<|a|\le 1$, $b\in\mathbb{C}$, then C_ϕ on B_σ^2 is unitarily equivalent to a weighted composition operator $\widehat{C}_\phi$ on $L^2[-\sigma,\sigma]$ defined by*

$$(\widehat{C}_\phi F)(t)=\frac{1}{|a|}\chi_{(-|a|\sigma,|a|\sigma)}(t)e^{\frac{ibt}{a}}F\left(\frac{t}{a}\right)$$

where $\chi_{(c,d)}$ denotes a characteristic function. Moreover we have $(\widehat{C}_\phi^*F)(t)=\overline{e^{ibt}}F(at)$.

Proof. We first assume $0<a\le 1$. For each $f\in B_\sigma^2$, we have

$$\begin{aligned}(C_\phi f)(z)&=f(az+b)\\ &=\frac{1}{\sqrt{2\pi}}\int_{-\sigma}^{\sigma}\widehat{f}(t)e^{i(az+b)t}\,\mathrm{d}t\\ &=\frac{1}{\sqrt{2\pi}}\int_{-\sigma}^{\sigma}\widehat{f}(t)e^{iazt}e^{ibt}\,\mathrm{d}t\\ &=\frac{1}{\sqrt{2\pi}}\int_{-a\sigma}^{a\sigma}\frac{1}{a}\widehat{f}\left(\frac{s}{a}\right)e^{isz}e^{ib(\frac{s}{a})}\,\mathrm{d}s\\ &=\frac{1}{\sqrt{2\pi}}\int_{-\sigma}^{\sigma}\frac{1}{a}\chi_{(-a\sigma,a\sigma)}(t)e^{\frac{ibs}{a}}\widehat{f}\left(\frac{s}{a}\right)e^{isz}\,\mathrm{d}s\\ &=\frac{1}{\sqrt{2\pi}}\int_{-\sigma}^{\sigma}(\widehat{C}_\phi\widehat{f})(s)e^{isz}\,\mathrm{d}s\\ &=(\mathfrak{F}^{-1}\widehat{C}_\phi\mathfrak{F}f)(z)\end{aligned}$$

which gives $\mathfrak{F}C_\phi=\widehat{C}_\phi\mathfrak{F}$. For $-1\le a<0$, first consider the symbol $\eta(z)=-z$. Then the simple change of variables $s=-t$ again gives

$$(C_\eta f)(z)=f(-z)=\frac{1}{\sqrt{2\pi}}\int_{-\sigma}^{\sigma}(\widehat{C}_\eta\widehat{f})(s)e^{isz}\,\mathrm{d}s=(\mathfrak{F}^{-1}\widehat{C}_\eta\mathfrak{F}f)(z)\tag{4}$$

and hence $\mathfrak{F}C_\eta=\widehat{C}_\eta\mathfrak{F}$. Now let $\psi(z)=-az+b$ and note that $C_\phi=C_\eta C_\psi$. Therefore

$$\widehat{C}_\eta\widehat{C}_\psi=(\mathfrak{F}C_\eta\mathfrak{F}^{-1})(\mathfrak{F}C_\psi\mathfrak{F}^{-1})=\mathfrak{F}C_\phi\mathfrak{F}^{-1}$$

and to verify that indeed $\widehat{C}_\phi=\widehat{C}_\eta\widehat{C}_\psi$, one easily sees that

$$(\widehat{C}_\eta\widehat{C}_\psi F)(t)=\frac{1}{-a}e^{-\frac{ibt}{-a}}\chi_{(a\sigma,-a\sigma)}(-t)F\left(\frac{-t}{-a}\right)=(\widehat{C}_\phi F)(t)$$

for all $F\in L^2[-\sigma,\sigma]$. Therefore $\mathfrak{F}C_\phi=\widehat{C}_\phi\mathfrak{F}$ for all $-1\le a<0$ as well. Since the Fourier transform $\mathfrak{F}\colon B_\sigma^2\to L^2[-\sigma,\sigma]$ is an isometric isomorphism, the proof is complete. The adjoint formula follows by the change of variables $t=s/a$ as follows:

$$\langle\widehat{C}_\phi^*F,G\rangle=\int_{-\sigma}^{\sigma}\overline{e^{ibt}}F(at)\overline{G(t)}\,\mathrm{d}t=\int_{-|a|\sigma}^{|a|\sigma}F(s)\overline{\frac{1}{|a|}e^{\frac{ibs}{a}}G\left(\frac{s}{a}\right)}\,\mathrm{d}s=\langle F,\widehat{C}_\phi G\rangle$$

for all $F,G\in L^2[-\sigma,\sigma]$. This completes the proof of the result. □

In [4, p. 2209] it is claimed that if $\phi=az+b$ with $0<a<1$, then the adjoint of C_ϕ is

$$(C_\phi^*f)(z)=\frac{1}{a}f\left(\frac{z-\overline{b}}{a}\right),\quad f\in B_\pi^2.$$

However this cannot be correct since if f has exponential type π then $f(z/a)$ must have type $\pi/a > \pi$. Hence C_ϕ^* is not well-defined on B_π^2. This unfortunately leads to an incomplete description of normal composition operators (compare [4, Proposition 2.6] with Table 1).

3. Cyclicity and complex symmetry of C_ϕ

One of the main leitmotifs of this article is to show how results about cyclicity can lead to results about complex symmetry and vice versa. This hinges on a simple observation. If $TJ = JT^*$ for some operator T and conjugation J, then T is cyclic if and only if T^* is cyclic. The conjugation J serves as a bijection between the cyclic vectors of T and T^*. We demonstrate this by first dealing with the case when $0 < |a| < 1$.

Proposition 2. *Let $\phi(z) = az + b$ where $a \in \mathbb{R}$, $0 < |a| < 1$ and $b \in \mathbb{C}$. Then C_ϕ is not cyclic whereas C_ϕ^* is cyclic on B_σ^2 for $\sigma > 0$. Hence C_ϕ is not complex symmetric.*

Proof. We know that C_ϕ on B_σ^2 is unitarily equivalent to $\widehat{C}_\phi$ on $L^2[-\sigma,\sigma]$ by Proposition 1. So we show that $\widehat{C}_\phi$ is not cyclic. For any $F \in L^2[-\sigma,\sigma]$ we have $\widehat{C}_\phi F = \chi_{(-|a|\sigma,|a|\sigma)} G$ for some $G \in L^2[-\sigma,\sigma]$. So every element in $\operatorname{span}\{\widehat{C}_\phi^n F : n = 0,1,\ldots\}$ must have the form $cF + H$ where H vanishes ouside $(-|a|\sigma,|a|\sigma)$. In other words, the closure of this span consists only of functions that coincide with a constant multiple of F outside $(-|a|\sigma,|a|\sigma)$. This clearly makes the cyclicity of $\widehat{C}_\phi$ and hence of C_ϕ impossible. For C_ϕ^* we show that every reproducing kernel k_w is a cyclic vector. This is because $(C_\phi^*)^n k_w = C_{\phi^{[n]}}^* k_w = k_{\phi^{[n]}(w)}$ and $\phi^{[n]}(w)$ converges to the attractive fixed point $\alpha := \frac{b}{1-a}$ of ϕ. So any $f \in B_\sigma^2$ orthogonal to the orbit of k_w under C_ϕ^* must vanish on a sequence with a limit point and hence $f \equiv 0$. Therefore the span of the orbit of every k_w must be dense in B_σ^2 and hence C_ϕ^* is cyclic. It then follows that C_ϕ is not complex symmetric by the discussion above. □

By Proposition 2 we need only consider the cases $a = \pm 1$ for the following result. Note that in both cases $(J_a f)(z) = \overline{f(-a\bar{z})}$ defines a conjugation on B_σ^2 for $\sigma > 0$.

Proposition 3. *Let $\phi(z) = az + b$ where $a = \pm 1$ and $b \in \mathbb{C}$. Then C_ϕ on B_σ^2 is:*

(1) *always complex symmetric;*
(2) *normal if and only if $a = 1, b \in \mathbb{C}$ or $a = -1, b \in \mathbb{R}$;*
(3) *self-adjoint if and only if $a = 1, b \in i\mathbb{R}$ or $a = -1, b \in \mathbb{R}$;*
(4) *unitary if and only if $b \in \mathbb{R}$.*

Moreover C_ϕ is J_a-symmetric on B_σ^2.

Proof. We first note that when $a = -1$ we have $C_\phi^2 = I$ and hence C_ϕ is complex symmetric since every operator that is algebraic of order 2 is complex symmetric by [6, Theorem 2]. In general by Proposition 1 we have

$$(\widehat{C}_\phi F)(t) = e^{\frac{ibt}{a}} F(t/a) \qquad \text{and} \qquad (\widehat{C}_\phi^* F)(t) = e^{\overline{ibt}} F(at)$$

for $F \in L^2[-\sigma,\sigma]$. So when $a = 1$ we see that $\widehat{C}_\phi = M_{e^{ibt}}$ is a multiplication operator which is normal. This shows that C_ϕ is always complex symmetric and gives (1). If $a = -1$, then $\widehat{C}_\phi \widehat{C}_\phi^* = \widehat{C}_\phi^* \widehat{C}_\phi$ precisely when $e^{-ibt} e^{-\overline{ibt}} = e^{\overline{ibt}} e^{ibt}$ or when $e^{-i2\operatorname{Im}(b)t} = e^{i2\operatorname{Im}(b)t}$, and this holds only when $b \in \mathbb{R}$. This gives (2) and one easily obtains (3) and (4) similarly. Finally we show that C_ϕ is J_a-symmetric by first noting that

$$(J_a C_\phi J_a f)(z) = \overline{(C_\phi J_a f)(-a\bar{z})} = \overline{(J_a f)(-\bar{z} - a\bar{b})} = f(az - a\bar{b}) = (C_\psi f)(z)$$

where $\psi(z) = az - a\bar{b}$. We claim that $C_\psi = C_\phi^*$ and this follows by Proposition 1 because

$$(\widehat{C}_\psi F)(t) = e^{-i\bar{b}t} F(t/a) = \overline{e^{ibt}} F(at) = (\widehat{C}_\phi^* F)(t)$$

for all $F \in L^2[-\sigma,\sigma]$. Therefore $J_a C_\phi J_a = C_\phi^*$ and completes the proof of the result. □

Our main result characterizes the cyclicity of C_ϕ and C_ϕ^* simultaneously using complex symmetry, and then shows that no C_ϕ is supercyclic on B_σ^2 for all $\sigma > 0$ via normality. The latter was proved for the case $\sigma = \pi$ in [4, Theorem 2.7] using a lengthier argument.

Theorem 4. *Let $\phi(z) = az + b$ where $a \in \mathbb{R}$, $0 < |a| \le 1$ and $b \in \mathbb{C}$. Then C_ϕ on B_σ^2 is:*

(1) *cyclic if and only if $a = 1$ with $b \in \mathbb{C}\setminus\mathbb{R}$ or $0 < |b| \le \pi/\sigma$, $b \in \mathbb{R}$;*
(2) *never supercyclic.*

Moreover C_ϕ^* is cyclic if and only if $0 < |a| < 1$ or when C_ϕ is cyclic.

Proof. The case $0 < |a| < 1$ has been dealt with in Proposition 2 showing that C_ϕ is not cyclic or supercyclic. So first note that when $a = -1$ then $C_\phi^2 f = f$ and hence C_ϕ cannot be cyclic or supercyclic as the orbit of any $f \in B_\sigma^2$ contains at most two elements f and $f \circ \phi$. So let $a = 1$. Then C_ϕ is normal and normal operators are never supercyclic [11, p. 564]. Therefore this proves (2). For cyclicity we first observe that $\widehat{C}_\phi = M_{e^{ibt}}$ is a multiplication operator on $L^2[-\sigma,\sigma]$. A consequence of the spectral theory for normal operators is that a multiplication operator M_Φ on $L^2(\mu)$ (where $\Phi \in L^\infty(\mu)$ and μ is a compactly supported measure on $\mathbb{C}$) is cyclic if and only if Φ is injective on a set of full measure (see [14, Theorem 1.1 and Proposition 1.3]). Therefore we need to determine when $\Phi(t) = e^{ibt}$ is injective almost everywhere on $[-\sigma,\sigma]$. For $b \in \mathbb{C}\setminus\mathbb{R}$ we see that $\left|\Phi(t)\right| = e^{-\mathrm{Im}(b)t}$ which is clearly injective on all of $[-\sigma,\sigma]$ and hence so is Φ. For the case when $b \in \mathbb{R}$ note that Φ is $2\pi/|b|$ periodic. It follows that Φ is injective on $(-\sigma,\sigma)$ (hence a.e. on $[-\sigma,\sigma]$) precisely when the period is greater than or equal to the length of the interval, that is, precisely when

$$\frac{2\pi}{|b|} \ge 2\sigma \qquad \text{equivalently} \qquad 0 < |b| \le \frac{\pi}{\sigma}.$$

The assertion about cyclicity of C_ϕ^* follows by Proposition 2 and complex symmetry. □

Finally we ask the following question: *are any of the reproducing kernels $(k_w)_{w\in\mathbb{C}}$ cyclic vectors for C_ϕ?* The answer reveals an interesting dichotomy and a connection with the question of completeness of exponential sequences in $L^2[-\sigma,\sigma]$.

Theorem 5. *Let $\phi(z) = z + b$ where $b \in \mathbb{C}$. Then the kernels $(k_w)_{w\in\mathbb{C}}$ in B_σ^2 are:*

(1) *all cyclic vectors for C_ϕ when $b \in \mathbb{C}\setminus\mathbb{R}$ or $0 < |b| < \pi/\sigma$, $b \in \mathbb{R}$;*
(2) *never cyclic for C_ϕ when $|b| = \pi/\sigma$.*

Proof. We first consider the case when $b \in \mathbb{C}\setminus\mathbb{R}$. Note that since $\phi^{[n]}(z) = z + nb$, one easily sees that $C_\phi^n k_w = k_{w-n\bar{b}}$ for $w \in \mathbb{C}$. So any function $f \in B_\sigma^2$ orthogonal to the span of the orbit of k_w must vanish at $w_n := w - n\bar{b}$ for all $n \ge 0$. But $(w_n)_{n\ge0}$ does not satisfy the Blaschke-type condition required for zero sets of functions in B_σ^2, that is

$$\sum_{n\ge0} \frac{\left|\mathrm{Im}(w_n)\right|}{1+|w_n|^2} \simeq \sum_{n\ge0} \frac{n}{1+n^2} = \infty.$$

Hence $f \equiv 0$ and $\mathrm{span}(C_\phi^n k_w)_{n\ge0}$ is dense in B_σ^2. So now let $b \in \mathbb{R}$. Since $(\mathfrak{F}k_w)(t) = e^{-i\overline{w}t}$ for $w \in \mathbb{C}$, it is enough to show that every exponential e^{iwt} is a cyclic vector for $\widehat{C}_\phi = M_{e^{ibt}}$. This is equivalent to the completeness of the sequence $(e^{i(bn+w)t})_{n\ge0}$ in $L^2[-\sigma,\sigma]$. But this sequence is the image of $(e^{ibnt})_{n\ge0}$ under $M_{e^{iwt}}$ which has dense range in $L^2[-\sigma,\sigma]$. Therefore it is sufficient to prove the completeness of $(e^{ibnt})_{n\geq 0}$ in $L^2[-\sigma,\sigma]$. For this we use a classical result of Carleman (see [16, p. 97]): if $(\lambda_n)_{n\geq 0}$ is a sequence of positive real numbers and

$$\liminf_{n\to\infty} \frac{n}{\lambda_n} > \frac{\sigma}{\pi},$$

then $(e^{i\lambda_n t})_{n\geq 0}$ is complete in $\mathscr{C}[-\sigma,\sigma]$ (space of continuous functions). This condition is clearly satisfied for $\lambda_n := bn$ when $0 < b < \pi/\sigma$. The case $-\pi/\sigma < b < 0$ follows since now $(e^{-ibnt})_{n\geq 0}$ is complete and then conjugating. If $|b| = \pi/\sigma$, then the exponentials $(e^{ibnt})_{n\geq 0}$ are 2σ-periodic as we saw in the proof of Theorem 4. It follows that $(e^{ibnt})_{n\in\mathbb{Z}}$ is an orthogonal basis for $L^2[-\sigma,\sigma]$. So $(e^{ibnt})_{n\geq 0}$ in not complete in $L^2[-\sigma,\sigma]$ and neither is its image under the injective operator $M_{e^{iwt}}$. This completes the proof. $\square$

The last result hints at the possibility of finding deeper connections between composition operators on model subspaces K_Θ of $H^2(\mathbb{C}_+)$ and non-harmonic Fourier analysis.

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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