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Weakly elliptic damping gives sharp decay

Un amortissement faiblement elliptique mène à une décroissance optimale

Lassi Paunonen [a], Nicolas Vanspranghe [a] and Ruoyu P. T. Wang [b]

[a] Mathematics Research Centre, Faculty of Information Technology and Communication Sciences, Tampere University, P.O. Box 692, 33101 Tampere, Finland

[b] Department of Mathematics, University College London, London, WC1H 0AY, United Kingdom

E-mails: lassi.paunonen@tuni.fi, nicolas.vanspranghe@tuni.fi, ruoyu.wang@ucl.ac.uk

Abstract. We prove that weakly elliptic damping gives sharp energy decay for the abstract damped wave semigroup, where the damping is not in the functional calculus. In this case, there is no overdamping. We show applications in linearised water waves and Kelvin–Voigt damping.

Résumé. On démontre qu'un amortissement faiblement elliptique mène à une décroissance optimale de l'énergie pour le semi-groupe d'ondes amorties abstraites associé, et ce, même lorsque cet amortissement n'est pas dans le calcul fonctionnel. Dans ce cas, il n'y pas de suramortissement. On présente des applications à une équation des vagues linéarisée et à l'amortissement de Kelvin–Voigt.

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1. Introduction

1.1. *Motivation*

Let $\Delta \geq 0$ be the Laplace–Beltrami operator on a compact manifold M of dimension $d \geq 1$ without boundary. There is recent interest in the study of decay of the damped water wave equation, linearised via paradifferential diagonalisation,

$$\left(\partial_t^2 + \Delta_x^{\frac{1}{2}} + \Delta_x^{\frac{1}{4}} a(x) \Delta_x^{\frac{1}{4}} \partial_t\right) u(t,x) = 0, \tag{1}$$

$$\left(u(0,x), \partial_t u(0,x)\right) = (u_0, u_1) \in W^{\frac{1}{2},2}(M) \times L^2(M), \tag{2}$$

describes the evolution of a fluid interface in the gravity-capillary water wave system subject to an external pressure, studied in [1–5,21]. We define the energy of the solution to (1) by

$$E(u,t) = \|\partial_t u\|^2_{L^2(M)} + \|\Delta^{\frac{1}{4}} u\|^2_{L^2(M)}. \tag{3}$$

We want to understand the decay of energy, when the dissipation coefficient $a(x) \geq 0 \in L^\infty(M)$ may vanish on a set of measure zero in M. Here the damping term $\Delta_x^{\frac{1}{4}} a(x) \Delta_x^{\frac{1}{4}}$, though not relatively compact, may still have some weak elliptic properties. This motivates us to study how elliptic damping gives sharp energy decay rates in generalised semigroup setting. As a corollary, in Example 11 we prove that the energy of (1) decays exponentially when $a(x)$ degenerates fast near its zeros.

Corollary 1. *Let $d \geq 2$. Assume $a(x) \geq 0 \in L^\infty(M)$, and $\big(a(x)\big)^{-1} \in L^p(M)$ for $p \in [d,\infty)$. Then there is $C > 0$ such that*

$$E(u,t) \leq e^{-Ct} E(u,0), \tag{4}$$

uniformly in $t > 0$ and u satisfying (1)*. For $p \in (1,d)$, we have*

$$E(u,t)^{\frac{1}{2}} \leq C\langle t\rangle^{-\frac{p}{d-p}} \Big(\|u_0\|_{W^{\frac{1}{2},2}(M)} + \|u_1\|_{W^{\frac{1}{2},2}(M)} + \big\|\Delta^{\frac{1}{2}} u_0 + \Delta^{\frac{1}{4}} a(x) \Delta^{\frac{1}{2}} u_1\big\|_{L^2}\Big), \tag{5}$$

uniformly in $t > 0$ and u solving (1) *with all initial data with finite right hand side.*

1.2. *Introduction*

Let $H = H_0$ be an infinite-dimensional Hilbert space and $P\colon H_1 \to H$ be a nonnegative self-adjoint operator with compact resolvent, defined on H_1, a dense subspace of H. The operator P admits a spectral resolution and a functional calculus

$$Pu = \int_0^\infty \rho^2 \, \mathrm{d}E_\rho(u), \qquad f(P)u = \int_0^\infty f(\rho^2)\, \mathrm{d}E_\rho(u), \tag{6}$$

where E_ρ is a projection-valued measure on H and $\operatorname{supp} E_\rho \subset [0,\infty)$, and f is a Borel measurable function on $[0,\infty)$ that formally yields an operator $f(P)$. For $s \in \mathbb{R}$, define the scaling operators and the interpolation spaces via

$$\Lambda^s u = \int_0^\infty (1+\rho^2)^s \, \mathrm{d}E_\rho(u), \qquad H_s = \Lambda^{-s}(H_0). \tag{7}$$

Those operators $\Lambda^{-s}\colon H \to H_s$ are bounded from above and below, and they commute with P. For $s > 0$, H_{-s} is isomorphic to the dual space of H_s with respect to H.

Let the observation space Y be a Hilbert space. We will consider damping of the form Q^*Q, where the control operator $Q^* \in \mathscr{L}(Y, H_{-\frac{1}{2}})$ and the observation operator $Q \in \mathscr{L}(H_{\frac{1}{2}}, Y)$. Note that Q^*Q is not necessarily a bounded operator on H. We consider an abstract damped second-order evolution equation:

$$(\partial_t^2 + P + Q^*Q\partial_t)u = 0. \tag{8}$$

It can be written as a first-order evolution system:

$$\partial_t \begin{pmatrix} u \\ \partial_t u \end{pmatrix} = \mathscr{A} \begin{pmatrix} u \\ \partial_t u \end{pmatrix}, \qquad \mathscr{A} = \begin{pmatrix} 0 & 1 \\ -P & -Q^*Q \end{pmatrix}. \tag{9}$$

Here $\mathscr{A}$, defined on $\big\{(u,v) \in H_{\frac{1}{2}} \times H_{\frac{1}{2}} : Pu + Q^*Qv \in H\big\}$, generates a strongly continuous semigroup $e^{t\mathscr{A}}$ on $\mathscr{H} = H_{\frac{1}{2}} \times H$. See [13,21] for further details.

1.3. *Main results*

The goal of this note is to understand the stability, that is the norm of $(\mathscr{A} + i\lambda)^{-1}$, and thus decay, of $e^{t\mathscr{A}}$ on the energy space $\dot{\mathscr{H}} = \mathscr{H} / \operatorname{Ker}\mathscr{A}$, when Q^*Q satisfies some ellipticity conditions but is not by itself in the functional calculus of P. We define the weak ellipticity and boundedness:

Definition 2 (m-ellipticity). *Let $m(\lambda)$ be a positive continuous function on $(0,\infty)_\lambda$. We say the observation operator $Q\colon H_{\frac{1}{2}} \to Y$ is $m(\lambda)$-*elliptic*, if for some $\chi \in C^0\big([0,\infty)\big)$ with $\chi(1) > 0$, there exist $C, \lambda_0, N > 0$ such that*

$$m(\lambda)\big\|\chi(\lambda^{-2}P)u\big\|_H^2 \le C\|Qu\|_Y^2 + o\Big(\min\{m(\lambda)\lambda^{4N}, \lambda\}\Big)\|\Lambda^{-N}u\|_H^2 \tag{10}$$

uniformly for all $u \in H_{\frac{1}{2}}$ and all $\lambda \in \mathbb{R}$ with $\lambda \ge \lambda_0$.

Definition 3 (m-boundedness). *Let $m(\lambda)$ be a positive continuous function on $(0,\infty)_\lambda$. We say the observation operator $Q\colon H_{\frac{1}{2}} \to Y$ is $m(\lambda)$-*bounded*, if for some $\chi \in C^0\big([0,\infty)\big)$ with $\chi(1) > 0$, there is $C > 0$ such that*

$$\big\|(1+\lambda^{-2}P)^{-\frac{1}{2}}Q^*Q\chi(\lambda^{-2}P)u\big\|_H^2 \le Cm(\lambda)\big\|(1+\lambda^{-2}P)^{\frac{1}{2}}u\big\|_H^2, \tag{11}$$

for all $u \in H_{\frac{1}{2}}$.

Remark 4.

(1) For readers familiar with semiclassical analysis, m-ellipticity (or m-boundedness) heuristically means $\big(m(h^{-1})\big)^{-\frac{1}{2}}Q$ (or $\big(m(h^{-1})\big)^{-1}Q^*Q$) being semiclassically elliptic (or bounded) over $\{h^2\rho^2 = 1\}$, the characteristic variety of $h^2P - 1$.

(2) Consider two positive continuous functions $m_-(\lambda) \le m_+(\lambda)$ on $(0,\infty)_\lambda$. If Q is m_+-elliptic, then Q is also m_--elliptic. If Q is m_--bounded, then Q is also m_+-bounded.

(3) Consider two observation operators $Q_\pm\colon H_{\frac{1}{2}} \to Y_\pm$ such that $\|Q_-u\|_{Y_-}^2 \le C\big\|Q_+u\big\|_{Y_+}^2$ uniformly for $u \in H_{\frac{1}{2}}$. If Q_- is m-elliptic, then Q_+ is also m-elliptic.

(4) Any observation operator Q bounded from $H_{\frac{1}{2}}$ to Y is a priori λ^2-bounded.

For classically elliptic and bounded operators, the m-ellipticity and m-boundedness are easy to verify via Theorem 22 proved at the end of this note.

Example 5 (Linearised water waves). In the setting of Section 1.1, $P = \Delta$, $H = L^2(M)$, $H_s = W^{s,2}(M)$ are the Sobolev spaces of order s for $s \in \mathbb{R}$, $Q = \sqrt{a(x)}\Delta^{\frac{1}{4}}\colon H_{\frac{1}{2}} \to Y = H$. When $a(x) \in L^\infty(M)$ is bounded from above and below by positive constants, Q is classically elliptic with respect to $\Lambda^{\frac{1}{2}}$:

$$\big\|\sqrt{a(x)}\Delta^{\frac{1}{4}}u\big\|_{L^2(M)} \ge C^{-1}\|\Lambda^{\frac{1}{2}}u\|_{L^2(M)} - C\|u\|_{L^2(M)} \tag{12}$$

and Theorem 22 implies Q is λ^2-elliptic and λ^2-bounded.

Our result is bifold. The first result is that m-elliptic damping gives an upper bound for semigroup stability.

Theorem 6 (Weak ellipticity gives stability). *Let $m(\lambda)$ be a positive continuous function and let Q be $m(\lambda)$-elliptic. Then there are $C, \lambda_0 > 0$ such that*

$$\big\|(\mathscr{A} + i\lambda)^{-1}\big\|_{\mathscr{L}(\mathscr{H})} \le C\max\Big\{\tfrac{1}{m(|\lambda|)}, 1\Big\} \tag{13}$$

uniformly for all $|\lambda| \ge \lambda_0$, $\lambda \in \mathbb{R}$.

When the unique continuation hypothesis holds (that is, $(P - \lambda^2)u = 0$ implies $Qu \ne 0$ for $\lambda > 0$), and $m(\lambda)$ is chosen to be 1, or the reciprocal of a function of positive increase, for example, λ^{-s} or $e^{-s\lambda}$ for some $s \ge 0$, one can use [21, Lemma 3.10] (based on semigroup equivalence results in [7,24]), to turn the stability results into exponential, polynomial and logarithmic energy decay for $e^{t\mathscr{A}}$ respectively.

The second result is that m-bounded damping gives a lower bound for the semigroup stability, which is asymptotic to the upper bound found in Theorem 6.

Theorem 7 (Weak boundedness gives sharpness). *Let $m(\lambda)$ be positive continuous function and let Q be $m(\lambda)$-bounded. Then there exist a sequence of $\lambda_k \to \infty$ and $C > 0$,*

$$\left\|(\mathscr{A} + i\lambda_k)^{-1}\right\|_{\mathscr{L}(\mathscr{H})} \geq \frac{1}{Cm\big(|\lambda_k|\big)}. \tag{14}$$

It implies that when Q is m-bounded, it is not possible to improve the bound in Theorem 6 to $o\big(1/m\big(|\lambda|\big)\big)$. Thus, we have proved that a m-elliptic and m-bounded damping gives sharp semigroup stability. We have a handy corollary below to show energy decay.

Corollary 8 (Weak ellipticity gives decay). *Let $Q\colon H_{\frac{1}{2}} \to Y$ be bounded from below by Λ^s for some $s \in (-\infty, 0)$, that is, there exists $C > 0$ such that*

$$\|\Lambda^s u\|_H \leq C\|Qu\|_Y \tag{15}$$

uniformly for all $u \in H_{\frac{1}{2}}/\operatorname{Ker} P$. Then there is $C > 0$ such that for all $t \geq 0$,

$$\|e^{t\mathscr{A}}\mathscr{A}^{-1}\|_{\mathscr{H}\to\dot{\mathscr{H}}} \leq C\langle t\rangle^{\frac{1}{4s}}. \tag{16}$$

When (15) *holds with $s = 0$, then there is $C > 0$ such that*

$$\|e^{t\mathscr{A}}\|_{\dot{\mathscr{H}}\to\dot{\mathscr{H}}} \leq e^{-Ct}. \tag{17}$$

Remark 9 (Strong monotonicity). Corollary 8 is interpreted that damping given by Q larger than Λ^s must give at least the same decay rate as that given by Λ^s. In particular, in this case there is no overdamping, contrary to the general case of weak monotonicity discussed in [21, Section 2.2] (see also [6,14,19,20,25]).

Theorems 6, 7 and Corollary 8 constitute improvements over [15, Theorems 12.1 and 12.2], where $Q^*Q = f(P)$ for some nonnegative continuous function f. Such an assumption is often too strong and restrictive. Our results do not require Q^*Q to be in the functional calculus of P and also accommodates compact errors. The compact errors are natural in the studies of elliptic estimates, and accommodating them in our theorems allows us to obtain new stability results: see Example 15. Corollary 8 also generalises the result of [23], the authors of which studied the case $\operatorname{Ker} P = 0$ without allowing compact errors in (10). The separation of $\operatorname{Ker} P$ from $P \geq 0$ is not trivial, when Q^*Q is no longer relatively compact. Furthermore, in contrast to [15,23], we are able to assess sharpness of the decay rates even for damping outside of the functional calculus.

Here we present concrete examples where we get new results.

Example 10 (Linearised water waves, bounded from below). In the setting of Section 1.1 and Example 10, assume $a(x)$ is bounded from above and below by positive constants. Note uniformly for all $u \in W^{\frac{1}{2},2}(M)/\operatorname{Span}\{1\}$,

$$\|\Lambda^{\frac{1}{2}}u\|_{L^2(M)} \leq C\left\|\sqrt{a(x)}\Delta^{\frac{1}{4}}u\right\|_{L^2(M)}. \tag{18}$$

Corollary 8 implies for some $C > 0$,

$$E(u,t) \leq e^{-Ct}E(u,0), \tag{19}$$

for all $t > 0$ and for all u that solves (1).

Example 11 (Linearised water waves, degenerate). We now consider that in the setting of Section 1.1 that $a(x) \geq 0$ may vanish on a set of measure zero. In order to control its degree of degeneracy, we assume $\big(a(x)\big)^{-1} \in L^p(M)$ for $p \in (1,\infty)$: the larger p is, the faster $a(x)$ vanishes near its zeros. Sobolev embedding implies that $\big(a(x)\big)^{-\frac{1}{2}}\colon L^2(M) \to W^{-\frac{d}{2p},2}(M)$ is bounded (see for example, [21, Lemma 2.17]). Thus

$$\|a^{\frac{1}{2}}\Delta^{\frac{1}{4}}u\|_{L^2} \geq C^{-1}\left\|(1+\Delta)^{-\frac{d}{4p}}\Delta^{\frac{1}{4}}u\right\|_{L^2} \geq C^{-1}\left\|\Lambda^{-\frac{1}{4}(\frac{d}{p}-1)}u\right\|_{L^2} - C\left\|\Lambda^{-\frac{1}{4}(\frac{d}{p}-1)-1}u\right\|_{L^2}. \tag{20}$$

Thus $Q = a^{\frac{1}{2}}\Delta^{\frac{1}{4}}$ is classically elliptic with respect to $\Lambda^{-\frac{1}{4}(\frac{d}{p}-1)}$. Apply Theorem 22 to see Q is $\lambda^{-(\frac{d}{p}-1)}$-elliptic. Furthermore, since a only vanishes on a set of measure zero, $a^{\frac{1}{2}}\Delta^{\frac{1}{4}}u = 0$ implies $\Delta^{\frac{1}{4}}u = 0$ on M, and thus $\Delta u = 0$. This implies the unique continuation holds. Apply Theorem 6 with [21, Lemma 3.10] to see the following.

(1) When $p \in [d,\infty)$ and $p \neq 1$, there is $C > 0$ such that uniformly for all $t > 0$,
$$E(u,t) \le e^{-Ct}E(u,0), \tag{21}$$
for all u solving (1).

(2) When $p \in (1,d)$, there is $C > 0$ such that uniformly for all $t > 0$,
$$E(u,t)^{\frac{1}{2}} \le C\langle t\rangle^{-\frac{p}{d-p}}\left(\|u_0\|_{W^{\frac{1}{2},2}(M)} + \|u_1\|_{W^{\frac{1}{2},2}(M)} + \left\|\Delta^{\frac{1}{2}}u_0 + \Delta^{\frac{1}{4}}a(x)\Delta^{\frac{1}{2}}u_1\right\|_{L^2}\right) \tag{22}$$
for all u solving (1) with initial data with finite right hand side.

(3) When $p = 1$ and $d \ge 2$, the Sobolev embedding works with $\left(a(x)\right)^{-\frac{1}{2}}: L^2(M) \to W^{-\frac{d}{2p}-0,2}(M)$. For each $\epsilon > 0$, there is $C_\epsilon > 0$ such that uniformly for all $t > 0$,
$$E(u,t)^{\frac{1}{2}} \le C_\epsilon\langle t\rangle^{-\frac{p}{d-p}+\epsilon}\left(\|u_0\|_{W^{\frac{1}{2},2}(M)} + \|u_1\|_{W^{\frac{1}{2},2}(M)} + \left\|\Delta^{\frac{1}{2}}u_0 + \Delta^{\frac{1}{4}}a(x)\Delta^{\frac{1}{2}}u_1\right\|_{L^2}\right) \tag{23}$$
for all u solving (1) with initial data with finite right hand side.

(4) When $p = 1$ and $d = 1$, for any $N > 0$, there is $C_N > 0$ such that uniformly for all $t > 0$,
$$E(u,t)^{\frac{1}{2}} \le C_N\langle t\rangle^{-N}\left(\|u_0\|_{W^{\frac{1}{2},2}(M)} + \|u_1\|_{W^{\frac{1}{2},2}(M)} + \left\|\Delta^{\frac{1}{2}}u_0 + \Delta^{\frac{1}{4}}a(x)\Delta^{\frac{1}{2}}u_1\right\|_{L^2}\right) \tag{24}$$
for all u solving (1) with initial data with finite right hand side.

As an example, consider $M = \mathbb{S}^1 = [-1,1]_x$ with endpoints identified. Consider the damping $a(x) = x^{2s}$ for $s \in \left[0,\frac{1}{2}\right)$, then $\left(a(x)\right)^{-1} \in L^{\frac{1}{2s}-0}(M) \subset L^{1+0}(M)$ and we always have exponential decay (21). Contextually, in [5], only polynomial decay has been shown with a smaller damping $Q = a^{\frac{1}{2}}$ that degenerates slowly near its zeros. See also [9,12,22] for related results on damped wave equations when the damping is allowed to vanish on a set of measure zero.

Remark 12. The exponential decay rates in Examples 10 and 11 are optimal, while the polynomial decay rates are possibly suboptimal. It may be of further interest to sharpen those rates using explicit microlocal structures on some special domains, but we choose not to go into that direction in this manuscript.

The next few examples are devoted to the damped wave equations with unbounded, Kelvin–Voigt, and pseudodifferential damping. Let M be a compact smooth manifold, $H = L^2(M)$, $P = \Delta \ge 0$ and $Q\colon W^{1,2}(M) \to Y$. The damped wave equation is
$$(\partial_t^2 + \Delta + Q^*Q\partial_t)u(t,x) = 0, \tag{25}$$
$$u(0,x) \in W^{1,2}(M), \quad \partial_t u(0,x) \in L^2(M), \tag{26}$$
where the damping operator Q will be specified in each example below. The energy of the solution is
$$E(u,t) = \|\partial_t u\|^2_{L^2(M)} + \|\nabla u\|^2_{L^2(M)}. \tag{27}$$

Example 13 (Damped wave equation). Let a function $a(x) \in L^\infty(M)$ with $\inf_{x\in M}\left|a(x)\right| > 0$, and let $Q = \sqrt{a(x)}\langle\Delta\rangle^s\colon W^{1,2}(M) \to L^2(M)$ for $s \in \left(-\infty,\frac{1}{2}\right]$. Here $\langle\cdot\rangle = \left(1+|\cdot|^2\right)^{\frac{1}{2}}$. Note that $[Q^*Q,P] \neq 0$ and hence Q^*Q is not in the functional calculus of P. Here Q is a classically elliptic operator, and Theorem 22 implies Q is λ^{4s}-elliptic. We consider the damped wave equation (25) with this Q. Corollary 8 further gives the energy decay rates.

(1) When $s \in \left[0, \frac{1}{2}\right]$, there is $C > 0$ such that uniformly for all $t > 0$,

$$E(u,t) \leq e^{-Ct} E(u,0), \tag{28}$$

for all u solving (25). This rate still holds when we replace $\langle\Delta\rangle^s$ by Δ^s.

(2) When $s \in (-\infty, 0)$, there is $C > 0$ such that uniformly for all $t > 0$,

$$E(u,t)^{\frac{1}{2}} \leq C\langle t\rangle^{\frac{1}{4s}} \left(\left\| u(0,x) \right\|_{W^{2,2}(M)} + \left\| \partial_t u(0,x) \right\|_{W^{1,2}(M)} \right) \tag{29}$$

for all u solving (25) with initial data in $W^{2,2}(M) \times W^{1,2}(M)$.

When $s \in \left[0, \frac{1}{2}\right]$ and $a(x) \in L^\infty(M)$, Theorem 22 implies Q is λ^{4s}-bounded, and Theorem 7 implies the rate in (1) are optimal. When $s \in (-\infty, 0)$, if we further impose the regularity assumption to $a(x) \in W^{-2s+0,\infty}(M)$ being $(-2s+0)$-Hölder, Theorem 22 implies Q is λ^{4s}-bounded and the rate in (2) is optimal.

Example 14 (Kelvin–Voigt damping). Let A be a bundle isomorphism on TM such that $A_x \in \mathrm{Iso}(T_x M)$, not necessarily continuous in $x \in M$. Assume

$$\operatorname{ess\,sup}_{x\in M} \|A_x\|_{\mathscr{L}(T_x M)} < \infty, \qquad \operatorname{ess\,sup}_{x\in M} \|A_x^{-1}\|_{\mathscr{L}(T_x M)} < \infty. \tag{30}$$

Then $Q = A\nabla\colon W^{1,2}(M) \to L^2(M, TM)$ is λ^2-bounded and λ^2-elliptic, where ∇ is the gradient. We consider the damped wave equation (25) with this Q. Apply Corollary 8 to see there is $C > 0$ such that for all $t > 0$,

$$E(u,t) \leq e^{-Ct} E(u,0), \tag{31}$$

for all u solving (25). This estimate is optimal. In the special case that $A_x(x, v) = \left(x, a(x)v\right)$ for $a(x) \in L^\infty(M)$ with $\operatorname{ess\,inf}_{x\in M}\left|a(x)\right| > 0$, we recovered the exponential decay for the Kelvin–Voigt damping of L^∞-regularity. Contextually, under additional C^1-regularity, it was shown in [8,11] that the same exponential rate holds when $a(x)$ may vanish on some open sets but still satisfies the geometric control condition. L^∞-regularity in this case only gives optimal polynomial decay: see [10].

Example 15 (Pseudodifferential damping). For $s \in \left(-\infty, \frac{1}{2}\right]$, consider $Q \in \Psi^{2s}(M)$, a (one-step polyhomogeneous) pseudodifferential operator of order $2s$: see [17, Definition 18.1.5]. Then Theorem 22 implies Q is λ^{4s}-bounded. If Q is classically elliptic, that is, its principal symbol $\left|\sigma_{2s}(Q)(x,\xi)\right|$ is uniformly bounded from below by $C^{-1}\langle\xi\rangle^{2s}$ on $\left\{|\xi| \geq C\right\}$ for some $C > 0$, then for any $N > 0$, we have the elliptic estimate

$$\|u\|_{W^{2s,2}(M)} \leq C\|Qu\|_{L^2(M)} + C_N\|u\|_{W^{-N,2}(M)}, \tag{32}$$

uniformly for all $u \in W^{1,2}(M)$.We consider the damped wave equation (25) with this Q. Theorem 22 implies Q is λ^{4s}-elliptic. Our Theorems 6 and 7 below imply the sharp estimate $\left\|(\mathscr{A} + i\lambda)^{-1}\right\|_{\mathscr{L}(\mathscr{H})} \leq C\max\{\lambda^{-4s}, 1\}$ for large λ. With an extra unique continuation hypothesis on Q, that is to assume, for any $\lambda \in \mathbb{R}$,

$$(\Delta - \lambda^2)u = 0, \quad Qu = 0 \quad \Longrightarrow \quad u = 0, \tag{33}$$

one can obtain the optimal results that the energy decays exponentially when $s \geq 0$, and polynomially with the rate $\langle t\rangle^{\frac{1}{4s}}$ when $s < 0$. Pseudodifferential damping models dissipation in anisotropic materials: the energy of waves is dissipated at different rates depending on the direction of propagation. The case $s = 0$ was studied in [18], and case $s \neq 0$ in [21]. See further in Theorem 22 and references [16,26].

2. Proof

Consider the semiclassical operator $P_h = h^2 P - ihQ^* Q - 1\colon H_{\frac{1}{2}} \to H_{-\frac{1}{2}}$, where $h \in (0, h_0)$ for some small $h_0 > 0$. It is Fredholm of index 0: see [21, Lemma 3.4]. In this section, we prove the upper bounds and lower bounds for P_h^{-1} in Propositions 16 and 21, respectively. We state and prove Theorem 22 at the end of this section. Throughout this section, we use the abbreviation $m = m(\lambda) = m(h^{-1})$. We start with an upper bound for P_h^{-1}.

Proposition 16 (Semiclassical resolvent estimate). *Let m be a positive continuous function on $(0,\infty)$ and let Q be m-elliptic, that is, for some $\chi \in C^0([0,\infty))$ with $\chi(1) > 0$, there exist $C, N > 0$ such that*

$$\left\|\chi(h^2 P)u\right\|^2 \le C\left(m(h^{-1})\right)^{-1} \|Qu\|_Y^2 + o\left(\min\left\{h^{-4N}, m(h^{-1})^{-1} h^{-1}\right\}\right) \|\Lambda^{-N} u\|^2, \tag{34}$$

uniformly for all $u \in H_{\frac{1}{2}}$ and all small $h > 0$. Then there is $C > 0$ such that

$$\|P_h^{-1}\|_{\mathscr{L}(H)} \le C \max\left\{\frac{h^{-1}}{m(h^{-1})}, 1\right\} \tag{35}$$

uniformly for all $h > 0$ small.

To prove Proposition 16, we need to estimate the compact error in (34).

Lemma 17 (Compact error estimates). *For $N > 0$, there exists $C > 0$ such that*

$$\|\Lambda^{-N} u\|^2 \le C h^{4N} \|u\|^2 + C\left|\|hP^{\frac{1}{2}} u\|^2 - \|u\|^2\right| \tag{36}$$

uniformly for all $u \in H_{\frac{1}{2}}$ and all small $h > 0$.

Proof. Consider that

$$\|\Lambda^{-N} u\|^2 = \int_{|h^2\rho^2 - 1| < \frac{1}{2}} (1+\rho^2)^{-2N} \langle \mathrm{d}E_\rho u, u\rangle + \int_{|h^2\rho^2-1| \ge \frac{1}{2}} (1+\rho^2)^{-2N} \langle \mathrm{d}E_\rho u, u\rangle. \tag{37}$$

We can estimate the second term

$$\int_{|h^2\rho^2-1|\ge\frac{1}{2}} (1+\rho^2)^{-2N} \langle \mathrm{d}E_\rho u, u\rangle \le 2\int_{|h^2\rho^2-1|\ge\frac{1}{2}} |h^2\rho^2 - 1| \langle \mathrm{d}E_\rho u, u\rangle \tag{38}$$

and therefore bound it by $C\left|\|hP^{\frac{1}{2}} u\|^2 - \|u\|^2\right|$. To bound the first term, we note that on $\{|h^2\rho^2 - 1| < \frac{1}{2}\}$, we have $1 + \rho^2 > 1 + \frac{1}{2}h^{-2}$ and thus $(1+\rho^2)^{-2N} < \left(1 + \frac{1}{2}h^{-2}\right)^{-2N} \le Ch^{4N}$ for small h. Thus we can estimate the first term

$$\int_{|h^2\rho^2-1|<\frac{1}{2}} (1+\rho^2)^{-2N} \langle \mathrm{d}E_\rho u, u\rangle \le Ch^{4N} \int_{|h^2\rho^2-1|<\frac{1}{2}} \langle \mathrm{d}E_\rho u, u\rangle \tag{39}$$

and bound it by $Ch^{4N}\|u\|^2$ as desired. □

The following lemma allows us to obtain a high-frequency unique continuation result.

Lemma 18 (High-frequency unique continuation). *Assume Q is m-elliptic. Then*

$$(\operatorname{Im} P_h)^\perp = \operatorname{Ker} P_h^* = \{0\} \tag{40}$$

and $P_h\colon H_{\frac{1}{2}} \to H_{-\frac{1}{2}}$ is invertible for all h small.

Proof. Assume $P_h^* u = (h^2 P + ihQ^* Q - 1)u = 0$ for some u. This implies

$$\|hP^{\frac{1}{2}} u\|^2 - \|u\|^2 = 0, \qquad \|Qu\|_Y^2 = 0, \qquad Qu = 0, \qquad (h^2 P - 1)u = 0. \tag{41}$$

We then have

$$E_{h^{-1}} u = E_{h^{-1}} h^2 P u = u, \tag{42}$$

and thus

$$\chi(h^2 P)u = \int_0^\infty \chi(h^2\rho^2)\, \mathrm{d}E_\rho u = \chi(1) u. \tag{43}$$

Now apply Lemma 17 to see (34) reduces to

$$\chi(1)^2\|u\|^2 \le o(1)\|u\|^2. \tag{44}$$

Thus uniformly for small $h>0$, this implies $u=0$ and $\operatorname{Ker} P_h^* = \{0\}$. From [21, Lemma 3.4], we know $P_h\colon H_{\frac{1}{2}} \to H_{-\frac{1}{2}}$ is Fredholm of index 0 for all h small. Thus $(\operatorname{Im} P_h)^{\perp} = \operatorname{Ker} P_h^* = \{0\}$ for all h small. □

Proof of Proposition 16. Pair $P_h u$ with u to observe that

$$\langle P_h u, u\rangle = \|hP^{\frac{1}{2}}u\|^2 - \|u\|^2 - ih\|Qu\|^2, \tag{45}$$

whose real and imaginary parts satisfy

$$\|hP^{\frac{1}{2}}u\|^2 - \|u\|^2 = \operatorname{Re}\langle P_h u, u\rangle, \qquad h\|Qu\|_Y^2 = -\operatorname{Im}\langle P_h u, u\rangle. \tag{46}$$

We can estimate them by

$$\left|\|hP^{\frac{1}{2}}u\|^2 - \|u\|^2\right| \le 4\epsilon^{-2}\|P_h u\|^2 + \epsilon^2\|u\|^2, \tag{47}$$

$$\|Qu\|_Y^2 \le 4\epsilon^{-2}h^{-2}\|P_h u\|^2 + \epsilon^2\|u\|^2. \tag{48}$$

The m-ellipticity (34) implies

$$\begin{aligned}\left\|\chi(h^2P)u\right\|^2 &\le Cm^{-1}\|Qu\|_Y^2 + e(h)\|\Lambda^{-N}u\|^2\\ &\le C\epsilon^{-2}m^{-2}h^{-2}\|P_h u\|^2 + \epsilon^2\|u\|^2 + e(h)\|\Lambda^{-N}u\|^2,\end{aligned} \tag{49}$$

where $e(h) = o\big(\min\{h^{-4N}, m(h^{-1})^{-1}h^{-1}\}\big)$ as in (34). We apply Lemma 17 to estimate the compact error:

$$e(h)\|\Lambda^{-N}u\|^2 \le o(1)\|u\|^2 + e(h)\left|\|hP^{\frac{1}{2}}u\|^2 - \|u\|^2\right| \le C\epsilon^{-2}e(h)^2\|P_h u\|^2 + \epsilon^2\|u\|, \tag{50}$$

for the last inequality of which we used a variant of (47). Noting $e(h)^2 = o(m^{-2}h^{-2})$ by assumption, we have

$$\left\|\chi(h^2P)u\right\|^2 \le C\epsilon^{-2}m^{-2}h^{-2}\|P_h u\|^2 + \epsilon^2\|u\|^2. \tag{51}$$

Now since $\left|\chi(s)\right|$ is uniformly bounded from below near $s=1$, the algebraic inequality

$$C^{-1} \le |h^2\rho^2 - 1| + \left|\chi(h^2\rho^2)\right|^2 \tag{52}$$

holds uniformly for all $h>0$, $\rho\in\mathbb{R}$. Thus

$$\|u\|^2 = \int_0^\infty \mathrm{d}\langle E_\rho u, u\rangle \le C\int_0^\infty |h^2\rho^2-1| + \left|\chi(h^2\rho^2)\right|^2 \mathrm{d}\langle E_\rho u, u\rangle \tag{53}$$

can be estimated by

$$\|u\|^2 \le C\Big(\left|\|hP^{\frac{1}{2}}u\|^2 - \|u\|^2\right| + \left\|\chi(h^2P)u\right\|^2\Big) \le C(1+m^{-2}h^{-2})\|P_h u\|^2 + \epsilon^2\|u\|^2. \tag{54}$$

We absorb the ϵ-term to observe

$$\|P_h^{-1}\|_{\mathscr{L}(H)} \le C\max\{m^{-1}h^{-1}, 1\}, \tag{55}$$

as desired. □

We now move on to prove the lower bound for P_h^{-1}. To do so, it is convenient to introduce the semiclassical interpolation spaces H_s^h. Define the semiclassical scaling operators

$$\Lambda_h^s u = \int_0^\infty (1+h^2\rho^2)^s\,\mathrm{d}E_\rho(u), \tag{56}$$

and interpolation spaces $H_s^h = \Lambda_h^{-s}(H)$ equipped with the norm $\|\cdot\|_{H_s^h} = \|\Lambda_h^s\cdot\|_H$.

Lemma 19 (Norm equivalence). *For $s \geq 0$ and any χ Borel measurable on $(0, \infty)$ whose support is away from 0, there exists $C > 0$ such that*

$$C^{-1} h^{2s} \|\Lambda^s u\| \leq \|\Lambda_h^s u\| \leq C \|\Lambda^s u\|, \tag{57}$$

$$\left\|\Lambda_h^s \chi(h^2 P) u\right\| \leq C h^{2s} \left\|\Lambda^s \chi(h^2 P) u\right\|, \tag{58}$$

$$h^{-2s} \left\|\Lambda^{-s} \chi(h^2 P) u\right\| \leq C \left\|\Lambda_h^{-s} \chi(h^2 P) u\right\|, \tag{59}$$

uniformly for all $u \in H_s$ and all $h > 0$ small.

Proof. It suffices to note the algebraic inequalities

$$C^{-1} h^{2s} (1 + \rho^2)^s \leq (1 + h^2 \rho^2)^s \leq C (1 + \rho^2)^s, \tag{60}$$

$$(1 + h^2 \rho^2)^s \chi(h^2 \rho^2) \leq C (h^2 \rho^2)^s \chi(h^2 \rho^2) \leq C h^{2s} (1 + \rho^2)^s \chi(h^2 \rho^2), \tag{61}$$

$$h^{-2s} (1 + \rho^2)^{-s} \chi(h^2 \rho^2) \leq C (h^2 + h^2 \rho^2)^{-s} \chi(h^2 \rho^2) \leq C (1 + h^2 \rho^2)^{-s} \chi(h^2 \rho^2), \tag{62}$$

for the last two lines of which we used that χ is supported away from 0. □

We will use the following lemma to compare different norms of P_h^{-1}: it is a semiclassical version of [21, Lemma 3.9].

Lemma 20 (Operator norm estimate). *Assume that $P_h \colon H_{\frac{1}{2}} \to H_{-\frac{1}{2}}$ is invertible for fixed $h > 0$. Then*

$$\|P_h^{-1}\|_{H \to H^h_{\frac{1}{2}}} \leq C \big(1 + \|P_h^{-1}\|_{\mathscr{L}(H)}\big). \tag{63}$$

Here C does not depend on h.

Proof. Given any $f \in H_{-\frac{1}{2}}$, there exists a unique $u \in H_{\frac{1}{2}}$ such that $P_h u = f$. Pair

$$\langle P_h u, u \rangle = \|h P^{\frac{1}{2}} u\|^2 - \|u\|^2 - i h \|Q u\|_Y^2, \tag{64}$$

whose real part implies

$$\|h P^{\frac{1}{2}} u\|^2 \leq C \epsilon^{-1} \|P_h u\|^2 + \|u\|^2 + \epsilon \|u\|^2. \tag{65}$$

Note that, after absorption of ϵ, we have

$$\|u\|^2_{H^h_{\frac{1}{2}}} = \|u\|^2 + \|h P^{\frac{1}{2}} u\|^2 \leq C \|P_h u\|^2 + C \|u\|^2, \tag{66}$$

that is

$$\|P_h^{-1} f\|_{H^h_{\frac{1}{2}}} \leq C \big(\|f\|_H + \|P_h^{-1} f\|_H\big), \tag{67}$$

yielding the desired operator norm bound. □

We now prove the lower bound for P_h^{-1}.

Proposition 21 (Resolvent lower bound). *Let m be a positive continuous function and let Q be m-bounded, that is, for some $\chi \in C^0\big([0, \infty)\big)$ with $\chi(1) > 0$, there is $C > 0$ such that*

$$\left\|Q^* Q \chi(h^2 P) u\right\|_{H^h_{-\frac{1}{2}}} \leq C m(h^{-1}) \|u\|_{H^h_{\frac{1}{2}}} \tag{68}$$

uniformly for all $u \in H_{\frac{1}{2}}$ and all real $h \leq C^{-1}$. Then there is $C > 0$ such that

$$\|P_h^{-1}\|_{\mathscr{L}(H)} \geq C^{-1} \big(m(h^{-1})\big)^{-1} h^{-1}, \tag{69}$$

along some sequence of $h \to 0$.

Proof. If $P_h\colon H_{\frac{1}{2}} \to H_{-\frac{1}{2}}$ fails to be invertible on a sequence of $h \to 0$, then (69) trivially holds due to [21, Lemmas 3.6 and 3.9]. Without loss of generality we assume P_h is invertible for all h small. Since P has compact resolvent, P is unbounded and has discrete spectrum. There then exists a sequence of nontrivial $u_h \in H_{\frac{1}{2}}$ as $h \to 0$, such that $(h^2P-1)u_h = 0$. Let $\chi \in C_c^\infty\big((0,\infty)\big)$ be a cutoff with $\chi(1) = 1$. Now we have $u_h = \chi(h^2P)u_h$ and

$$\|P_h^* u_h\|_{H^h_{-\frac{1}{2}}} = h\big\|Q^*Q\chi(h^2P)u_h\big\|_{H^h_{-\frac{1}{2}}} \le Cmh\|u_h\|_{H^h_{\frac{1}{2}}} \le Cmh\|u_h\|. \tag{70}$$

Recalling that $H^h_{\frac{1}{2}}$ and $H^h_{-\frac{1}{2}}$ are in duality, we then have

$$\|P_h^{-1}\|_{H\to H^h_{\frac{1}{2}}} = \big\|(P_h^*)^{-1}\big\|_{H^h_{-\frac{1}{2}}\to H} \ge C^{-1}m^{-1}h^{-1}. \tag{71}$$

Apply the norm inequality in Lemma 20 while noting m is bounded from below to see

$$\|P_h^{-1}\|_{\mathscr{L}(H)} \ge C^{-1}m^{-1}h^{-1} \tag{72}$$

as desired. □

Proof of Theorems 6 and 7. It remains to convert those resolvent estimates to upper and lower bounds for $\big\|(\mathscr{A}+i\lambda)^{-1}\big\|$. We have the following characterisation from [21, Lemma 3.7]: for any m bounded from above,

$$\|P_h^{-1}\|_{\mathscr{L}(H)} \le C\big(m(h^{-1})\big)^{-1}h^{-1} \tag{73}$$

uniformly for h small is equivalent to

$$\big\|(\mathscr{A}+i\lambda)^{-1}\big\|_{\mathscr{L}(\mathscr{H})} \le \frac{C}{m(\lambda)}, \tag{74}$$

as desired. Apply [21, Lemma 3.9] to obtain the lower bound of $(\mathscr{A}+i\lambda)^{-1}$. □

Proof of Corollary 8. Theorem 6 already gives bounds on $(\mathscr{A}+i\lambda)^{-1}$. With [21, Lemma 3.10], it remains to show that the unique continuation property holds to conclude the energy decay. That is, we want to show for any $\lambda > 0$, $(P-\lambda^2)u = 0$ implies $Qu \neq 0$. When $(P-\lambda^2)u = 0$, [21, Lemma 3.3(4)] implies $\Lambda^s u \neq 0$. Now (15) implies $Qu \neq 0$ as desired. □

We conclude the note with a lemma to conveniently turn classical ellipticity and boundedness of Q into weak ellipticity (34) and weak boundedness (68).

Theorem 22 (Classical estimates). *Let $h = \lambda^{-1}$. The following are true:*

(1) *If Q is classically elliptic with respect to Λ^s for $s \in \big(-\infty, \frac{1}{2}\big]$, that is, there are $C, N > 0$ such that $-N < s$ and*

$$\|\Lambda^s u\|^2 \le C\|Qu\|_Y^2 + C\|\Lambda^{-N}u\|^2, \tag{75}$$

for all $u \in H_{\frac{1}{2}}$, then Q is λ^{4s}-elliptic: for some $\chi \in C^\infty\big([0,\infty)\big)$ with $\chi(1) > 0$,

$$h^{-4s}\big\|\chi(h^2P)u\big\|^2 \le C\|Qu\|_Y^2 + C\|\Lambda^{-N}u\|^2. \tag{76}$$

Note here the compact error is of desired size $C = o(h^{-4s-4N}) = o(h^{-1})$.

(2) *If Q is bounded by Λ^s for $s \in \big[0, \frac{1}{2}\big]$, that is, there is $C > 0$ such that*

$$\|Qu\|_Y^2 \le \|\Lambda^s u\|_H^2 \tag{77}$$

for all $u \in H_s$. Then Q is λ^{4s}-bounded, that is,

$$\big\|\Lambda_h^{-\frac{1}{2}}Q^*Q\chi(h^2P)u\big\| \le h^{-4s}C\|\Lambda_h^{\frac{1}{2}}u\|. \tag{78}$$

(3) *If Q is bounded by Λ^s for $s \in (-\infty, 0)$, then Q is λ^{2s}-bounded. If additionally Q^*Q is* spectrally local, *in the sense that there are cutoffs $\phi, \psi \in C^\infty[0,\infty)$ with $\phi(s) \equiv 1$ near $s = 0$, $\phi(s) \equiv 1$ near $s = 1$, $\phi\psi = 0$ such that for all h small and all $u \in H$,*

$$\left\|\phi(h^2P)Q^*Q\psi(h^2P)u\right\|_H \le Ch^{-4s}\|u\|_H. \tag{79}$$

Then Q is λ^{4s}-bounded.

(4) *When $s \in (-\infty, 0)$, if Q^*Q is bounded by Λ^{2s} on H_{2s}, that is*

$$\|Q^*Qu\|_H \le C\|\Lambda^{2s}u\|_H \tag{80}$$

uniformly for $u \in H_{2s}$. Then Q^*Q is spectrally local as in (3)*, and Q is λ^{4s}-bounded.*

Proof. We again use the semiclassicalisation $h = \lambda^{-1}$.

(1). Assume Q is classically elliptic with respect to Λ^s. When $0 \le s \le \frac{1}{2}$, we have for large λ, that

$$\left\|(h^2P)^s\right\|^2 \le Ch^{4s}\|\Lambda^s u\|^2 \le Ch^{4s}\|Qu\|_Y^2 + Ch^{4s}\|\Lambda^{-N}u\|^2 \tag{81}$$

as desired. When $s < 0$, consider a nonnegative cutoff $\psi \in C_c^\infty\big((0,\infty)\big)$ with $\psi(1) = 1$. Then from Lemma 19

$$\left\|\psi(h^2P)\Lambda_h^s u\right\|^2 \le Ch^{4s}\|\Lambda^s u\|^2 \le Ch^{4s}\|Qu\|_Y^2 + Ch^{4s}\|\Lambda^{-N}u\|^2 \tag{82}$$

as desired.

(2). Assume Q is bounded by Λ^s. Note then $\Lambda^{-s}Q^* : Y \to H$ is bounded and we have

$$\|\Lambda^{-s}Q^*Qu\| \le C\|\Lambda^s u\|. \tag{83}$$

Step 2a. When $0 \le s \le \frac{1}{2}$, with Lemma 19, we have

$$\|\Lambda_h^{-\frac{1}{2}}Q^*Qu\| \le C\|\Lambda_h^{-s}Q^*Qu\| \le Ch^{-2s}\|\Lambda^{-s}Q^*Qu\| \le Ch^{-2s}\|\Lambda^s u\| \le Ch^{-4s}\|\Lambda_h^s u\|. \tag{84}$$

Step 2b. When $s < 0$, consider a nonnegative cutoff $\psi \in C_c^\infty\big((0,\infty)\big)$ with $\psi(1) = 1$. Note $Q^* : Y \to H$ is bounded and we have with Lemma 19,

$$\left\|\Lambda_h^{-\frac{1}{2}}Q^*Q\psi(h^2P)u\right\| \le C\left\|Q^*Q\psi(h^2P)u\right\| \le C\left\|\Lambda^s\psi(h^2P)u\right\| \le Ch^{-2s}\|\Lambda_h^s u\| \tag{85}$$

showing that Q is λ^{2s}-bounded as desired.

Step 2c. When $s < 0$, assume additionally there are cutoffs $\phi, \psi \in C^0\big([0,\infty)\big)$ with $\phi(s) \equiv 1$ near $s = 0$, $\psi(s) \equiv 1$ near $s = 1$, $\phi\psi = 0$ such that for all h small and all $u \in H$,

$$\left\|\phi(h^2P)Q^*Q\psi(h^2P)u\right\|_H \le Ch^{-4s}\|u\|_H. \tag{86}$$

Now consider

$$\Lambda_h^{-\frac{1}{2}}Q^*Q\psi(h^2P)u = \Lambda_h^{-\frac{1}{2}}(1-\phi)(h^2P)Q^*Q\psi(h^2P)u + \Lambda_h^{-\frac{1}{2}}\phi(h^2P)Q^*Q\psi(h^2P)u, \tag{87}$$

the second term has the size $Ch^{-4s}\|u\|$ from the assumption. We can estimate the first term. Note $1-\phi$ is supported away from 0. Apply Lemma 19 to see

$$\begin{aligned}\left\|\Lambda_h^{-\frac{1}{2}}(1-\phi)(h^2P)Q^*Q\psi(h^2P)u\right\| &\le \left\|\Lambda_h^{-s}(1-\phi)(h^2P)Q^*Q\psi(h^2P)u\right\| \\ &\le Ch^{-2s}\left\|\Lambda^{-s}(1-\phi)(h^2P)Q^*Q\psi(h^2P)u\right\| \\ &\le Ch^{-2s}\left\|Q\psi(h^2P)u\right\|_Y \\ &\le Ch^{-2s}\left\|\Lambda^s\psi(h^2P)u\right\| \\ &\le Ch^{-4s}\|u\|.\end{aligned} \tag{88}$$

We then have

$$\left\|\Lambda_h^{-\frac{1}{2}}Q^*Q\psi(h^2P)u\right\| \le Ch^{-4s}\|u\| \tag{89}$$

yielding the desired λ^{4s}-boundedness.

(3). Assume $s < 0$ and Q^*Q is bounded by Λ^{2s}. Pick any cutoffs ϕ, ψ as described in Step 2c. Then

$$\left\|\phi(h^2P)Q^*Q\psi(h^2P)u\right\| \le \left\|Q^*Q\psi(h^2P)u\right\| \le C\left\|\Lambda^{2s}\psi(h^2P)u\right\| \le Ch^{-4s}\|u\| \tag{90}$$

uniformly for all H as desired. Thus Q^*Q is spectrally local and we apply (3) to see Q is λ^{4s}-bounded. □

Remark 23. In the case $s < 0$, Theorem 22(3) may be suboptimal without the spectral locality of Q^*Q. The spectral locality (79) forbids the communication between zero sections $\{\rho = 0\}$ and semiclassical characteristics $\{\rho = h^{-1}\}$. Note that if $Q^*Q = f(P)$ is indeed within the functional calculus of P, then Q^*Q is automatically spectrally local:

$$\phi(h^2P)Q^*Q\psi(h^2P)u = \int_0^\infty \phi(\rho^2)f(\rho^2)\psi(\rho^2)\,\mathrm{d}E_\rho u = 0, \tag{91}$$

whenever $\phi\psi = 0$. When P is a self-adjoint nonnegative pseudodifferential operator of positive order on a compact smooth manifold without boundary, and Q^*Q is the multiplication by a smooth function, the spectral locality also holds with the right of (91) replaced by h^∞, when ϕ and ψ have disjoint supports: see [16, equation (E.2.5)]. In practice, it is easier to check the classical boundedness of Q^*Q and use Theorem 22(4) instead: see Examples 13, 15.

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