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A short proof of the $C^{1,1}$ regularity for the eikonal equation

Une preuve simple de la régularité $C^{1,1}$ pour l'équation eikonale

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In the memory of my PhD advisor Haïm Brezis

Abstract. We give a short and self-contained proof of the interior $C^{1,1}$ regularity of solutions $\varphi\colon \Omega \to \mathbb{R}$ to the eikonal equation $|\nabla\varphi| = 1$ in an open set $\Omega \subset \mathbb{R}^N$ in dimension $N \geq 1$ under the assumption that φ is pointwise differentiable in Ω.

Résumé. Nous présentons une preuve courte et auto-contenue de la régularité intérieure $C^{1,1}$ des solutions $\varphi\colon \Omega \to \mathbb{R}$ de l'équation eikonale $|\nabla\varphi| = 1$ dans un ouvert $\Omega \subset \mathbb{R}^N$ en toute dimension $N \geq 1$ sous l'hypothèse que φ est différentiable en tout point de Ω.

Keywords. Eikonal equation, characteristics, regularity.

Mots-clés. Équation eikonale, caractéristiques, régularité.

2020 Mathematics Subject Classification. 35F21, 35B65.

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1. Introduction

Haïm Brezis always liked short and new proofs of known results. The aim of this note is to give such a short and self-contained proof of the following result known in the theory of Hamilton–Jacobi equations.

Theorem 1. *Let $\Omega \subset \mathbb{R}^N$ be an open set in dimension $N \geq 1$ and $\varphi\colon \Omega \to \mathbb{R}$ be a pointwise differentiable solution to the eikonal equation $|\nabla\varphi| = 1$ in Ω. Then $\nabla\varphi$ is locally Lipschitz in Ω.*

The usual (standard) proof of this result is based on the following steps (see e.g. Lions [7], Cannarsa–Sinestrari [2]): first, one checks that φ (and $-\varphi$) is a viscosity solution to the eikonal equation (see [2, Definition 5.2.1]); second, one proves that φ is both semiconcave and semiconvex with linear modulus (see [2, Theorem 5.3.7]); third, one proves that φ is C^1 (see [2, Theorem 3.3.7]) and finally, that φ is locally $C^{1,1}$ in Ω (see [2, Corollary 3.3.8]).

Our approach is based on the geometry of characteristics associated to the eikonal equation. More precisely, if $x_0 \in \Omega$, we say that $X := X_{x_0}$ is a characteristic of a solution φ passing through x_0 in some time interval $t \in [-T, T]$ if

$$\begin{cases} \dot{X}(t) = \nabla\varphi\big(X(t)\big) & \text{for } t \in [-T, T], \\ X(0) = x_0. \end{cases} \tag{1}$$

Then the beautiful proof of Caffarelli–Crandall [1, Lemma 2.2] shows in a short and self-contained manner that every point $x_0 \in \Omega$ has a characteristic X_{x_0} that is a straight line along which $\nabla\varphi$ is constant and φ is affine. Finally, we give a geometric argument on the structure of characteristics yielding the locally Lipschitz regularity of $\nabla\varphi$ in Ω.

The regularity result in Theorem 1 is optimal: such solution φ of the eikonal equation is not C^2 in general (see e.g. [5, Proposition 1]). We mention that a more general regularizing effect (i.e., $\nabla\varphi$ is locally Lipschitz away from vortex point singularities) is proved under a weaker assumption $\nabla\varphi \in W^{1/p,p}$ for $p \in [1,3]$, see [3,5]. Similar results are obtained in the context of the Aviles–Giga model which can be seen as a regularization of the eikonal equation (see [4,6]).

2. Proof of the main result

The first step is to show that each point $x_0 \in \Omega$ has a characteristic $X := X_{x_0}$ that is a straight line in direction $\nabla\varphi(x_0)$. Moreover, $\nabla\varphi$ is constant while φ is affine along this characteristic. This fact yields $\varphi \in C^1(\Omega)$. In order to have a self-contained proof of Theorem 1, we repeat here the very nice argument of Caffarelli–Crandall [1, Lemma 2.2] based on a maximum type principle for the eikonal equation.

Lemma 2. *Let $\Omega \subset \mathbb{R}^N$ be an open set and $\varphi\colon \Omega \to \mathbb{R}$ be a pointwise differentiable solution of the eikonal equation $|\nabla\varphi| = 1$ in Ω. Then for every $x_0 \in \Omega$, $X(t) = x_0 + t\nabla\varphi(x_0)$ is a characteristic of* (1) *and*

$$\nabla\varphi\big(X(t)\big) = \nabla\varphi(x_0), \quad \varphi\big(X(t)\big) = \varphi(x_0) + t, \quad \forall\, t \in [-T, T],$$

for some $T > 0$. As a consequence, $\varphi \in C^1(\Omega)$.

Proof. This proof follows the lines of [1, Lemma 2.2]. Let $R > 0$ be such that $\overline{B}_R(x_0) \subset \Omega$ and consider

$$M_r = \max_{\overline{B}_r(x_0)} \varphi, \quad m_r = \min_{\overline{B}_r(x_0)} \varphi, \quad \forall\, r \in [0, R].$$

Claim 3. *$M_r = \varphi(x_0) + r$ and $m_r = \varphi(x_0) - r$ for every $r \in [0, R]$.*

Proof. For $r \in [0, R]$, we pick some maximum point $x_r^+ \in \overline{B}_r(x_0)$ such that $\varphi(x_r^+) = M_r$. First, we show that $r \in [0, R] \mapsto M_r$ is a nondecreasing 1-Lipschitz function. Indeed, for $R \geq r > \tilde{r}$, as $|x_r^+ - x_0| \leq r$, we can find a vector $e \in \mathbb{R}^N$ such that $|e| \leq r - \tilde{r}$ and $|x_r^+ + e - x_0| \leq \tilde{r}$, i.e., $x_r^+ + e \in \overline{B}_{\tilde{r}}(x_0)$; this yields

$$0 \leq M_r - M_{\tilde{r}} \leq \varphi(x_r^+) - \varphi(x_r^+ + e) \leq |e| \leq r - \tilde{r}$$

because φ is 1-Lipschitz. Second, we prove that $\frac{\mathrm{d}M_r}{\mathrm{d}r} = 1$ a.e. in $(0, R)$ because for $r \in (0, R)$ and for small $h > 0$, as $x_r^+ + h\nabla\varphi(x_r^+) \in \overline{B}_{r+h}(x_0)$, we have

$$\liminf_{h\to 0} \frac{M_{r+h} - M_r}{h} \geq \liminf_{h\to 0} \frac{\varphi\big(x_r^+ + h\nabla\varphi(x_r^+)\big) - \varphi(x_r^+)}{h} = \big|\nabla\varphi(x_r^+)\big|^2 = 1.$$

As $M_0 = \varphi(x_0)$, we conclude $M_r = \varphi(x_0) + r$. Up to changing φ in $-\varphi$, one also gets $m_r = \varphi(x_0) - r$ for $r \in [0, R]$. □

To conclude the proof of Lemma 2, pick some minimum point $x_R^- \in \overline{B}_R(x_0)$ such that $\varphi(x_R^-) = m_R = \varphi(x_0) - R$ (by Claim 3). As φ is 1-Lipschitz, we have, again by Claim 3:

$$2R = \varphi(x_R^+) - \varphi(x_R^-) \le |x_R^+ - x_R^-| \le 2R,$$

which means that $[x_R^+, x_R^-]$ is a diameter in $\overline{B}_R(x_0)$. Note that x_R^+ (resp. x_R^-) is the unique maximum (resp. minimum) of φ in $\overline{B}_R(x_0)$ because if $\tilde{x}_R^+$ is another maximum, then it has to be antipodal to x_R^-, that is, $\tilde{x}_R^+ = x_R^+$ (the same for the uniqueness of x_R^-). In particular, $e_* = \frac{x_R^+ - x_0}{R} \in \mathbb{S}^{N-1}$. Define $g\colon [-R, R] \to \mathbb{R}$ by $g(r) = \varphi(x_0 + re_*) - \varphi(x_0)$. Then g is 1-Lipschitz and $g(\pm R) = \varphi(x_R^\pm) - \varphi(x_0) = \pm R$ (by Claim 3). So $g(r) = r$ for every $r \in (-R, R)$ yielding $1 = g'(r) = e_* \cdot \nabla\varphi(x_0 + re_*)$ for every r. Thus, $\nabla\varphi(x_0 + re_*) = e_*$ for every $r \in [-R, R]$, in particular, $e_* = \nabla\varphi(x_0)$, i.e., $X(r) = x_0 + r\nabla\varphi(x_0)$ is a characteristic of (1) and

$$\nabla\varphi\big(X(r)\big) = \nabla\varphi(x_0), \quad \varphi\big(x_0 + r\nabla\varphi(x_0)\big) = \varphi(x_0) + r, \quad \forall\, r \in [-R, R].$$

In particular, the (unique) maximum and minimum of φ in $\overline{B}_R(x_0)$ are achieved at the points $x_R^\pm = x_0 \pm R\nabla\varphi(x_0)$.

It remains to prove that $\nabla\varphi$ is continuous in Ω. Indeed, let $x_n \to x_0$ in Ω and $\overline{B}_R(x_n) \subset \Omega$ for large n. Up to a subsequence, we may assume that $\nabla\varphi(x_n) \to e \in \mathbb{S}^{N-1}$. By above, we know that $\varphi\big(x_n + R\nabla\varphi(x_n)\big) = \varphi(x_n) + R$. Passing to the limit, we obtain $\varphi(x_0 + Re) = \varphi(x_0) + R$, meaning that $x_0 + Re$ is the maximum of φ in $\overline{B}_R(x_0)$. By uniqueness of the maximum point x_R^+, we conclude that $e = \nabla\varphi(x_0)$. The uniqueness of the limit e for such subsequences yield the convergence of the whole sequence $\big(\nabla\varphi(x_n)\big)_n$ to $\nabla\varphi(x_0)$. □

Proof of Theorem 1. Let B be a ball, $\overline{B} \subset \Omega$ and we consider $d \in \big(0, \frac{\mathrm{dist}(B, \partial\Omega)}{5}\big)$. We will prove the following.

Claim 4. *There exists a universal constant $C > 0$ such that*

$$\big|\nabla\varphi(x) - \nabla\varphi(y)\big| \leqslant \frac{C}{d}|x - y|, \quad \forall\, x, y \in B \text{ with } |x - y| < \frac{d}{10}.$$

Proof. Let X_x and X_y be the characteristics passing through x and y constructed in Lemma 2 (that are lines in direction $\nabla\varphi(x)$ and $\nabla\varphi(y)$). If X_x and X_y coincide inside B (in particular, $\nabla\varphi(x) = \pm\nabla\varphi(y)$ by Lemma 2), then Lemma 2 implies that $\nabla\varphi(x) = \nabla\varphi(y)$ and the claim is trivial in that case. Otherwise, $\nabla\varphi(x) \neq \pm\nabla\varphi(y)$ and X_x and X_y cannot intersect inside Ω (as $\nabla\varphi$ is continuous in Ω by Lemma 2). Let $|x - y| < \frac{d}{10}$ and x' be the projection of y on X_x. Clearly,

$$\ell = |x' - y| \le |x - y| < \frac{d}{10},$$

$\mathrm{dist}(x', \partial\Omega) > 5d - \ell$ and $\nabla\varphi(x') = \nabla\varphi(x)$. Up to changing φ in $-\varphi$, we may assume that $\varphi(x') \le \varphi(y)$ and up to an additive constant, we also may assume that $0 = \varphi(x') \le \varphi(y) = a$. As $|\nabla\varphi| = 1$ on the segment $[x'y]$, we deduce that $0 \le a = \varphi(y) - \varphi(x') \le |x' - y| = \ell$. Let z be the point on the characteristic X_y reached at time $t = d$ in direction $\nabla\varphi(y)$, i.e., $|z - y| = d$, $\nabla\varphi(y) = \frac{z - y}{d}$ and $\varphi(z) = a + d$; in particular, $z \in \Omega$. Let w be the point on the characteristic $X_{x'}$ reached at time $t = -d$ in direction $\nabla\varphi(x')$, i.e., $|x' - w| = d$, $\nabla\varphi(x') = \frac{x' - w}{d}$ and $\varphi(w) = -d$; in particular, $w \in \Omega$. We deduce that

$$|z - w| = \int_{[zw]} |\nabla\varphi|\, \mathrm{d}\mathscr{H}^1 \ge \left|\int_{[zw]} \nabla\varphi \cdot \frac{z - w}{|z - w|}\, \mathrm{d}\mathscr{H}^1\right| = \big|\varphi(z) - \varphi(w)\big| = 2d + a \ge 2d.$$

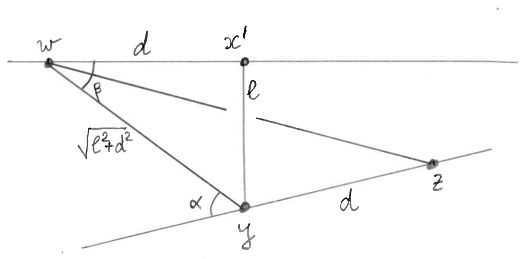

Figure 1. Geometry of characteristics X_x (containing x') and X_y.

By Pythagoras's formula in the triangle $x'yw$ we have $|y-w|^2 = \ell^2 + d^2$. Denoting α the angle between $\overrightarrow{wy}$ and $\nabla\varphi(y)$, the cosine formula in the triangle wyz yields

$$\begin{aligned}-\cos\alpha &= \frac{|y-w|^2 + |y-z|^2 - |z-w|^2}{2|y-z|\cdot|y-w|}\\ &= \frac{2d^2+\ell^2-|z-w|^2}{2d\sqrt{d^2+\ell^2}}\\ &\le -\frac{2d^2-\ell^2}{2d\sqrt{d^2+\ell^2}}\\ &= -1 + O\left(\frac{\ell^2}{d^2}\right).\end{aligned}$$

As $\alpha \in \left(0, \frac{\pi}{2}\right)$, it yields $\sin^2\alpha = 1-\cos^2\alpha \le 2(1-\cos\alpha) = O\left(\frac{\ell^2}{d^2}\right)$. So, $\sin\alpha = O\left(\frac{\ell}{d}\right)$. Denoting $\beta \in \left(0, \frac{\pi}{2}\right)$ the angle between $\overrightarrow{wy}$ and $\nabla\varphi(x)$, we compute in the triangle $x'yw$:

$$\sin\beta \le \tan\beta = \frac{|y-x'|}{|x'-w|} = \frac{\ell}{d}.$$

In particular, $0 \le \alpha+\beta \le \frac{\pi}{2}$ if $\ell < \frac{d}{10}$. Finally, denoting γ the angle between $\nabla\varphi(x)$ and $\nabla\varphi(y)$, the triangle inequality yields

$$\sin\gamma \le \sin(\alpha+\beta) \le \sin\alpha + \sin\beta = O\left(\frac{\ell}{d}\right).$$

We conclude

$$\left|\nabla\varphi(x) - \nabla\varphi(y)\right| = 2\sin\frac{\gamma}{2} = O(\sin\gamma) = O\left(\frac{\ell}{d}\right) \le \frac{C}{d}|x-y|$$

for some universal $C > 0$. □

The conclusion of Theorem 1 follows. □

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Declaration of interests

The author does not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and has declared no affiliations other than their research organizations.

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