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Uniform regularity and weight estimates for the Poisson problems when rounding off the conical points

Régularité uniforme et estimations de poids pour les problèmes de Poisson lors de l'arrondissement des points coniques

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Abstract. We establish uniform solvability estimates for the Poisson problems associated to a suitably bounded family $\{\Omega_n\}_{n\in\mathfrak{I}}$ of domains in $\mathbb{R}^d$. The main example is that of a suitable sequence of smooth domains that "converges" to a *domain with conical points* by rounding off the conical points. We give full details for the case of a straight polygonal domain approximated by a sequence of smooth domains rounding off its corners. The method of proof relies on a conformal modification of the metric, with respect to which the union of our domains becomes a manifold with boundary and relative bounded geometry.

Résumé. Nous établissons des estimées de résolubilité uniforme pour les problèmes de Poisson associés à une famille $\{\Omega_n\}_{n\in\mathfrak{I}}$ de domaines de $\mathbb{R}^d$ bornée de manière appropriée. L'exemple principal est celui d'une suite appropriée de domaines lisses qui « converge » vers un *domaine à points coniques* en arrondissant les points coniques. Nous donnons les détails complets dans le cas d'un domaine polygonal rectiligne approché par une suite de domaines lisses arrondissant ses angles. La méthode de la démonstration repose sur une modification conforme de la métrique, pour laquelle l'union de nos domaines devient une variété à bord à géométrie relativement bornée.

Keywords. Strongly elliptic operators, Poincaré inequality, polygonal domain, Babuška–Kondratiev spaces, Sobolev spaces, manifolds with bounded geometry.

Mots-clés. Opérateurs fortement elliptiques, inégalité de Poincaré, domaine polygonal, espaces de Babuška–Kondratiev, espaces de Sobolev, variétés à géométrie bornée.

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1. Introduction

Let $\Omega_n \subset \mathbb{R}^d$ be a family of *bounded domains,* where n belongs to a certain index set $\mathfrak{I}$, and let us consider the family of *Poisson problems*

$$\Delta u_n = f_n, \quad \text{where } u_n \in H_0^1(\Omega_n) := \{u \in H^1(\Omega_n) \mid u_n = 0 \text{ on } \partial\Omega_n\}. \tag{1}$$

Ignoring, for the moment, the precise form of our assumptions, we have the following.

Theorem 1. *Under Assumption 3, there exists $\eta_U > 0$ with the following property: if $a < \eta_U$ and $a \le 1$, then there exists $C_a > 0$ such that, for any $n \in \mathfrak{I}$ and any $f_n \in L^2(\Omega_n)$, the solutions u_n of the Poisson problems* (1) *satisfy*

$$\|u_n\|_{H^{1+a}(\Omega_n)} \le C_a \|f_n\|_{L^2(\Omega_n)}.$$

For a single domain, it is well known that one can find η_U and C_a with these properties. The main point of the above result is therefore that the bounds η_U and C_a are the same for all the domains Ω_n. The proof of this result and the bound η_U are obtained from a similar result in weighted (or *Babuška–Kondratiev*) spaces, namely Theorem 4, which holds also for higher regularity spaces. Under the additional Assumption 5, we also provide an explicit estimate for η_U (Theorem 6). Families of Poisson problems were also studied by Antoine Henrot and Michel Pierre in their book [11], obtaining various estimates of a different kind.

The Assumptions 3 and 5 appearing in our main results are non-trivial. Thus, in order to prove that our theorems are not empty, we illustrate them with a concrete example for $\mathfrak{I} = \mathbb{N} \cup \{\infty\} = \{1, 2, \ldots\} \cup \{\infty\}$ as follows. First, for any open subset $\omega \subset S^{d-1}$ of the unit sphere in $\mathbb{R}^d$, we let λ_ω be the smallest eigenvalue of (minus) the Dirichlet Laplacian $-\Delta' := -\Delta_{S^{d-1}}$ on $H_0^1(\omega)$ and $\mathbb{R}_+\omega$ be the (open, straight) cone with *spherical cap* ω. If $p \in \partial\Omega$ is such that there exists a neighborhood G of p such that $G \cap \Omega = G \cap [p + (\mathbb{R}_+\omega_p)]$ for some $\omega_p \subset S^{d-1}$, then we say that p is a *straight conical point of* Ω. A point $p \in \partial\Omega$ will be called a *Diff-conical point* if there exists a neighborhood G of p such that the pair $(G, G \cap \Omega)$ is diffeomorphic to the pair $(B_\delta(0), B_\delta(0) \cap \mathbb{R}_+\omega_p)$ [15]. It is possible that more general classes of "conical points" can be used, but we do not address this question in this paper.

Let us assume then that we are given an arbitrary (bounded) domain $\Omega_\infty \subset \mathbb{R}^d$ with N *straight conical points,* let $\omega_k \subset S^{d-1}$ be the spherical cap domains associated to its conical points (as explained in the previous paragraph). We then construct an explicit sequence $\Omega_1 \supset \Omega_2 \supset \cdots \supset \Omega_n \supset \cdots$ of *smooth* domains that "converges" to Ω_∞ such that the family $\{\Omega_n\}_{n \in \mathfrak{I} = \mathbb{N} \cup \{\infty\}}$ satisfies our Assumptions 3 and 5 and for which Theorem 1 is true with $\eta_U = \min_k\{\lambda_{\omega_k}\}$, $k = 1, \ldots, N$. Informally, this amounts to "rounding off" the conical points of Ω_∞. This construction generalizes to domains with Diff-conical points, but is significantly more involved than in the case of straight conical points (at least in dimension ≥ 3).

If $d = 2$ and if Ω_∞ is a *straight* polygonal domain, then our proof is complete (that is, all the details are available either here or in the preprint [8]). In this case, let α_{MAX} be the maximum angle of Ω_∞. Then we can take $\eta_U = \frac{\pi}{\alpha_{\mathrm{MAX}}} = \min_k\{\lambda_{\omega_k}\}$ and this value of η_U is known to be optimal. This result extends the results obtained in [6,12] for a very specific rounding procedure.

Theorem 1 is based on several other results that are interesting on their own. Our first such result is in terms of weighted (or "Babuška–Kondratiev") spaces. (These spaces were studied, for instance, in [1,2,4,7,13–15].) We let ∇^M be the Levi-Civita connection on tensors on M (it acts as the de Rham differential on functions). We shall write $\nabla = \nabla^M$ when there is no danger of confusion, as in the next definition.

Definition 2. *Let G be an open subset of a Riemannian manifold* (M, g) *and* $f: G \to (0,\infty)$ *be a measurable function. The* Babuška–Kondratiev space associated to G, f, $m \in \mathbb{Z}_+ := \{0, 1, 2, \dots\}$, and $a \in \mathbb{R}$, *is defined by*

$$\mathcal{K}_a^m(G; f) := \Big\{ u\colon G \to \mathbb{C} \;\Big|\; f^{k-a} \nabla^k u \in L^2(G; T^{*\otimes k}M, g),\ k \le m \Big\}.$$

For $f = 1$ and $G \subset M = \mathbb{R}^d$, we recover the usual Sobolev spaces: $\mathcal{K}_a^m(G; 1) = H^m(G)$. We will make this choice (i.e., weight $f = 1$) if there are no conical points, a case that corresponds to $N = 0$ in the Assumption 3 below (our result is new even in this case). By "a domain" we will mean an *open, connected, bounded set.* We let $\mathrm{dist}_M(x, y)$ denote the geodesic distance between x and y in the Riemannian manifold M. For our construction to yield the desired result, we shall need a number of assumptions, the first group of which follows next.

Assumption 3. We are given $N \in \mathbb{Z}_+$, $R \in (0, 1]$, and an index set $\mathfrak{I}$ such that:

(i) For each $n \in \mathfrak{I}$, the set $\Omega_n \subset \mathbb{R}^d$ is open, $\mathcal{V}_n = \{p_{1n}, p_{2n}, \dots, p_{Nn}\} \subset \mathbb{R}^d \smallsetminus \Omega_n$, and $\bigcup_{n \in \mathfrak{I}} \Omega_n$ is a bounded subset of $\mathbb{R}^n$. (Thus the sets $\mathcal{V}_n$ are empty precisely when $N = 0 \in \mathbb{Z}_+$.)

(ii) $R \le \inf_{n \in \mathfrak{I}, i \ne j} |p_{in} - p_{jn}|$, with $R = 1$ if $N = 0$.

(iii) $\chi\colon [0, \infty) \to [0, R/6]$ is a fixed smooth function such that $\chi(t) = t$, for $t \le R/8$, $0 \le \chi'(t) \le \chi(t)/t$ for $t > 0$, and $\chi'(t) = 0$, for $t \ge R/5$.

(iv) $\phi_n(x) := \chi\big(\mathrm{dist}_{\mathbb{R}^d}(x, \mathcal{V}_n)\big)$ and $\widehat{g}_n := \phi_n^{-2}\, \mathrm{d}x^2$.

(v) Let $\phi = \phi_n$ on $\{n\} \times \mathbb{R}^d$ and $\widehat{g} := \phi^{-2}\, \mathrm{d}x^2$, then $(U, \widehat{g}) := \bigcup_{n \in \mathfrak{I}} (\{n\} \times \Omega_n, \widehat{g}_n)$ is *a manifold with boundary and relative bounded geometry* (Definition 7).

(vi) If $d = 2$, we also assume that U has *finite width* (Definition 11).

The function $\phi_n\colon \mathbb{R}^d \to [0, R/6]$ is thus a *smooth function* equivalent (bounded quotients from above and away from zero) to $|x - \mathcal{V}_n| := \mathrm{dist}_{\mathbb{R}^d}(x, \mathcal{V}_n)$, the distance function to $\mathcal{V}_n$ in $\mathbb{R}^n$. (The roles of (ii) and (iii) is to ensure that ϕ_n is smooth outside $\mathcal{V}_n$.) We endow $\mathfrak{I}$ with the discrete topology.

We will consider the *Laplacian* Δ on $\mathbb{R}^d$, acting on various function spaces. It will always be defined on functions vanishing at the boundary, that is, we consider the *Dirichlet Laplacian* Δ_D. In addition to this index "D," we may decorate the Laplacian with some additional indices that will specify its domain (as in the next theorem). On U, we shall consider the weighted spaces $\mathcal{K}_a^m(U; \phi)$ associated to the flat metric and the weight ϕ (recall that $\phi(n, x) = \phi_n(x) = \chi\big(\mathrm{dist}_{\mathbb{R}^d}(x, \mathcal{V}_n)\big)$ on $\{n\} \times \Omega_n$) and that $\mathcal{K}_a^m(U; \phi) := \big\{ u\colon U \to \mathbb{C} \;\big|\; \phi^{|\alpha| - a} \partial^\alpha u \in L^2(U; \mathrm{d}x^2),\ |\alpha| \le m \big\}$. A first step in the proof of Theorem 1 is the following theorem, in which we consider arbitrary regularity Sobolev exponents $m \in \mathbb{Z}_+$.

Theorem 4. *Under Assumption 3, there exists* $\eta_U > 0$ *such that, for every* $|a| < \eta_U$ *and every* $m \in \mathbb{Z}_+$, *the Dirichlet Laplacian* Δ_D *defines an isomorphism*

$$\Delta_{D,m,a}\colon \mathcal{K}_{a+1}^{m+1}(U; \phi) \cap \big\{ u\big|_{\partial U} = 0 \big\} \longrightarrow \mathcal{K}_{a-1}^{m-1}(U; \phi).$$

By decomposing the Hilbert spaces $\mathcal{K}_a^m(U; \phi) \simeq \bigoplus_{n \in \mathfrak{I}} \mathcal{K}_a^m(\Omega_n; \phi_n)$ (Hilbert-space direct sum), we obtain estimates that are *independent of* $n \in \mathfrak{I}$ *for the Laplacians on the domains* Ω_n. This observation and an interpolation argument then give Theorem 1. To obtain an estimate on the bound η_U of the last two theorems, we need an additional assumption, which, unlike Assumption 3, was not considered in [8]. Let $B_r(x)$ denote the open ball of radius r centered at some x in some given metric space.

Assumption 5. If $N > 0$, then we further assume that:

(i) there exist $\delta > R$ and domains $\omega_{kn} \subset S^{d-1}$, $\overline{\omega}_{kn} \ne S^{d-1}$, such that, for all $n \in \mathfrak{I}$ and $1 \le k \le N$, $\Omega_n \cap B_\delta(p_{kn}) \subset p_{kn} + \mathbb{R}_+ \omega_{kn}$;

(ii) $\lambda_{\min} := \inf_{k,n} \{\lambda_{\omega_{kn}}\} > 0$.

The set $p_{kn} + (\mathbb{R}_+\omega_{kn})$ is, of course, nothing but the translation of the cone $\mathbb{R}_+\omega_{kn}$ with basis ω_{kn}. We then have the following bound on η_U.

Theorem 6. *Under Assumptions 3 and 5, one can take $\eta_U = \lambda_{\min}$ in Theorems 1 and 4.*

Let Ω_∞ be a domain with N conical points and let $\omega_k = \omega_{k\infty}$ its associated spherical cap domains, $k = 1,\dots,N$, (that is, the spherical cap domains defining the tangent cones at the conical points). Our results are non-empty in the following sense (see Proposition 18 for details). Assume $N > 0$ and $\eta < \min_k\{\lambda_{\omega_k}\}$. Then there exists a decreasing sequence of smooth domains Ω_n, $n \in \mathbb{N}$, approaching Ω_∞ (so $\overline{\Omega}_\infty \subset \Omega_n$), such that our assumptions are satisfied for $\mathfrak{I} := \mathbb{N} \cup \{\infty\}$ and the family $\{\Omega_n\}_{n\in\mathbb{N}\cup\{\infty\}}$ *and, moreover,* $\lambda_{\min} \ge \eta$. Therefore, for this sequence, we can choose $\eta_U = \eta$ (hence, we can arrange so that η_U is close to $\min_k\{\lambda_{\omega_k}\}$). In case Ω_∞ has *straight* conical points, then we can further arrange that our sequence Ω_n is such that $\omega_{kn} = \omega_k$ for all n and, therefore, in this case, we can take $\eta_U = \min_k\{\lambda_{\omega_k}\}$. Consequently, if $d = 2$ and the limit domain Ω_∞ is a straight polygonal domain with angles α_k, then we can choose our approximating sequence Ω_n such that $\eta_U = \frac{\pi}{\alpha_{\mathrm{MAX}}}$, where $\alpha_{\mathrm{MAX}} := \max \alpha_k$ is the maximum angle of Ω_∞. It is known (and easy) that this value is optimal. If $N = 0$, then the spaces involved do not depend on a (they reduce to the usual Sobolev spaces) and the result is true for any $\eta_U > 0$ (but our result is still new and non-trivial even in this case).

2. Preliminary material

Our uniform estimates are obtained from related estimates on manifolds with bounded geometry and on their close cousins, the manifolds with boundary and relative bounded geometry. Our presentation follows [4]. Let (M, g) is a *smooth Riemannian manifold.* Recall that ∇^M denotes its Levi-Civita connection and $r_{\mathrm{inj}}(M) \ge 0$ denotes its injectivity radius.

Definition 7. *A (boundaryless) Riemannian manifold (M, g) is said to have* bounded geometry *if $r_{\mathrm{inj}}(M) > 0$ and its curvature $R^M := (\nabla^M)^2$ and all its covariant derivatives $(\nabla^M)^k R^M$ are bounded.*

Example 8. Let $\mathcal{V}_n = \{p_{1n}, p_{2n},\dots, p_{Nn}\} \subset \mathbb{R}^d$, $\phi_n(x) := \chi\big(\mathrm{dist}_{\mathbb{R}^d}(x,\mathcal{V}_n)\big)$, $\widehat{g}_n := \phi_n^{-2}\,\mathrm{d}x^2$, $\phi = \phi_n$ on $\{n\}\times\mathbb{R}^d$, $n \in \mathfrak{I}$, and $\widehat{g} := \phi^{-2}\,\mathrm{d}x^2$ be as in Assumption 3. Then the *disjoint union*

$$(\widehat{M}, \widehat{g}) := \bigcup_{n\in\mathfrak{I}} \{n\} \times (\mathbb{R}^d \smallsetminus \mathcal{V}_n, \widehat{g}_n)$$

has bounded geometry [8]. Note that the set U of Assumption 3 is contained isometrically as an open subset of $\widehat{M}$ and that ϕ_n is *smooth* on $\mathbb{R}^d \smallsetminus \mathcal{V}_n$.

Let $H \subset M$ be a hypersurface possessing a unit normal vector field ν. We let

$$\exp^\perp(x, t) := \exp_x^M(t\nu_x), \tag{2}$$

where $\exp_x$ is the geodesic exponential map (defined on a neighborhood of the zero vector in T_xM). Recall that the *second fundamental form* II^H of H in M is defined by $\mathrm{II}^H(X,Y)\nu := \nabla^M_X Y - \nabla^H_X Y$.

Definition 9. *We say that H is a* bounded geometry hypersurface in (M, g) *if*

(i) *(M, g) has bounded geometry and H is a closed hypersurface of M;*
(ii) *all covariant derivatives $(\nabla^H)^k \mathrm{II}^H$, $k \ge 0$, are bounded;*
(iii) *there is $\epsilon > 0$ such that $\exp^\perp\colon H \times (-\epsilon,\epsilon) \to M$ is a diffeomorphism onto its image (see (2)).*

We are ready now to recall the definition of a central concept in this paper.

Definition 10. *We shall say that M_0* is a manifold with boundary and relative bounded geometry *if M_0 is isometrically contained in a (boundaryless) Riemannian manifold M with bounded geometry such that ∂M_0 is a bounded geometry hypersurface in M.*

Manifolds with boundary and *relative* bounded geometry were introduced in [18] (they were called "manifolds with boundary and bounded geometry" there; the definition given here is from [4]). For our results, when $d = 2$, we shall also need the "finite width" condition of [4].

Definition 11. *Let (M_0, g) be a Riemannian manifold with boundary ∂M_0 and relative bounded geometry. If the function $M_0 \ni x \mapsto \mathrm{dist}_{M_0}(x, \partial M_0)$ is bounded on M_0, then we shall say that (M_0, g) has* finite width.

Next, to recall the definition of Sobolev spaces, we allow the Riemannian manifold (M, g) to have boundary in this paragraph. We endow all vector bundles derived from the tangent bundle with the Levi-Civita connection, denoted generically ∇. Then

$$W^{m,p}(M) := \{u \mid \nabla^k u \in L^p(M; T^{*\otimes k}M, g),\ k \le m\}, \quad m \in \mathbb{Z}_+, \tag{3}$$

and $W^{\infty,p}(M) := \bigcap_{m\in\mathbb{Z}_+} W^{m,p}(M)$, $p \in [1,\infty]$. We let $H^m(M) = W^{m,2}(M)$, $H^1_0(M) := \{u \in H^1(M) \mid u|_{\partial M} = 0\}$, and $H^{-1}(M) := H^1_0(M)^*$, as usual. See also [2,10,13] and the references therein. See also [5,16,17] for the classical theory.

To complete our review of background material, recall that a second order differential operator P is *coercive* on $V \subset H^1(M)$ if there exists $\gamma_P > 0$ such that

$$\mathrm{Re}(Pu, u) \ge \gamma_P \|u\|_{H^1(M)}, \quad u \in V. \tag{4}$$

The Lax–Milgram Lemma and the regularity results on manifolds with bounded geometry then give the following result (see [3,4]).

Theorem 12. *Let M_0 be a Riemannian manifold with boundary and relative bounded geometry (Definition 7). Let P be a differential operator on M_0 with coefficients in $W^{\infty,\infty}(M_0)$ that is coercive on $H^1_0(M_0)$. Then P induces isomorphisms*

$$P\colon H^{m+1}(M_0) \cap H^1_0(M_0) \longrightarrow H^{m-1}(M_0), \quad m \in \mathbb{Z}_+.$$

3. Uniform estimates

This section is similar to [8, Section 3], to which we refer for the proofs omitted here. We have the following result that is classical, see [1–3,8] for further details, references, and extensions.

Lemma 13. *Let and $U := \bigcup_{n\in\mathfrak{I}} \{n\} \times \Omega_n \subset \widehat{M} := \bigcup_{n\in\mathfrak{I}} \{n\} \times (\mathbb{R}^d \smallsetminus \mathcal{V}_n)$ be as in Assumption 3, as usual. In particular, we endow both U and $\widehat{M}$ with the metric $\widehat{g} = \phi^{-2}\,\mathrm{d}x^2$, where $\phi \in \mathscr{C}^\infty(\widehat{M})$ is given by $\phi(n,x) = \chi\big(\mathrm{dist}_{\mathbb{R}^d}(x, \mathcal{V}_n)\big)$. Then:*

(i) *multiplication by ϕ^b induces isomorphisms*

$$\phi^b\colon \mathcal{K}^m_a(U;\phi) \longrightarrow \mathcal{K}^m_{a+b}(U;\phi),$$

whose norms depend continuously on a and b;

(ii) *the assumption that $(U, \widehat{g})$ is a manifold with boundary and relative bounded geometry implies that*

$$H^m(U;\widehat{g}) = \mathcal{K}^m_{d/2}(U;\phi);$$

(iii) *moreover, we have $\mathcal{K}^m_a(U;\phi) \simeq \bigoplus_{n\in\mathfrak{I}} \mathcal{K}^m_a(\Omega_n;\phi_n)$ (Hilbert space direct sum).*

We shall need the following regularity type result that is the reformulation of a regularity result from [3,4] after the identification in (ii) above.

Proposition 14. *Let D be a second order, strongly elliptic differential operator with smooth coefficients on a domain $G \subset (M, g)$, let $f\colon G \to \mathbb{R}_+$ be an admissible weight, and let*

$$D_m := D\colon \mathcal{K}^{m+1}_{a+1}(G;f) \cap \{u|_{\partial G} = 0\} \longrightarrow \mathcal{K}^{m-1}_{a-1}(G;f).$$

If $(G, f^{-2}g)$ is a domain with boundary and bounded geometry and D_0 is invertible then, for any $m \in \mathbb{N}$, D_m is also invertible.

The following lemma is the case $a = 0$ of Theorem 4. For $d = 2$, it was proved in [8] using the finite width condition. For $d > 2$, it is proved using Hardy's inequality instead of the finite width condition (see [9] for details).

Lemma 15. *If Assumption 3 holds, then, for any $m \in \mathbb{Z}_+$, the Dirichlet Laplacian induces isomorphisms*

$$\Delta_{D,m} := \Delta\colon \mathcal{K}_1^{m+1}(U;\phi) \cap \left\{u\big|_{\partial U} = 0\right\} \longrightarrow \mathcal{K}_{-1}^{m-1}(U;\phi).$$

Proposition 14 and a perturbation argument applied to the continuous family $\phi^a \Delta_D \phi^{-a}\colon \mathcal{K}_1^1(U;\phi) \cap \left\{u\big|_{\partial U} = 0\right\} \to \mathcal{K}_{-1}^{-1}(U;\phi)$ then yield Theorem 4, which, in turn, yields Theorem 1, as in [8].

4. The bound η_U

Recall that S^{d-1} denotes the unit sphere in $\mathbb{R}^d$. Let $\rho\colon \mathbb{R}^d \to [0,\infty)$ denote the distance to the origin, as usual, and let $(\rho, x') \in (0,\infty) \times S^{d-1}$ be the *generalized spherical coordinates* of the point $x = \rho x' \in \mathbb{R}^d \smallsetminus \{0\}$. The proof of the following lemma is a simple calculation (see [9] for its proof and further details).

Lemma 16. *Let χ be as in Assumption 3 and let $|\cdot|$ denote the norm on $\mathbb{R}^d$.*

(i) *Let $\xi := \chi(\rho)^{-1}\nabla\chi(\rho)\colon \mathbb{R}^d \smallsetminus \{0\} \to \mathbb{R}^d$. Then $|\xi| \le \frac{1}{\rho}$ and, for all $a \in \mathbb{C}$,*

$$\operatorname{Re}\Big[\nabla\big(\chi(\rho)^a u\big)\cdot\nabla\big(\chi(\rho)^{-a}u\big)\Big] = |\nabla u|^2 - |au\xi|^2, \quad u \in \mathcal{C}_c^\infty\big(\mathbb{R}^d \smallsetminus \{0\}\big).$$

(ii) *The function $\chi(t)/t$ is non-increasing on* $(0,\infty)$.

Recall that λ_ω is the smallest eigenvalue of (minus) the spherical Laplacian $-\Delta' := -\Delta_{S^{d-1}}$ on $H_0^1(\omega)$. We next concentrate in the neighborhood of a singular point. We can assume that this singular point is the origin.

Lemma 17. *We use the notation of Lemma 16. Let $\omega \subset S^{d-1}$ be open and assume that $G \subset \mathbb{R}^d$ is a domain such that $G \cap B_t(0) \subset \mathbb{R}_+\omega$. Let χ be as in Assumption 3*(iii)*, as usual. Then, for every $u \in \mathcal{C}_c^\infty(G)$ and $a \in \mathbb{C}$, $|a|^2 \le \lambda_\omega$, we have*

$$\int_{G\cap B_t(0)} \left[|\nabla u|^2 - |au\xi|^2\right] \mathrm{d}x \ge \frac{\left(\lambda_\omega - |a|^2\right)\chi^2(t)}{t^2} \int_{G\cap B_t(0)} \frac{\left|u(x)\right|^2}{\chi(\rho)^2}\,\mathrm{d}x.$$

Proof. Let us replace G with $G \cap B_t(0)$. Let ∇' be the gradient on the sphere S^{d-1} and $v \in H_0^1(\omega)$. Then

$$\int_\omega |\nabla' v|^2\,\mathrm{d}x' \ge \lambda_\omega \int_\omega v^2\,\mathrm{d}x'.$$

We extend u with 0 outside G, we integrate and estimate the resulting integrals using the above inequality for $v = u$ on $\rho\omega$, for every fixed $\rho \in (0,t)$, as follows:

$$\begin{aligned}\int_G |\nabla u|^2\,\mathrm{d}x &= \int_G \left[(\partial_\rho u)^2 + \rho^{-2}|\nabla' u|^2\right] \mathrm{d}x\\ &\ge \int_0^t \int_\omega \rho^{-2}|\nabla' u|^2 \rho^{d-1}\,\mathrm{d}\rho\,\mathrm{d}x'\\ &\ge \lambda_\omega \int_0^t \int_\omega |u|^2 \rho^{d-3}\,\mathrm{d}\rho\,\mathrm{d}x'\\ &= \lambda_\omega \int_G \frac{|u|^2}{\rho^2}\,\mathrm{d}x.\end{aligned}$$

Finally, the last estimate and the inequality $|\xi| \le \rho^{-1}$ of Lemma 16 give

$$\int_G \left[|\nabla u|^2 - |au\xi|^2\right] \mathrm{d}x \ge \left(\lambda_\omega - |a|^2\right) \int_G \frac{|u|^2}{\rho^2}\,\mathrm{d}x \ge \frac{\left(\lambda_\omega - |a|^2\right)\chi^2(t)}{t^2} \int_G \frac{|u|^2}{\chi(\rho)^2}\,\mathrm{d}x.$$

□

We can now prove Theorem 6.

Proof of Theorem 6. We will use the notation of Assumption 5. In particular, $\lambda_{\min} := \inf_{k,n\in\mathfrak{I}}\{\lambda_{\omega_{kn}}\}$, for all $k = 1,\dots,N$ and $n \in \mathfrak{I}$. Let then $a \in (-\lambda_{\min}, \lambda_{\min})$. To prove the theorem, we need to show that

$$\Delta_a \colon \mathcal{K}^{m+1}_{a+1}(U;\phi) \cap \{u\big|_{\partial U} = 0\} \longrightarrow \mathcal{K}^{m-1}_{a-1}(U;\phi)$$

is invertible. By Proposition 14, we may assume that $m = 0$. Lemma 13 gives that Δ_a is similar to $\phi^{-a}\Delta_a\phi^a \colon \mathcal{K}^1_1(U;\phi) \cap \{u\big|_{\partial U} = 0\} \to \mathcal{K}^{-1}_{-1}(U;\phi)$. It is enough hence to show that the latter is invertible. To this end, we shall prove that it is coercive.

Using the direct sum decomposition $\mathcal{K}^s_a(U;\phi) \simeq \bigoplus_{n\in\mathfrak{I}} \mathcal{K}^s_a(\Omega_n;\phi_n)$ of Lemma 13, it is enough to prove that each

$$\phi_n^{-a}\Delta_{n,a}\phi_n^a \colon \mathcal{H} := \mathcal{K}^1_1(\Omega_n;\phi_n) \cap \{u\big|_{\partial\Omega_n} = 0\} \longrightarrow \mathcal{K}^{-1}_1(\Omega_n;\phi_n) = \mathcal{H}^*$$

is coercive on $\mathcal{H}$ with coercivity constant independent of n. To this end, we shall use Lemma 17 and the usual Poincaré inequality.

Let us fix then n and let $u \in \mathscr{C}^\infty_c(\Omega_n)$. Let us consider the vector field $\xi := \chi(\rho)^{-1}\nabla\chi(\rho)$ of Lemma 16 with the origin replaced with p_{kn}, for each $k = 1,\dots,N$. The resulting vector field is $\phi_n^{-1}\nabla\phi_n$ and has support contained in the union of the balls $B_\delta(p_{kn})$, where δ is as in Assumption 5 (the support condition is due to the fact that ϕ_n is constant outside the disjoint balls $B_\delta(p_{kn})$, $k = 1,\dots,N$). Lemma 16 gives $\operatorname{Re}\left[\nabla(\phi_n^a u)\cdot\nabla(\phi_n^{-a}u)\right] = |\nabla u|^2 - a^2|u\xi|^2$ on Ω_n.

For the given choice of a, let us choose also $\alpha > 1$ such that $|\alpha a| < \lambda_{\min}$, which is possible, since we have assumed that $|a| < \lambda_{\min}$. Part (i) of Assumption 5 allows us to use Lemma 17 with $t = \delta$ and $G = \Omega_n$ (after a translation such that p_{kn} is in the origin). Using also $\lambda_{\omega_{kn}} \geq \lambda_{\min}$ and $\phi = \phi_n = \chi(\rho)$ on $B_\delta(p_{kn})$, we obtain (setting $G_{kn} := \Omega_n \cap B_\delta(p_{kn})$)

$$\int_{G_{kn}} \left[|\nabla u|^2 - (\alpha a)^2|u\xi|^2\right] \mathrm{d}x \geq \frac{\left(\lambda_{\min} - (\alpha a)^2\right)\chi^2(\delta)}{\delta^2} \int_{G_{kn}} \frac{|u|^2}{\phi_n^2}\,\mathrm{d}x. \tag{5}$$

We also remark that the usual Poincaré inequality holds on the domains Ω_n with a constant C_{Poinc} independent of $n \in \mathfrak{I}$, because $\bigcup_{n\in\mathfrak{I}}\Omega_n$ is bounded by Assumption 3(i). Therefore, for each $n \in \mathfrak{I}$, we obtain the inequality

$$\int_{\Omega_n} |\nabla u|^2\,\mathrm{d}x \geq C_{\mathrm{Poinc}} \int_{\Omega_n} |u|^2\,\mathrm{d}x. \tag{6}$$

Because $|a| < \lambda_{\min}$, we can assume that we have chosen $\alpha > 1$ such that $(\alpha a)^2 < \lambda_{\min}$, we multiply (5) by α^{-2} and (6) by $1 - \alpha^{-2}$, and add them up to obtain (recall that $G_{kn} := \Omega_n \cap B_\delta(p_{kn})$)

$$\begin{aligned}
\operatorname{Re}(\phi_n^{-a}\Delta\phi_n^a u, u) &= \operatorname{Re}\int_{\Omega_n}\left[\nabla(\phi_n^a u)\cdot\nabla(\phi_n^{-a}u)\right]\mathrm{d}x \\
&= \int_{\Omega_n}\left[|\nabla u|^2 - a^2|u\xi|^2\right]\mathrm{d}x \\
&\geq \sum_{k=1}^N \alpha^{-2}\int_{G_{kn}}\left[|\nabla u|^2 - (\alpha a)^2|u\xi|^2\right]\mathrm{d}x + (1-\alpha^{-2})\int_{\Omega_n}|\nabla u|^2\,\mathrm{d}x \\
&\geq \sum_{k=1}^N \frac{(\alpha^{-2}\lambda_{\min} - a^2)\chi^2(\delta)}{\delta^2}\int_{G_{kn}}\frac{|u|^2}{\phi_n^2}\,\mathrm{d}x + (1-\alpha^{-2})C_{\mathrm{Poinc}}\int_{\Omega_n}|u|^2\,\mathrm{d}x \\
&\geq c\int_{\Omega_n}\frac{|u|^2}{\phi_n^2}\,\mathrm{d}x
\end{aligned} \tag{7}$$

for some $c > 0$ independent of n. This completes the proof. □

5. Approximating a domains with conical points

Let $\Omega_\infty \subset \mathbb{R}^d$ be a bounded domain with N conical points $\mathcal{V}_\infty = \{p_1, p_2, \ldots, p_N\}$. This means that Ω_∞ has a smooth boundary, except at the points $p_{k\infty} = p_k$, and that, near each of these points, our domain is approximated to second order by a smooth cone. More precisely, for each $k = 1, \ldots, N$, we assume that near p_k, Ω_∞ is approximated by the cone $p_k + \mathbb{R}_+\omega_k$, where $\omega_k \subset S^{d-1}$. The cone $p_k + \mathbb{R}_+\omega_k$ is thus the tangent cone to our domain at the point p_k; the domain ω_k will be called the *spherical cap domain* associated to p_k. If Ω_∞ coincides with $p_k + \mathbb{R}_+\omega_k$ near p_k, we shall say that p_k is a *straight conical point* of Ω_∞. Given the domain $\Omega_\infty \subset \mathbb{R}^d$ with N conical points, we will construct an explicit sequence of smooth, bounded domains Ω_n, $n \in \mathbb{N}$, such that the family $\{\Omega_n\}_{n\in\mathbb{N}\cup\{\infty\}}$ satisfies all conditions of Assumptions 3 and 5, with suitable ω_{kn} and $\eta_U = \lambda_{\min} := \min_{k,n\in\mathfrak{I}}\{\lambda_{\omega_{kn}}\}$ close to $\min_k\{\lambda_{\omega_k}\}$, as follows.

Proposition 18. *Let Ω_∞ be a domain with N conical points and let $\omega_k = \omega_{k\infty}$ its associated spherical cap domains, $k = 1, \ldots, N$. We assume that $\overline{\omega}_k \neq S^{d-1}$ for all k. Then there exists a decreasing sequence of smooth, bounded domains Ω_n, $n \in \mathbb{N}$, such that Assumptions 3 and 5 are satisfied for $\mathfrak{I} := \mathbb{N}\cup\{\infty\}$ and the family $\{\Omega_n\}_{n\in\mathbb{N}\cup\{\infty\}}$. Moreover, using the notation of those assumptions, we can choose our sequence (Ω_n) such that $\overline{\Omega}_j \subset \Omega_n$ if $j, n \in \mathbb{N}\cup\{\infty\}$, $j > n$, ω_{kn} is independent of n for $n \in \mathbb{N}$, and $\lambda_{\min} := \min_{k,n\in\mathfrak{I}}\{\lambda_{\omega_{kn}}\}$ is close to $\min_k\{\lambda_{\omega_k}\}$ as follows.*

(a) *Let $\eta < \min_k\{\lambda_{\omega_k}\}$. Then we can choose our sequence (Ω_n) such that $\lambda_{\min} \geq \eta$.*

(b) *Let us assume that Ω_∞ has* straight *conical points. Then we can choose our sequence (Ω_n) such that $\omega_{kn} = \omega_k$, and hence $\lambda_{\min} = \min_k\{\lambda_{\omega_k}\}$.*

Therefore, for the sequence (Ω_n) constructed in the above proposition, we can choose $\eta_U = \eta$ as close as desired to $\min_k\{\lambda_{\omega_k}\}$. In the case (b), that is, if Ω_∞ has *straight* conical points, we obtain a better result, in the sense that for the sequence (Ω_n) constructed in the above proposition we can choose $\eta_U = \lambda_{\min} = \min_k\{\lambda_{\omega_k}\}$. If, furthermore, $d = 2$ and the limit domain Ω_∞ is a *straight polygonal domain* with angles α_k, then $\lambda_{\omega_k} = \frac{\pi}{\alpha_k}$. Let $\alpha_{\mathrm{MAX}} := \max\{\alpha_k\}$ be the maximum angle of Ω_∞, then we can take $\eta_U = \frac{\pi}{\alpha_{\mathrm{MAX}}}$. Figure 1 shows an example of such a sequence if Ω_∞ is a rectangle in the plane.

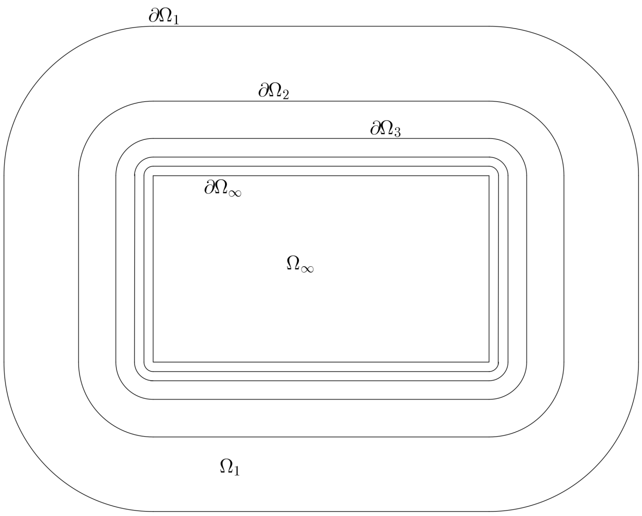

Figure 1. A sketch of the first five domains (Ω_n, $n = 1, \ldots, 5$) when the limit domain Ω_∞ is a rectangle.

The construction is slightly simpler if $d = 2$ and Ω_∞ is a *straight polygonal domain*, so we shall start with this case. The most important step is to construct Ω_1, the first of our domains. Let $t > 0$ and consider, for each edge of Ω_∞ a straight line parallel to it and at distance t on the exterior side of that edge. The intersection points of two successive such lines will be the points p_{k1} (we take the lines and points in cyclical order). We choose t small enough such for each k, the point p_{k1} is at distance $< \frac{R}{16}$ to $p_{k\infty}$ and such that the resulting polygon $p_{11}p_{21}\cdots p_{N1}$ contains the given polygon Ω_∞ and does not have self-intersections. For each $k \in \{1,2,\dots,N\}$, let Q_k be a point on the edge $[p_{(k-1)1}p_{k1}]$ and Q'_k be a point on $[p_{k1}p_{(k+1)1}]$ such that

$$c\,\mathrm{dist}_{\mathbb{R}^d}(p_{k1}, p_{k\infty}) = \mathrm{dist}_{\mathbb{R}^d}(p_{k1}, Q_k) = \mathrm{dist}_{\mathbb{R}^d}(p_{k1}, Q'_k) \neq 0, \tag{8}$$

for some $c \le 1$ fixed, and hence $Q_k \neq Q'_k$. See Figure 2. (Recall that we let $p_{(N+1)1} = p_{11}$ and $p_{01} = p_{N1}$, by the cyclical order assumption.) We then join Q_k and Q'_k with a smooth curve γ_{k1} contained in the polygon $p_{11}p_{12}\cdots p_{N1}$, inside the disk of radius $\mathrm{dist}_{\mathbb{R}^d}(p_{k1}, p_{k\infty})$ with center p_{k1}, but not containing p_{k1} and separating p_{k1} from Ω_∞. The resulting curve (formed by the curves γ_{k1} and parts of the edges $[p_{(k-1)1}p_{k1}]$) is the boundary of a smooth domain Ω_1. Then Ω_1 is at positive distance to $\mathcal{V}_1$ and $\Omega_\infty \subset \Omega_1$. We also obtain that the smooth curve γ_{k1} that we have included is at distance $\le R/8$ to p_{k1} (so that $\phi_1(\rho) = \rho$ on this curve γ_{k1}). Next, to define Ω_n, for $n > 1$, we proceed similarly, but replacing t with $2^{-n}t$ and making sure that, for each k, the curve γ_{kn} is homothetic to γ_{1k} with a ratio of 2^{-n}. To check that U has relative bounded geometry, it is enough to check that its boundary ∂U is a bounded geometry hypersurface in $\widehat{M}$. Near the curves γ_k (and their homothetic images), the conditions are satisfied because there homotheties are isometries of $(\widehat{M}, \widehat{g})$. Away from these curves, the boundaries $\partial\Omega_n$ consist of straight lines, where the conditions are again satisfied (this is a very particular case of a result in [8], namely Lemma 4.9).

The proof in d dimensions for domains with straight conical points is similar, but, in addition to the curves γ_{kn} and the points at distance t to the boundary, we need to include an intermediate region smoothly joining these parts of the boundary. If we further drop the assumption of *straight conical points*, we again proceed similarly, but we need to first choose domains ω'_k containing $\overline{\omega}_k$ with $\lambda_{\omega'_k} \ge \eta$. Our example provided a sequence approximating our domain Ω_∞ *from the outside*. A similar construction can be devised that will approximate our domain *from the inside*.

The domains obtained in Proposition 18, that is, our domains Ω_n and their "limit" Ω_∞ satisfy Assumptions 3 and 5, so they also satisfy our three theorems, Theorems 1, 4, and 6.

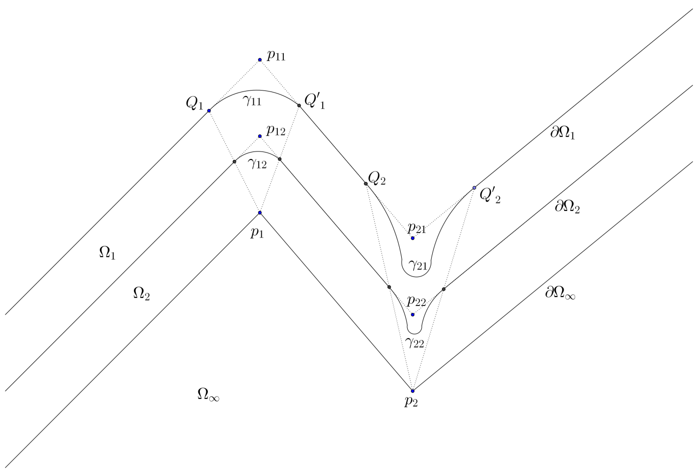

Figure 2. Some local details of our construction for $n = 1$ and $n = 2$.

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Declaration of interests

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